TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 11, Pages 4515–4543
S 0002-9947(99)02153-4
Article electronically published on July 19, 1999

DENSITIES OF IDEMPOTENT MEASURES AND LARGE DEVIATIONS

MARIANNE AKIAN

ABSTRACT. Considering measure theory in which the semifield of positive real numbers is replaced by an idempotent semiring leads to the notion of idempotent measure introduced by Maslov. Then, idempotent measures or integrals with density correspond to supremums of functions for the partial order relation induced by the idempotent structure. In this paper, we give conditions under which an idempotent measure has a density and show by many examples that they are often satisfied. These conditions depend on the lattice structure of the semiring and on the Boolean algebra in which the measure is defined. As an application, we obtain a necessary and sufficient condition for a family of probabilities to satisfy the large deviation principle.

INTRODUCTION

A probability or a positive measure is in some loose sense a continuous morphism from a Boolean σ-algebra $(\mathcal{A}, \cup, \cap)$ of subsets of some set Ω, to the semifield $(\mathbb{R}^+, +, \times)$. If we replace $(\mathbb{R}^+, +, \times)$ by an idempotent semiring[1] (or dioid) [6] $(\mathbb{D}, \oplus, \otimes)$, we obtain the notion of idempotent measure. This notion has been introduced by Maslov in [20] where idempotent integrals were also constructed.

If we consider the particular semifield $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$, measure or probability theory (resp. Wiener processes, linear second order elliptic equations) is replaced by optimization theory (resp. Bellman processes, particular Bellman equations) and some of the notions may be transferred from the first domain to the second one. Illustrations and applications of this correspondence may be found in Maslov [20], Maslov and Samborskii [22], Del Moral, Thuillet, Rigal and Salut [12], Del Moral [11], Quadrat [32], Bellalouna [7], Akian, Quadrat and Viot [3, 4], and Akian [1, 2]. Moreover, idempotent $\mathbb{R}_{\max}$-measures are particular fuzzy measures (see for instance Pap [27, 28, 29]).

Large deviations provide another correspondence between classical and idempotent probabilities: $\mathbb{R}_{\max}$-probabilities appear not only as formal analogues (by a change of semiring), but also as "limits" of classical probabilities. The notion of capacity introduced in Norberg and Vervaat [24], O'Brien and Vervaat [26] and

Received by the editors June 16, 1995 and, in revised form, April 17, 1997.
1991 *Mathematics Subject Classification.* Primary 28B15, 49J52; Secondary 06B35, 60F10.
Key words and phrases. Idempotent semiring, dioid, max-plus algebra, continuous lattice, idempotent measure, optimization, large deviations.

[1] A *semiring* $(\mathbb{D}, \oplus, \otimes)$ satisfies all the properties of a ring except the existence of opposites for the $\oplus$ law, that is $\oplus$ is commutative, associative, has a neutral element $\mathbb{0}$; $\otimes$ is associative, has a unit $\mathbb{1}$, distributes over $\oplus$ and $\mathbb{0}$ is absorbing ($\mathbb{0} \otimes a = a \otimes \mathbb{0} = \mathbb{0}$ for all $a \in \mathbb{D}$). A semiring is *idempotent* if the $\oplus$ law is idempotent ($a \oplus a = a$ for all $a \in \mathbb{D}$), it is *commutative* if the $\otimes$ law is commutative, and it is a semifield if nonzero elements have an inverse for the $\otimes$ law.

O'Brien [25], which is similar to that of Choquet [10], brings together regular probabilities and sup-measures (defined in [26, 25]). Then, the large deviation principle is nothing but a weak convergence of capacities, where the limit is a sup-measure. Sup-measures coincide with $\mathbb{R}_{\max}$-probabilities with density. Probability notions with respect to sup-measures, also called deviabilities, are studied in Puhalskii [30, 31].

Whereas Maslov treats (in [20]) idempotent measure theory in general ordered (not necessarily idempotent) semirings $\mathbb{D}$, and general measure spaces Ω, some improvements may be done. First, at least for the construction of measures, $\mathbb{D}$ does not need to be a metric space but only a dually continuous lattice, which is an order property. Second, in the idempotent case, the existence of a density has not been clarified. This point has been neglected in most of the studies on this domain, except in [18, 19, 17, 21], where Kolokoltsov and Maslov prove the existence of a "density" for linear forms, which in some particular cases implies the existence of a density for idempotent measures. The present paper is essentially devoted to this last problem. In particular, using continuous lattices techniques, we find general conditions for an idempotent measure to have a density.

Let us consider again the dioid $\mathbb{R}_{\max}$ with zero (neutral element for the "addition" max) $\mathbb{0} = -\infty$ and unit (neutral element for the "multiplication" $+$) $\mathbb{1} = 0$. Addition corresponds to finite maximization, "integration" corresponds to taking infinite supremum. The equivalent of the Lebesgue measure on $(\Omega, \mathcal{A})$, where $\Omega = \mathbb{R}$ and $\mathcal{A}$ is the Borel sets algebra, is the "uniform idempotent measure" $\lambda(A) = \mathbb{1}$ for all $A \subset \Omega$, $A \neq \emptyset$ and $\lambda(\emptyset) = \mathbb{0}$. Then, the "integral" of a continuous function f is $\lambda(f) = \sup_{\omega\in\Omega} f(\omega)$. Now, the function $\mathbb{K}(A) = \sup_{\omega\in A} c(\omega)$ defines an idempotent measure with density c with respect to the "Lebesgue measure". The integral of a measurable function f with respect to the measure $\mathbb{K}$, as defined by Maslov, is $\mathbb{K}(f) = \oint_\Omega f(\omega) \otimes \mathbb{K}(d\omega) = \sup_{\omega\in\Omega} f(\omega) + c(\omega)$. Thus, the integral of a function with respect to a measure with density has a simple expression. We may then ask if there exist, as in classical measure theory, (interesting) measures which have no density. As a first answer, let us note that the most natural measures without density in classical measure theory have a density in $\mathbb{R}_{\max}$. Indeed, the upper semi-continuous (u.s.c.) function

$$\delta_m(\omega) = \mathbb{1} \text{ if } \omega = m, \quad \delta_m(\omega) = \mathbb{0} \text{ otherwise}$$

is the density of the "Dirac measure" at point m:

$$\delta_m(f) = \sup_{\omega\in\Omega} f(\omega) + \delta_m(\omega) = f(m).$$

However, we may exhibit the following measure without density:

$$\mathbb{K}(A) = \operatorname*{ess\,sup}_{\omega\in A} c(\omega),$$

where c is a continuous function and the essential supremum is taken with respect to the (classical) Lebesgue measure. This measure satisfies the conditions of Definition 2.5 below on $(\Omega, \mathcal{A})$. Since $\mathbb{K}(\{\omega\}) = \mathbb{0} = -\infty$ for all $\omega \in \Omega$, $\mathbb{K}$ has no density. However, the restriction of $\mathbb{K}$ to the algebra of open sets has c as density. Then, $\oint_\Omega f(\omega) \otimes \mathbb{K}(d\omega) = \sup_{\omega\in\Omega} f(\omega) + c(\omega)$ for any lower semi-continuous (l.s.c.) function f [20]. The nonexistence of a density of $\mathbb{K}$ on the entire algebra of Borel sets is in general not relevant and every measure seems to have a density in a sufficiently large algebra of subsets.

From the previous examples, we see that the order relation $\leq$ plays an important role in the semiring $\mathbb{R}_{\max}$. More generally, if $(\mathbb{D}, \oplus, \otimes)$ is an idempotent semiring, the idempotent law $\oplus$ defines a partial order relation $\preceq$ such that $(\mathbb{D}, \preceq)$ is a sup-semilattice. Properties of measures and integrals are related with lattice properties of $\mathbb{D}$ that we will use throughout this paper. We thus begin by recalling and extending in Section 1 definitions and properties of continuous lattices. We follow the presentation of Gierz, Hoffman, Keimel, Lawson, Mislove and Scott [15], up to subsidiary extensions. Idempotent measures are introduced in Section 2. In Section 3, we prove that any idempotent measure on a suitable algebra $\mathcal{A}$ of subsets of a space Ω has necessarily a density. This includes Polish spaces with the algebra $\mathcal{A}$ of their open sets. For the proof, we construct the maximal extension of the idempotent measure to the algebra of all subsets of Ω and prove that the value of this extension on singletons is a density of the initial measure. In Section 4, we recall in a general context the theorem of Maslov which proves the uniqueness of idempotent integrals of "semi-measurable" functions. This theorem is a consequence of the construction of the idempotent integral of Maslov, that we extend to idempotent semirings $\mathbb{D}$ which are continuous lattices. Moreover, in order to relate our results on density of idempotent measures with the existing ones on density of idempotent linear forms, we prove a "probabilistic" version of Riesz representation theorem.

Our approach (the restriction of idempotent measures to open sets) was initially motivated by the large deviation principle as defined by Varadhan [33]. The purpose of large deviations is to obtain asymptotics of probability families P_ε, of the form $\mathbb{K}(A) = \lim_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(A)$, where $\mathbb{K}(A) = -\inf_{\omega\in A} I(\omega)$ with I a l.s.c. function. Thus, $\exp \mathbb{K}$ is a sup-measure [26, 25] or a deviability [30, 31], which implies that $\mathbb{K}$ is an idempotent $\mathbb{R}_{\max}$-measure with density $-I$. Generalizing this concept of large deviation by using general idempotent measures, we give (in Section 5) necessary and sufficient conditions for the large deviation principle to be satisfied and prove that when it exists, I may be calculated by using open sets only. This last result is indeed related to Theorems 4.1.11 and 4.1.18 of Dembo and Zeitouni [13].

1. Continuous lattices

In this section, we give a short presentation of definitions, results and examples concerning continuous lattices. Apart from some minor extensions (on locally complete and locally continuous lattices), these results may be found in [15], with proofs.

Let us first recall some classical terminology. Let L be a set endowed with a partial order $\preceq$.

Definition 1.1. $(L, \preceq)$ is a *semilattice* (resp. a *sup-semilattice*, resp. a *lattice*) if every nonempty finite set admits a greatest lower bound or infimum (resp. a least upper bound or supremum, resp. an infimum and a supremum). It is a *complete lattice* if every (possibly empty) set admits an infimum (or equivalently if every set admits a supremum). The symbol $\top$ denotes the top element or supremum of L, and $\perp$ denotes the bottom element or infimum of L.

In the previous definition, we use the convention that the infimum (resp. the supremum) of the empty set is the top element (resp. the bottom element) of the lattice L. The supremum is denoted by sup or $\vee$ and the infimum by inf or $\wedge$.

In the sequel, we will equip dioids with a semilattice structure, as follows.

Example 1.2. Let $(\mathbb{D},\oplus)$ be a commutative idempotent monoid, that is $\oplus$ is associative, commutative and idempotent: $a\oplus a=a$ for any $a\in\mathbb{D}$, and has a neutral element, denoted by $\mathbb{0}$. We denote by $\preceq$ the partial order relation associated with the idempotent $\oplus$ operation: $a\preceq b\Leftrightarrow a\oplus b=b$. Then, $a\oplus b$ is the least upper bound of a and b and $a\succeq\mathbb{0}$ for any $a\in\mathbb{D}$. Thus, $(\mathbb{D},\preceq)$ is a sup-semilattice with a bottom element $\mathbb{0}$. Conversely, a partial order $\preceq$ such that $(\mathbb{D},\preceq)$ is a sup-semilattice with a element $\mathbb{0}$, defines an idempotent commutative associative law $\oplus$ with a neutral element $\mathbb{0}$ on $\mathbb{D}$, setting $a\oplus b=\sup(a,b)$.

In the sequel, we say that a commutative idempotent monoid $(\mathbb{D},\oplus)$ satisfies a lattice property (for instance, completeness) when the associated sup-semilattice $(\mathbb{D},\preceq)$ satisfies the same property.

The monoid structure $(\mathbb{R}\cup\{-\infty\},\max)$ of the semifield $\mathbb{R}_{\max}$ is not complete. It can be extended in the complete monoid $(\overline{\mathbb{R}}=[-\infty,+\infty],\max)$, but then the semifield structure is lost for the semiring $\overline{\mathbb{R}_{\max}}=(\overline{\mathbb{R}},\max,+)$ (see Section 2 for the exact definition). Therefore, in the sequel, we do not impose the completeness of the monoid $(\mathbb{D},\oplus)$, but only the following property.

Definition 1.3. The lattice $(L,\preceq)$ is *locally complete* if it satisfies one of the following equivalent conditions :

1. every nonempty lower bounded set admits an infimum;
2. every nonempty upper bounded set admits a supremum;
3. there exists a complete lattice denoted $\overline{L}$ with bottom element $\perp$ and top element $\top$, such that L is a sublattice of $\overline{L}$, $\overline{L}=L\cup\{\perp,\top\}$, $\inf L=\perp$ and $\sup L=\top$.

Remark 1.4. A commutative idempotent monoid $(\mathbb{D},\oplus)$ has automatically the bottom element $\perp=\mathbb{0}$. Thus, it is locally complete iff every nonempty set admits an infimum or equivalently every upper bounded set admits a supremum.

The following definition concerns the continuity of complete lattices.

Definition 1.5.
- $D\subset L$ is a *directed set* if any finite subset of D has an upper bound in D.
- We denote by $\preceq^{\rm op}$ the *opposite order* of L : $a\preceq^{\rm op} b\Leftrightarrow b\preceq a$. If $(L,\preceq)$ is a lattice, then $L^{\rm op}$ denotes the lattice $(L,\preceq^{\rm op})$.
- D is a *filtered set* of L if D is a directed set of $L^{\rm op}$.
- The *way below* relation $\ll$ is defined by $a\ll b$ if and only if for all directed sets D of L, such that $b\preceq\sup D$, there exists $x\in D$ such that $a\preceq x$.
- The complete lattice L is *continuous* if

$$x=\sup\{y\in L,\ y\ll x\}\quad\text{for any}\quad x\in L. \tag{1}$$

- I is a *basis* of the continuous lattice L if I is a sup-subsemilattice of L, containing $\perp$, and such that

$$x=\sup\{y\in I,\ y\ll x\}\quad\text{for any}\quad x\in L. \tag{2}$$

- The way below relation for $\preceq^{\rm op}$ is denoted by $\ll^{\rm op}$, and L is said to be *dually continuous* if $L^{\rm op}$ is continuous.

Example 1.6. In a totally ordered lattice L, $a\prec b$ or $a=\perp$ implies $a\ll b$ (by definition $a\prec b\Leftrightarrow(a\preceq b$ and $a\neq b))$. If $L=\overline{\mathbb{R}}$ with the order $\leq$, then $a\ll b$ is equivalent to $(a<b$ or $a=-\infty)$, and L is continuous (and dually continuous). If

$L = \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, then $a \ll b$ is equivalent to ($a \leq b$ and $a \neq +\infty$), and L is continuous (and dually continuous).

Example 1.7. If L is a complete lattice, then L^n with the componentwise order relation is a complete lattice and $a = (a_1, \ldots, a_n) \ll b = (b_1, \ldots, b_n)$ in L^n iff $a_i \ll b_i$ for $i = 1, \ldots n$. Therefore, L continuous implies L^n continuous. In particular $((\overline{\mathbb{R}})^n, \leq)$ is a continuous and dually continuous lattice.

Now, by eliminating the top element of $\overline{\mathbb{R}}$, we obtain $L = \mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ which is a locally complete lattice such that $\overline{L}$ is continuous and dually continuous. We can also prove that L locally complete implies L^n locally complete. However, $\overline{L^n} = [-\infty, +\infty)^n \cup \{+\infty \stackrel{\text{def}}{=} (+\infty, \ldots, +\infty)\}$ endowed with the componentwise order relation $\leq$ is a dually continuous lattice but not a continuous lattice. Indeed, $a \ll b$ iff $a = \bot = (-\infty, \ldots, -\infty)$ (because $+\infty = \sup(L \times \{-\infty\} \times \cdots \times \{-\infty\})$). For the opposite order, however, all behave as in $(\overline{\mathbb{R}})^n$. If we also eliminate $-\infty$, that is if we consider the locally complete lattice $L = \mathbb{R}$, $\overline{L^n}$ is neither continuous nor dually continuous.

Example 1.8. Another usual example of complete lattice is the set $\mathcal{P}(X)$ of subsets of a set X with the $\subset$ order relation. The set of open sets $\mathcal{O}(X)$ (resp. the set of closed sets $\mathcal{C}(X)$) of a topological space X (resp. the set of closed convex sets $\mathrm{Con}(X)$ of a topological vector space X) is also a complete lattice with bottom element $\emptyset$ and top element X, even if it is not a sub-complete-lattice of $\mathcal{P}(X)$.

Let us suppose that X is a Hausdorff topological space. In $\mathcal{C}(X)$, $A \ll B$ iff A is a finite subset of B composed of isolated points (in B). Thus, $\mathcal{C}(X)$ and $\mathrm{Con}(X)$ (when X is a vector space) are not continuous and $\mathcal{O}(X)$ is not dually continuous. In $\mathcal{O}(X)$, $A \ll B$ when $\overline{A}$ is compact $\subset B$, which is often noted $A \subset\subset B$. If X is locally compact, the two conditions are equivalent, then $\mathcal{O}(X)$ is continuous and $\mathcal{C}(X)$ dually continuous. If K is a compact convex subset of a Hausdorff locally convex topological vector space X, $\mathrm{Con}(K)$ is a dually continuous lattice.

Example 1.7 shows that if we consider different sublattices of the same complete lattice L, which are complete, the way below relation defined in these sublattices may be different ($\overline{(\mathbb{R}_{\max})^n}$ is a sublattice of $(\overline{\mathbb{R}})^n$). This comes from the fact that these sublattices are not necessarily stable by infinite sup of L. However, if we generalize the way below relation to locally complete lattices in the following manner, this type of boundary effect disappears.

Definition 1.9. In a locally complete lattice L, the *way below* $\ll$ relation is defined by : $a \ll b$ if and only if for all upper bounded directed sets D of L, such that $b \preceq \sup D$, there exists $x \in D$ such that $a \preceq x$.

Note that a directed set is necessarily nonempty (the empty set is a finite subset of D, then it has an upper bound in D), which makes $\sup D$ well defined in the previous statement. For a locally complete lattice L, $a \ll b$ is equivalent to $a \ll b$ in any sublattice of L containing a and b and of the form $[c, d] = \{x \in L,\ c \preceq x \preceq d\}$, with $c \preceq d \in L$. It is also equivalent to $a \ll b$ in any sublattice of $L \cup \{\bot\}$ of the form $[\bot, d]$ with $d \in L$. Using this way below relation, the definition of continuous lattices and basis may be generalized to locally complete lattices (in this case a basis does not necessarily contain $\bot$). Locally complete lattices which are continuous are said to be *locally continuous*. Under this definition, $(\mathbb{R}_{\max})^n$ and $\mathbb{R}^n$ become locally continuous and locally dually continuous lattices. Moreover, a locally complete lattice L is continuous iff every sublattice of L of the form $[c, d]$ is continuous. This is also equivalent to the continuity of the locally complete lattice $L \cup \{\bot\}$.

If L is a locally complete lattice, we may extend the definition of the way below relation to $\overline{L}$ by taking $a \ll b$ in $\overline{L}$ iff ($a \ll b$ in L, or $a \in L \cup \{\bot\}$ and $b = \top \notin L$, or $a = \bot$ and $b \in \overline{L}$). This relation is not equal to that defined directly in the complete lattice $\overline{L}$. For instance, if $L = (\mathbb{R}_{\max})^n$ or $\mathbb{R}^n$, this way below relation is the restriction of that of $(\overline{\mathbb{R}})^n$. If L is a locally continuous lattice and if I is a basis of L, (1) and (2) are still valid for $x = \top$ and for $x = \bot$ with $I \cup \{\bot\}$ instead of I.

The following characterization is the main ingredient of the proofs of Section 3 on extensions and densities of idempotent measures.

Theorem 1.10 ([15, Th. I.2.3])**.** *For a complete lattice L, the following conditions are equivalent*:

1. *L is continuous.*
2. *Let $\{D(j),\ j \in J\}$ be a family of directed sets of L. Let M be the set of all functions $f : J \to L$, with $f(j) \in D(j)$ for all $j \in J$. Then, the following identity holds*:

$$\inf_{j\in J} \sup D(j) = \inf_{j\in J} \left(\sup_{x\in D(j)} x \right) = \sup_{f\in M} \left(\inf_{j\in J} f(j) \right). \tag{3}$$

3. *Let $\{D(j),\ j \in J\}$ be any family of subsets of L. Let N be the set of all functions $f : J \to \mathrm{fin}L$, the set of finite subsets of L, with $f(j) \in \mathrm{fin}D(j)$ for all $j \in J$. Then, the following identity holds* :

$$\inf_{j\in J} \sup D(j) = \sup_{f\in N} \left(\inf_{j\in J} \sup f(j) \right). \tag{4}$$

Theorem 1.10 is still valid if L is only a locally complete lattice when the sets $D(j)$ considered in points 2 and 3 are supposed to be nonempty and upper bounded (that is $\sup D(j) \in L$).

The classical definition of lower semi-continuous (l.s.c.) functions with values in $\mathbb{R}$ may be generalized to functions with values in any (locally) complete lattice L.

Definition 1.11. A function $f : X \to L$ is *l.s.c.* if

$$f(x) \preceq \liminf_{y\to x} f(y) \stackrel{\text{def}}{=} \sup_{U\in\mathcal{U},\, U\ni x} \inf_{y\in U} f(y) \quad \forall x \in X,$$

where $\mathcal{U}$ is the set of open sets of the topological space X.

The Scott topology defined below allows one to characterize (in a (locally) continuous lattice) semi-continuity in terms of topology.

Definition 1.12. Let L be a (locally) complete lattice. We say that $U \subset L$ is *Scott open* (open for the Scott topology) if it satisfies the two following conditions:

1. $U = \uparrow U \stackrel{\text{def}}{=} \{x \in L,\ \exists y \in U,\ y \preceq x\}$.
2. $\sup D \in U$ implies $D \cap U \neq \emptyset$ for all (upper bounded) directed sets $D \subset L$.

Proposition 1.13 ([15, Prop. II.1.10])**.** *Let L be a (locally) continuous lattice.*

The Scott topology on L is the weakest topology such that the sets $\{x \in L,\ a \ll x\}$ are open.

A function $f : X \to L$ is l.s.c. iff it is continuous for the Scott topology of L.

The Scott topology is clearly not separated (Hausdorff). Let us generalize the definition of continuity. One possible way is to say that a function $f : X \to L$ is continuous if it is both l.s.c. and *upper semi-continuous* (u.s.c.), that is l.s.c. for the opposite order. The topology on L that defines this continuity is the *order topology.* It contains the common refinement of the two Scott topologies defined on L and $L^{\rm op}$, that we will call *bi-Scott topology.* On *bi-continuous* lattices (lattices which are both continuous and dually continuous), the two topologies (order and bi-Scott) coincide. The Lawson topology defined below is stronger than the Scott topology and weaker than the bi-Scott topology. It is relevant in lattices L which are only continuous. Moreover, it allows us to define a weaker (topological) continuity notion than the previous one, which still generalizes the continuity notion defined for real valued functions.

Definition 1.14. The *Lawson topology* denoted ΛL is defined as the common refinement of the Scott topology and the *lower topology*, that is the topology generated by sets $[a,\top]^c = \{x \in L,\ a \not\leq x\}$. A function $f : X \to L$ is *continuous* if it is both l.s.c. and continuous for the lower topology.

Proposition 1.15. *A function f from a topological space X to a* (*locally*) *continuous lattice L is continuous iff it is continuous for the Lawson topology of L. If $f : X \to L$ is both l.s.c. and u.s.c., then f is continuous. The converse proposition is false in general.*

Proof. The first assertion is a consequence of Proposition 1.13. For the second assertion, we only have to prove that an u.s.c. function is continuous for the lower topology, that is the sets $\{x \in X, a \not\leq f(x)\}$ are open for all $a \in L$, which is obvious. Let us give a counter example to the converse proposition. Let $X = [0,1]$ be endowed with the usual topology and let $L = \mathcal{O}([0,1])$ be the set of its open sets. In Example 1.8, we have seen that L is continuous but not dually continuous. Let us consider $f : X \to L,\ x \mapsto \{x\}^c$, where A^c denotes the complementary set of A. Then, f is l.s.c.:

$$\liminf_{y\to x} f(y) = \bigcup_{U\in L,\ U\ni x} \operatorname{int}\left(\bigcap_{y\in U}\{y\}^c\right) = \left(\bigcap_{U\in L,\ U\ni x}\overline{U}\right)^c = \{x\}^c = f(x),$$

where $\overline{A}$ and $\operatorname{int}(A)$ respectively denote the closure and the interior of A. Moreover, $f^{-1}(\{U \in L,\ A \not\subset U\}) = \{x \in X,\ A \not\subset \{x\}^c\} = A$ is open for all $A \in L$, which implies that f is (Lawson) continuous. However f is not u.s.c.:

$$\limsup_{y\to x} f(y) = \operatorname{int}\left(\bigcap_{U\in L,\ U\ni x}\left(\bigcup_{y\in U}\{y\}^c\right)\right) = [0,1] \not\subset \{x\}^c = f(x). \qquad \square$$

Example 1.16. The lattice $L = (\mathbb{R}_{\max})^n$ is a locally bi-continuous lattice, then the order topology is equivalent to the bi-Scott or *bi-Lawson* topology, that is the common refinement of the two Lawson topologies defined on L and $L^{\rm op}$. In this case, it also coincides with the Lawson topology.

Proposition 1.17 ([15, Th. III.1.10, Cor. III.4.10])**.** *For a continuous lattice L, ΛL is a compact Hausdorff space.*

Moreover, L has a countable basis iff ΛL is a compact metric space.

Remark 1.18. If L is a bi-continuous lattice, the bi-Scott topology is equal to the bi-Lawson topology. Then, if both L and L^{op} have a countable basis, the bi-Scott topology is metrizable (but not necessarily compact).

Remark 1.19. If L is only a locally continuous lattice, ΛL is still Hausdorff, and the induced topology on $[c,d]$ with $c \preceq d \in L$ is the Lawson topology on the continuous lattice $[c,d]$. If L has a countable basis I, then $I \cup \{\perp\}$ is a basis of $L \cup \{\perp\}$ and ΛL is a topological subspace of $\Lambda(L \cup \{\perp\})$ which is the countable union of the compact metric spaces $\Lambda[\perp, c]$ with $c \in I$.

2. Idempotent measures

Let $\mathcal{A}$ be a Boolean algebra or a Boolean σ-algebra of subsets of a set X. A probability P on $(\Omega, \mathcal{A})$ is such that

i) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$ and $P(\emptyset) = 0$.

In addition $P(\Omega) = 1$, and $P(A \cap B) = P(A) \times P(B)$ if A and B are independent. Thus, a probability is, in a loose sense, a morphism from the "complemented" semiring $(\mathcal{A}, \cup, \cap)$ (in the sense that $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$), with $\emptyset$ as zero and Ω as unit, to the symmetrizable semifield $(\mathbb{R}^+, +, \times)$. Since the field structure of $\mathbb{R}$ allows us to write $P(A^c) = 1 - P(A)$, the continuity of a probability can equivalently be defined by one of the two properties:

ii) $P(A_n) \underset{n\to+\infty}{\nearrow} P(A)$ if $A_n \underset{n\to+\infty}{\nearrow} A$ with A_n and A in $\mathcal{A}$,

iii) $P(A_n) \underset{n\to+\infty}{\searrow} P(A)$ if $A_n \underset{n\to+\infty}{\searrow} A$ with A_n and A in $\mathcal{A}$.

If we replace $(\mathbb{R}^+, +, \times)$ by an idempotent semiring $(\mathbb{D}, \oplus, \otimes)$, we loose "opposites" for the additive law $\oplus$ and as a consequence : a) the entire structure of Boolean algebra is no longer needed in order to get a "morphism", b) properties ii) and iii) are not equivalent, moreover iii) is rarely satisfied and is not preserved after extension of a probability P to a larger algebra (see Examples 2.1 and 2.3 below).

Example 2.1. Let $\mathcal{A}$ be the set of Borel sets of $\Omega = \mathbb{R}$ and let us consider $P(A) = \sup_{\omega\in A} c(\omega)$ where c is any function from $\mathbb{R}$ to $\mathbb{R}_{\max}$. Then, P satisfies property i) where addition is replaced by the max operator and property ii). Indeed, we will see in Section 3 that the restriction to open sets of any idempotent $\mathbb{R}_{\max}$-probability on $(\Omega, \mathcal{A})$ has this form. If P satisfies also property iii) on $\mathcal{A}$, then $P((a - 1/n, a + 1/n) \setminus \{a\})$ and $P([-n, n]^c)$ decrease towards $P(\emptyset) = \mathbb{0} = -\infty$. This implies that the set $\{\omega \in \mathbb{R},\ c(\omega) \geq b\}$ is finite for all $b \in \mathbb{R}$ and thus c has countable support (as atomic classical probabilities).

Definition 2.2. A set $\mathcal{A}$ of subsets of a given set Ω is a *Boolean semi-algebra* if it is a sublattice of $(\mathcal{P}(\Omega), \subset)$, that is if it contains Ω and $\emptyset$ and it is stable by the finite union and intersection operations. It is a *semi-σ-algebra* if in addition it is stable by the countable union operation.

Example 2.3. Let us consider the compact metric space $\Omega = [0,1]$. The set $\mathcal{A}$ of closed sets is a Boolean semi-algebra. Now if P is as in Example 2.1, P satisfies condition iii) on $\mathcal{A}$. However, the semi-σ-algebra generated by $\mathcal{A}$ contains open sets for which property iii) is false in general.

Let us consider $(\mathbb{D}, \oplus, \otimes)$ an idempotent semiring with $\mathbb{0}$ and $\mathbb{1}$ as neutral elements for the $\oplus$ and $\otimes$ operations respectively. We denote by $\preceq$ the partial order relation associated with the idempotent $\oplus$ operation (see Example 1.2). We denote by "sup" or $\oplus$ (resp. by "inf" or $\wedge$) the supremum (resp. the infimum) operation. In all this paper, we suppose that $\mathbb{D}$ is locally complete. If the top element $\top$ of $\overline{\mathbb{D}}$ does not belong to $\mathbb{D}$, the law $\otimes$ may be extended to $\overline{\mathbb{D}}$ so that $(\overline{\mathbb{D}}, \oplus, \otimes)$ becomes a semiring ($\top \otimes a = a \otimes \top = \top$ if $a \neq \mathbb{0}$ and $\top \otimes \mathbb{0} = \mathbb{0} \otimes \top = \mathbb{0}$). Examples of such idempotent semirings are $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$, $(\mathbb{R}^+, \max, \times)$, $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \min, +)$, (with $+\infty$, resp. $+\infty$, resp. $-\infty$, as top elements), and also $(\mathbb{R}_{\max})^n$, $(\mathbb{R}_{\min})^n$, ...

Remark 2.4. The second law $\times$ or $\otimes$ is only necessary in the construction of integrals or the definition of independence but not in the construction of measures. In particular the results of the following section depend only on the first law $\oplus$, and thus, on the lattice structure of $\mathbb{D}$.

Definition 2.5. An *idempotent* $\mathbb{D}$*-measure* on a Boolean semi-algebra $\mathcal{A}$ of subsets of Ω is a mapping $\mathbb{K}$ from $\mathcal{A}$ to $\overline{\mathbb{D}}$ such that

1. $\mathbb{K}(\emptyset) = \mathbb{0}$,
2. $\mathbb{K}(A \cup B) = \mathbb{K}(A) \oplus \mathbb{K}(B)$ for any A, B in $\mathcal{A}$,
3. $\mathbb{K}(A_n) \underset{n\to+\infty}{\nearrow} \mathbb{K}(A)$ if $A_n \underset{n\to+\infty}{\nearrow} A$, $A_n \in \mathcal{A}\ \forall n \in \mathbb{N}$ and $A \in \mathcal{A}$ (σ-additivity).

An idempotent $\mathbb{D}$-measure $\mathbb{K}$ is *finite* if $\mathbb{K}(\Omega) \in \mathbb{D}$. It is an *idempotent probability* if $\mathbb{K}(\Omega) = \mathbb{1}$.

Point 3 means that $\mathbb{K}(A_n)$ is nondecreasing and $\bigoplus_n \mathbb{K}(A_n) = \mathbb{K}(A)$, when A_n is nondecreasing such that $\bigcup_n A_n = A$. Therefore, $\mathbb{K}(A_n)$ converges towards $\mathbb{K}(A)$ for the order, Scott and Lawson topologies defined on $\mathbb{D}$ or $\mathbb{D}^{\mathrm{op}}$.

Remark 2.6. It follows immediately from point 2 of the definition that any idempotent measure $\mathbb{K}$ is monotone: $\mathbb{K}(A) \preceq \mathbb{K}(B)$ if $A \subset B$. In particular, if $\mathbb{K}$ is a probability, it takes its values in the subset $[\mathbb{0}, \mathbb{1}] = \{x \in \mathbb{D}, \mathbb{0} \preceq x \preceq \mathbb{1}\} = \{x \in \mathbb{D},\ x \preceq \mathbb{1}\}$ of $\mathbb{D}$.

By the idempotence property, we have:

Proposition 2.7. *A mapping $\mathbb{K}$ from $\mathcal{A}$ to $\overline{\mathbb{D}}$ is an idempotent $\mathbb{D}$-measure on $\mathcal{A}$ iff*

$$\mathbb{K}\left(\bigcup_{i\in I} A_i\right) = \bigoplus_{i\in I} \mathbb{K}(A_i)$$

for any finite or countable family $\{A_i,\ i \in I\}$ of elements of $\mathcal{A}$.

An idempotent measure with values in $\mathbb{R}_{\max}$ (resp. $\mathbb{R}_{\min}$) will be called a *reward* (resp. *cost*) measure. It is finite if and only if $\mathbb{K}(\Omega) < +\infty$ (resp. $\mathbb{K}(\Omega) > -\infty$) and it is a reward (resp. cost) probability if $\mathbb{K}(\Omega) = 0$. Note that the order relation associated to the "min" law is the opposite of the classical order $\leq$ of $\mathbb{R}$. Therefore, even if we are more interested with cost measures, that is with minimization problems, it is easier to consider reward measures since monotony properties and extensions constructions coincide with those of the classical Probability theory.

An *idempotent* $\mathbb{D}$*-probability space* (also called a *decision space*) $(\Omega, \mathcal{A}, \mathbb{K})$ is composed of a nonempty set Ω, a semi-σ-algebra $\mathcal{A}$ of subsets of Ω and an idempotent $\mathbb{D}$-probability $\mathbb{K}$.

Let us introduce the notion of density of an idempotent $\mathbb{D}$-measure. Consider a function c from Ω to $\overline{\mathbb{D}}$ and define for any subset A of Ω,

$$\mathbb{K}(A) = \sup\{c(\omega), \omega \in A\}. \tag{5}$$

It is easy to check that $\mathbb{K}$ is an idempotent $\mathbb{D}$-measure on $\mathcal{P}(\Omega)$.

Definition 2.8. An idempotent measure $\mathbb{K}$ has a *density* if (5) holds for some function c. In this case, any function c satisfying (5) is called a density of $\mathbb{K}$.

3. Idempotent measures extensions and densities

In [20], Maslov shows that there can be several extensions of the same idempotent measure from a Boolean algebra to the least σ-algebra containing it. For instance, the "Lebesgue measure" on $(\Omega, \mathcal{A})$ with values in $\mathbb{R}_{\max}$, where $\Omega = \mathbb{R}$, and $\mathcal{A}$ is the algebra of finite unions of intervals with rational bounds is defined by

$$\mathbb{K}(A) = 0 \qquad \text{if } A \neq \emptyset, \qquad \mathbb{K}(\emptyset) = -\infty.$$

It can be extended to the Borel sets σ-algebra as follows:

$$\mathbb{K}(A) = \sup_{x \in A} c(x)$$

with 1) $c(x) \equiv 0$ (which leads to the maximal extension) or 2) $c(x) = 0$ when x is rational and $c(x) = -\infty$ (or any number less than 0) when x is irrational. Indeed nonempty elements of $\mathcal{A}$ necessarily contain rationals. But clearly, densities 1) and 2) do not lead to the same value of $\mathbb{K}$.

However, the maximal extension always exists and plays an important role (see Section 4). Here we recall the definition of the maximal extension which only involves the Boolean semi-algebra structure of the initial set $\mathcal{A}$ of subsets of Ω. Although this construction seems natural (it is equivalent to that of classical measure theory), it implicitly uses the dual continuity of the locally complete lattice $\mathbb{D}$ or at least of the complete lattice $[\mathbb{0}, \mathbb{K}(\Omega)]$. The same property will still be necessary to prove that $\mathbb{K}$ has a density.

Let us consider an idempotent measure $\mathbb{K}$ on a Boolean semi-algebra $\mathcal{A}$ of subsets of Ω. We denote by $\mathcal{G}$ the set of countable unions of elements of $\mathcal{A}$; $\mathcal{G}$ is the least semi-σ-algebra containing $\mathcal{A}$. We define on $\mathcal{G}$ the extension $\mathbb{K}^+$ of $\mathbb{K}$:

$$\mathbb{K}^+(G) = \sup_n \mathbb{K}(A_n) \;\text{ if }\; G = \bigcup_{n \in \mathbb{N}} A_n \;\text{ with }\; A_n \in \mathcal{A}.$$

This definition is well posed: since $\mathbb{K}$ is σ-additive and $\mathcal{A}$ is stable by finite intersections and unions, the supremum is independent of the choice of the sets A_n. Hence, $\mathbb{K}^+$ is the unique extension of $\mathbb{K}$ to $\mathcal{G}$.

Now, for any subset A of Ω we define

$$\mathbb{K}^*(A) = \inf_{G \in \mathcal{G}, G \supset A} \mathbb{K}^+(G).$$

Proposition 3.1 ([20, Ch. VIII, Th. 4.1])**.** *Suppose that* $([\mathbb{0}, \mathbb{K}(\Omega)], \preceq)$ *is a dually continuous lattice. Then,* $\mathbb{K}^*$ *is the maximal extension of* $\mathbb{K}$ *to the set of all subsets of* Ω.

Proof. We recall the proof in order to point out the use of dual continuity. Let us first prove that $\mathbb{K}^*$ is maximal. For any semi-σ-algebra $\mathcal{B}$ containing $\mathcal{A}$ and any extension $\mathbb{K}'$ of $\mathbb{K}$ to $\mathcal{B}$ we have $\mathcal{B} \supset \mathcal{G}$ and $\mathbb{K}' = \mathbb{K}^+$ on $\mathcal{G}$. Then, for any $B \in \mathcal{B}$ and $G \in \mathcal{G}$ such that $G \supset B$ we have $\mathbb{K}'(B) \preceq \mathbb{K}'(G) = \mathbb{K}^+(G)$, thus $\mathbb{K}'(B) \preceq \mathbb{K}^*(B)$.

In order to prove that $\mathbb{K}^*$ is an idempotent measure on $\mathcal{P}(\Omega)$ we only have to show that, for any finite or countable family $\{A_i,\ i \in I\}$ of subsets of Ω, $\mathbb{K}^*(\bigcup_i A_i) = \sup_i \mathbb{K}^*(A_i)$. The monotony of $\mathbb{K}^*$ is obvious from the definition. Then, $\mathbb{K}^*(\bigcup_i A_i) \succeq \sup_i \mathbb{K}^*(A_i)$. For the other inequality, we have

$$\begin{aligned} \sup_i \mathbb{K}^*(A_i) &= \sup_i \left(\inf_{G\in\mathcal{G},\, G\supset A_i} \mathbb{K}^+(G) \right) \\ &= \inf_{G_i\in\mathcal{G},\, G_i\supset A_i\ \forall i\in I} \left(\sup_i \mathbb{K}^+(G_i) \right) \\ &= \inf_{G_i\in\mathcal{G},\, G_i\supset A_i\ \forall i\in I} \mathbb{K}^+ \left(\bigcup_i G_i \right) \\ &\succeq \mathbb{K}^* \left(\bigcup_i A_i \right), \end{aligned} \tag{6}$$

which leads to the requested equality. In (6), we have used an inversion formula of the sup and inf operations of the same type as (3) but for the opposite order $\preceq^{\rm op}$ ($\{\mathbb{K}^+(G),\ G \in \mathcal{G},\ G \supset A_i\}$ is a filtered set), which holds in a dually continuous lattice only. As $\mathbb{K}$ takes its values in $[\mathbb{0}, \mathbb{K}(\Omega)]$, the dual continuity of this sublattice is only needed. ☐

Let us note that as $\Omega \in \mathcal{A}$, $\mathbb{K}^*$ is necessarily a probability if $\mathbb{K}$ is so.

Example 3.2. If $\mathbb{K}$ is the "Lebesgue measure" on $\mathcal{A} : \mathbb{K}(A) = \mathbb{1}$ for $A \neq \emptyset$ and $\mathbb{K}(\emptyset) = \mathbb{0}$, then $\mathbb{K}^*$ is the "Lebesgue measure" on $\mathcal{P}(\Omega) : \mathbb{K}^*(A) = \mathbb{1}$ for any nonempty set A. The function $c(\omega) \equiv \mathbb{1}$ is the density of $\mathbb{K}^*$ and the maximal density of $\mathbb{K}$.

Example 3.3. If $\mathcal{U}$ is the set of open sets of a Hausdorff topological space X, the lattice $(\mathcal{U}, \subset)$ is in general not dually continuous (see Example 1.8). Let us consider the dioid $\mathbb{D} = (\mathcal{U}, \cup, \cap)$ with neutral elements $\mathbb{0} = \emptyset$ and $\mathbb{1} = X$ and take $\Omega = X$, $\mathcal{A} = \mathcal{U}$ and $\mathbb{K}(A) = A$ for any $A \in \mathcal{A}$. Clearly, $\mathbb{K}$ is an idempotent $\mathbb{D}$-probability, $\mathcal{G} = \mathcal{U}$ and thus $\mathbb{K}^+ = \mathbb{K}$. Now, for any subset A of Ω, $\mathbb{K}^*(A) = \operatorname{int}(A)$, the interior of A. Then, if X has accumulation points, $\mathbb{K}^*$ is not an idempotent measure and is even not additive.

Example 3.4. Consider now the dioid $\mathbb{D} = (\mathcal{C}, \cup, \cap)$, where $\mathcal{C}$ is the set of closed sets of a compact subspace K of X and $\mathbb{K}(A) = \overline{A \cap K}$ for all $A \in \mathcal{A} = \mathcal{U}$. Clearly, $\mathbb{K}$ is an idempotent $\mathbb{D}$-probability (note that if $\mathcal{F}$ is a subset of $\mathcal{C}$, then $\sup \mathcal{F} = \overline{\bigcup_{F\in\mathcal{F}} F}$). Since $(\mathcal{C}, \subset)$ is dually continuous, Proposition 3.1 shows that $\mathbb{K}^*$ is an idempotent measure. Indeed, we find by calculation $\mathbb{K}^*(A) = \overline{A \cap K}$. Moreover, the function $c(x) \overset{\rm def}{=} \mathbb{K}^*(\{x\}) = \{x\} \cap K$ is the density of $\mathbb{K}^*$.

As $\mathbb{K}^*$ is defined on all subsets of Ω, we find a good candidate to the density function of $\mathbb{K} : c^*(\omega) = \mathbb{K}^*(\{\omega\})$. Let us denote

$$\overline{\mathbb{K}}(A) = \sup\{c^*(\omega), \omega \in A\}.$$

Since $\mathbb{K}^*$ is monotone, we have $\overline{\mathbb{K}}(A) \preceq \mathbb{K}^*(A)$ for any subset A of Ω.

Proposition 3.5. *If $\mathbb{K}$ has a density on $\mathcal{A}$, then c^* is the maximal density of $\mathbb{K}$ on $\mathcal{A}$.*

Proof. Let c be a density of $\mathbb{K}$. We have

$$c^*(\omega) = \inf_{G\in\mathcal{G}, G\ni\omega} \mathbb{K}^+(G) = \inf_{A\in\mathcal{A}, A\ni\omega} \mathbb{K}(A) \succeq c(\omega).$$

Thus, $\mathbb{K}(A) \preceq \overline{\mathbb{K}}(A) \preceq \mathbb{K}^*(A) = \mathbb{K}(A)$ for any A in $\mathcal{A}$. □

Example 3.6. The idempotent measure of Example 3.3 has no density. Indeed, Proposition 3.5 shows that $c^*(x) = \emptyset = \mathbb{0}$ would be its maximal density, but $\mathbb{K} \not\equiv \mathbb{0}$.

Proposition 3.5 implies that if $\mathbb{K}$ has a density c, then $\mathbb{K}^+$ has c^* or c as density on $\mathcal{G}$. However, in order to prove that c^* is a density of $\mathbb{K}^*$, we need the stability of $\mathcal{G}$ by any union operation (even not countable). This is the case if $\mathcal{A}$ is the set of open sets of a topological space Ω. In this case, we have for all $A \subset \Omega$

$$\begin{aligned}\overline{\mathbb{K}}(A) &= \sup_{\omega\in A} c^*(\omega) \\ &= \sup_{\omega\in A}\left(\inf_{G\in\mathcal{G}, G\ni\omega} \mathbb{K}^+(G)\right) \\ &= \inf_{(G_\cdot\in\mathcal{G}^A, G_\omega\ni\omega\ \ \forall\omega\in A)} \left(\sup_{\omega\in A} \mathbb{K}^+(G_\omega)\right).\end{aligned}$$

Note that this last equality is of the same type as (3) for the opposite order $\preceq^{\mathrm{op}}$ and thus requires the dual continuity of the lattice $[\mathbb{0}, \mathbb{K}(\Omega)]$. Now if $\mathbb{K}^+$ has a density and $\bigcup_{\omega\in A} G_\omega \in \mathcal{G}$, then $\sup_{\omega\in A} \mathbb{K}^+(G_\omega) = \mathbb{K}^+(\bigcup_{\omega\in A} G_\omega)$. Since $\bigcup_{\omega\in A} G_\omega \supset A$, we obtain $\mathbb{K}^+(\bigcup_{\omega\in A} G_\omega) \succeq \mathbb{K}^*(A)$ and $\overline{\mathbb{K}}(A) \succeq \mathbb{K}^*(A)$. As the other inequality is always true, $\mathbb{K}^*$ has c^* as density.

In conclusion:

Proposition 3.7. *If $[\mathbb{0}, \mathbb{K}(\Omega)]$ is a dually continuous lattice, Ω is a topological space, $\mathcal{A}$ is the set of open sets of Ω and $\mathbb{K}$ has a density on $\mathcal{A}$, then $\mathbb{K}^*$ has c^* as density on $\mathcal{P}(\Omega)$.*

Remark 3.8. If $\mathbb{K}$ is a reward (resp. cost) measure with density c on the set $\mathcal{A}$ of open sets of a topological space Ω, then c^* is the upper semi-continuous (u.s.c.) (resp. lower semi-continuous (l.s.c.)) envelope of c. Indeed,

$$c^*(\omega) = \inf_{A\ni\omega, A\in\mathcal{A}} \left(\sup_{y\in A} c(y)\right),$$

which is the definition of the u.s.c. (or l.s.c., if $\preceq$ corresponds to $\geq$) envelope.

We prove now that under some conditions on the Boolean semi-algebra $\mathcal{A}$, any idempotent measure has a density.

Theorem 3.9. *Consider a Boolean semi-algebra $\mathcal{A}$ of subsets of Ω such that the following property holds*:

> *for any $A \in \mathcal{A}$ and any cover $A \subset \bigcup_{i\in I} A_i$ by elements of $\mathcal{A}$, there exists a countable subcover of A : $A \subset \bigcup_{i\in J} A_i$ ($J \subset I$ and J countable).*

Then, for any idempotent $\mathbb{D}$-measure $\mathbb{K}$ on $\mathcal{A}$ such that $[\mathbb{0}, \mathbb{K}(\Omega)]$ is a dually continuous lattice, c^* is a density of $\mathbb{K}$ in $\mathcal{A}$ (and a density of $\mathbb{K}^+$ in $\mathcal{G}$).

Proof. As $\mathcal{G}$ is the set of countable unions of elements of $\mathcal{A}$, $\mathcal{G}$ satisfies the same property as $\mathcal{A}$. Now we prove that c^* is a density of $\mathbb{K}^+$ in $\mathcal{G}$.

We still have $\mathbb{K}^+(G) = \mathbb{K}^*(G) \succeq \overline{\mathbb{K}}(G)$ for any $G \in \mathcal{G}$. On the other hand, using again property (3), we have for any $A \in \mathcal{G}$

$$\overline{\mathbb{K}}(A) = \inf_{(G_\cdot \in \mathcal{G}^A, G_\omega \ni \omega \ \ \forall \omega \in A)} \left(\sup_{\omega \in A} \mathbb{K}^+(G_\omega) \right).$$

We can extract from the cover $A \subset \bigcup_{\omega \in A} G_\omega$, with A and G_ω in $\mathcal{G}$, a countable subcover: $A \subset \bigcup_{i \in I} G_{\omega_i}$. As I is countable, $\bigcup_{i \in I} G_{\omega_i} \in \mathcal{G}$ and $\mathbb{K}^+(A) \preceq \mathbb{K}^+(\bigcup_{i \in I} G_{\omega_i}) = \sup_{i \in I} \mathbb{K}^+(G_{\omega_i}) \preceq \sup_{\omega \in A} \mathbb{K}^+(G_\omega)$. Then, $\overline{\mathbb{K}}(A) \succeq \mathbb{K}^+(A)$ for any A in $\mathcal{G}$. □

Corollary 3.10. *Consider a topological space Ω such that the set of open sets $\mathcal{A}$ satisfies the conditions of Theorem* 3.9. *Then, any idempotent $\mathbb{D}$-measure $\mathbb{K}$ on $\mathcal{A}$, such that $[\mathbb{0}, \mathbb{K}(\Omega)]$ is a dually continuous lattice, has c^* as density on $\mathcal{A}$, and $\mathbb{K}^*$ has c^* as density on $\mathcal{P}(\Omega)$.*

Remark 3.11. A regular topological space Ω that satisfies the assumptions of Corollary 3.10 is a hereditarily Lindelöf space (all subsets A of Ω are Lindelöf, that is for every open cover of A, there exist a countable subcover) or equivalently a perfectly normal Lindelöf space (a normal Lindelöf space such that all open sets are $\mathcal{F}_\sigma$ sets, that is countable unions of closed sets) [14].

Corollary 3.12. *Consider a set Ω and a Boolean semi-algebra $\mathcal{A}$ of Ω. Suppose that there exists a countable subset $\mathcal{B}$ of $\mathcal{A}$, such that one of the following equivalent conditions holds*:

- *for any $\omega \in A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\omega \in B \subset A$ ($\mathcal{B}$ is a "basis of neighborhoods")*,
- *any set $A \in \mathcal{A}$ is a union of elements of $\mathcal{B}$: $A = \bigcup_{i \in I} B_i$.*

Then, $\mathcal{G}$ is stable by any union operation and thus defines a topology on Ω with a countable basis of neighborhoods. Moreover, $\mathcal{A}$ satisfies the assumptions of Theorem 3.9 *and thus the conclusion of Corollary* 3.10 *holds.*

Remark 3.13. Let Ω be a Hausdorff topological space, $\mathcal{A}$ the set of its open sets and $\mathcal{K}$ the set of its compact sets. By definition, compact sets satisfy a stronger property than that of Theorem 3.9: for any cover of $C \in \mathcal{K}$ by elements of $\mathcal{A}$, there exists a finite subcover of C. Therefore, the proof of Theorem 3.9 yields $\mathbb{K}^*(C) = \sup_{x \in C} c^*(x)$ for all $C \in \mathcal{K}$, that is c^* is a density of the restriction of $\mathbb{K}^*$ to $\mathcal{K}$. This property only uses the additivity of $\mathbb{K}$ on $\mathcal{A}$ (the σ-additivity condition is not needed). Combined with the capacity property introduced in [24, 26, 25], that we recall in Definition 3.14, it leads to the existence of a density for $\mathbb{K}$ and $\mathbb{K}^*$ (see Proposition 3.15 below).

Definition 3.14. Let Ω be a Hausdorff topological space, $\mathcal{G}$ the set of its open sets and $\mathcal{K}$ the set of its compact sets. A $\mathbb{D}$-*capacity* on Ω is a map $\mathbb{K}$ from $\mathcal{P}(\Omega)$ into $\overline{\mathbb{D}}$ such that:

1. $\mathbb{K}(\emptyset) = \mathbb{0}$,
2. $\mathbb{K}(A) \preceq \mathbb{K}(B)$ for all $A \subset B \subset \Omega$,
3. $\mathbb{K}$ is outer continuous:

$$\mathbb{K}(C) = \inf_{G \in \mathcal{G},\, G \supset C} \mathbb{K}(G) \quad \text{for all } C \in \mathcal{K},$$

4. $\mathbb{K}$ is inner continuous:

$$\mathbb{K}(A) = \sup_{C\in\mathcal{K},\, C\subset A} \mathbb{K}(C) \quad \text{for all } A \subset \Omega.$$

When $\mathbb{D} = (\mathbb{R}^+, \max)$, the definition of a $\mathbb{D}$-capacity coincides with that of a capacity [24, 26, 25]. An idempotent $\mathbb{D}$-probability $\mathbb{K}^*$ on $\mathcal{P}(\Omega)$ which is a capacity is necessarily a sup-measure [26, 25], for which the existence of a density is stated in [25, Prop. 1.2c] (the proof can be found in [26, page 49]). Using Remark 3.13 and Proposition 3.7, this result can be generalized to dually continuous lattices.

Proposition 3.15. *Let Ω be a Hausdorff topological space, $\mathcal{A}$ the set of its open sets, $\mathcal{K}$ the set of its compact sets. Let $\mathbb{K}$ be an idempotent $\mathbb{D}$-measure on $\mathcal{A}$ such that $[\mathbb{0}, \mathbb{K}(\Omega)]$ is a dually continuous lattice. The following conditions are equivalent:*

1. $\mathbb{K}$ *has a density on* $\mathcal{A}$,
2. $\mathbb{K}^*$ *has* c^* *as density on* $\mathcal{P}(\Omega)$,
3. $\mathbb{K}^*$ *is inner continuous*,
4. $\mathbb{K}^*$ *is a* $\mathbb{D}$*-capacity on* Ω.

Moreover, the equivalence is still true when $\mathbb{K}$ is only supposed to be a morphism between the monoids $(\mathcal{A}, \cup)$ and $(\mathbb{D}, \oplus)$, that is if it satisfies conditions 1 *and* 2 *of Definition* 2.5.

Therefore, Theorem 3.9 provides conditions on the topology of Ω under which the maximal extension $\mathbb{K}^*$ of an idempotent measure on $\mathcal{A}$ is a capacity (or is inner regular). It is similar to the regularity and tightness result for classical probabilities in Polish (complete separable, that is with a dense countable subset, and metrizable) spaces. Below are examples of applications of Theorem 3.9. Let us first generalize the notion of tightness to idempotent measures.

Definition 3.16. Let Ω be a Hausdorff topological space, $\mathcal{A}$ the set of its open sets and $\mathcal{K}$ the set of its compact sets. An idempotent $\mathbb{D}$-measure $\mathbb{K}$ on $\mathcal{A}$ is *tight* if

$$\inf_{C\in\mathcal{K}} \mathbb{K}(C^c) = \mathbb{0}$$

where C^c denotes the complementary set of C.

Example 3.17. A separable metrizable space (in particular a Polish space) has a countable basis of neighborhoods. Thus, the conclusion of Corollary 3.10 holds. This includes any separable Banach space E endowed with the strong topology, thus almost all classical functional spaces: $L^p(\Omega)$ for $1 \le p < +\infty$ and Ω an open set of $\mathbb{R}^n$, $W^{k,p}(\Omega)$.... But contrarily to the classical case, completeness is not necessary to prove the inner regularity of $\mathbb{K}^*$. The proof is also simpler.

Example 3.18. Any Banach space E such that its dual space E' is separable has a countable basis of neighborhoods for the weak topology, and any dual space E' of a separable Banach space E has a countable basis of neighborhoods for the weak-$*$ topology. Thus, the result holds for $L^p(\Omega)$ endowed with the weak topology if $1 < p < +\infty$, for $L^\infty(\Omega)$ endowed with the weak-$*$ topology...

Example 3.19. If Ω is a Hausdorff topological space such that $\Omega = \bigcup_{n\in\mathbb{N}} C_n$ with C_n compact metrizable, then the set of open sets $\mathcal{A}$ satisfies the assumptions of Theorem 3.9 (any open set is the countable union of compact sets) and thus the conclusion of Corollary 3.10 holds. Example 3.18 may also be treated along these lines.

Example 3.20. A tight idempotent $\mathbb{D}$-measure $\mathbb{K}$ on the set $\mathcal{A}$ of open sets of a metrizable space Ω has a density when $[\mathbb{0},\mathbb{K}(\Omega)]^{\mathrm{op}}$ is a continuous lattice with a countable basis I. Indeed, let I_0 be the set of $y \in I$ such that $y \ll^{\mathrm{op}} \mathbb{0}$. For all $y \in I_0$, there exists $C_y \in \mathcal{K}$ such that $y \preceq^{\mathrm{op}} \mathbb{K}(C_y^c)$. Let $\Omega' = \bigcup_{y\in I_0} C_y$. We have $\mathbb{K}^*(\Omega'^c) \preceq \mathbb{K}(C_y^c) \preceq y$ for all $y \in I_0$, then $\mathbb{K}^*(\Omega'^c) = \mathbb{0}$ (I is a basis). Therefore, $\mathbb{K}^*(A) = \mathbb{K}^*(A \cap \Omega')$ for all $A \subset \Omega$. From Example 3.19, the restriction of $\mathbb{K}^*$ to the open sets of Ω' has necessarily the density $c(\omega) = \inf_{A\in\mathcal{A},\ A\cap\Omega'\ni\omega} \mathbb{K}^*(A\cap\Omega') = \inf_{A\in\mathcal{A},\ A\ni\omega} \mathbb{K}^*(A) = c^*(\omega)$, $\omega \in \Omega'$. Here, $c^*(\omega) = \mathbb{K}^*(\{\omega\})$ for $\omega \in \Omega$. Therefore, $\mathbb{K}(A) = \mathbb{K}^*(A\cap\Omega') = \sup_{\omega\in A\cap\Omega'} c^*(\omega) = \sup_{\omega\in A} c^*(\omega)$ for any $A \in \mathcal{A}$. Thus, $\mathbb{K}^*$ has c^* as density.

Example 3.21. We may find non-separable complete metric spaces in which the conclusion of Theorem 3.9 is false. Let us consider Ω a non-separable normed vector space (such as $L^\infty((0,1))$), and denote by $B(\omega,r)$ the open ball of center ω and radius r. For any idempotent semiring $\mathbb{D}$, we define on the set of open sets of Ω, the following idempotent $\mathbb{D}$-measure:

$$\mathbb{K}(A) = \begin{cases} \mathbb{0} & \text{if } \exists\ (\omega_n) \in \Omega^{\mathbb{N}} \text{ such that } A \subset \bigcup_{n\in\mathbb{N}} B(\omega_n, 1), \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

By the definition, we obtain $c^*(\omega) \preceq \mathbb{K}(B(\omega,1)) = \mathbb{0}$, thus $c^*(\omega) = \mathbb{0}$ for any $\omega \in \Omega$. Nevertheless, $\mathbb{K}(\Omega) = \mathbb{1}$ which implies that $\mathbb{K}$ has no density (otherwise c^* would be a density). Indeed, if $\Omega \subset \bigcup_n B(\omega_n, 1)$ then, by linearity, $\Omega \subset \bigcup_n B(\frac{\omega_n}{m}, \frac{1}{m})$ for any positive integer m. This implies that the countable set $\{\frac{\omega_n}{m},\ \ m \in \mathbb{N}^*,\ n \in \mathbb{N}\}$ is dense in Ω, which contradicts the non-separability of Ω.

Since the property imposed on the sets of $\mathcal{A}$, in Theorem 3.9, is satisfied by any countable union of compact sets when $\mathcal{A}$ is composed of open sets, we have the following corollary of Theorem 3.9.

Corollary 3.22. *Let Ω be a Hausdorff topological space such that Ω is a countable union of compact sets, and let $\mathcal{A}$ be the set of $\mathcal{F}_\sigma$ open sets, defined as the open sets which are countable unions of closed sets. Then, $\mathcal{A}$ is a semi-σ-algebra which satisfies the conditions of Theorem* 3.9*. Thus, any idempotent $\mathbb{D}$-measure $\mathbb{K}$ such that $[\mathbb{0},\mathbb{K}(\Omega)]$ is a dually continuous lattice has c^* as density on $\mathcal{A}$.*

However, $\mathcal{A}$ is in general not stable by any infinite union operation, thus $\mathbb{K}^*$ may not have c^* as density, as shown in Example 3.24 below. Let us note that $\mathcal{A}$ plays the same role as the Baire sets σ-algebra in classical probability theory: this is the semi-σ-algebra making continuous functions semi-measurable (see Section 4).

Example 3.23. If Ω is a Hausdorff topological space, any tight idempotent $\mathbb{D}$-measure on the set $\mathcal{A}$ of $\mathcal{F}_\sigma$ open sets, such that $[\mathbb{0},\mathbb{K}(\Omega)]^{\mathrm{op}}$ is a continuous lattice with countable basis, has c^* as density. The proof is similar to that of Example 3.20. The intersection with Ω' of a $\mathcal{F}_\sigma$ open set of Ω is a $\mathcal{F}_\sigma$ open set of Ω'. From Corollary 3.22, the restriction of $\mathbb{K}^*$ to the $\mathcal{F}_\sigma$ open sets of Ω' has a density c' on Ω'. Then, $c(\omega) = c'(\omega)$ on Ω' and $c(\omega) = \mathbb{0}$ on Ω'^c is a density of $\mathbb{K}$. Moreover, $\mathbb{K}^*$ has c' as density on the semi-σ-algebra of all $\mathcal{F}_\sigma$ sets. Indeed, the intersection of Ω' with a $\mathcal{F}_\sigma$ set is a countable union of compact sets, for which $\mathbb{K}^*$ has c' as density.

Example 3.24. Let $\Omega = [0,1]^{\mathbb{R}} = \mathcal{F}(\mathbb{R},[0,1])$ be endowed with the product (simple convergence) topology. The topological space Ω is compact but not metrizable. Thus, even if, in general, an idempotent measure has a density on the $\mathcal{F}_\sigma$ open sets semi-σ-algebra $\mathcal{A}$, it may not have a density on the entire open sets σ-algebra $\mathcal{A}'$. Also its maximal extension to all sets may not have a density. For instance, let us consider on $\mathcal{A}$ or $\mathcal{A}'$, the following idempotent $\mathbb{D}$-probability:

$$\mathbb{K}(A) = \begin{cases} \mathbb{0} & \text{if } \exists\, (x_n) \in \mathbb{R}^{\mathbb{N}} \text{ such that } A \subset \{f \in \Omega,\ \inf_n f(x_n) < 1/2\}, \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

By calculation, we find for both $\mathcal{A}$ and $\mathcal{A}'$ semi-σ-algebras the same value of c^* on Ω:

$$c^*(f) = \begin{cases} \mathbb{0} & \text{if } \exists\, x_0 \in \mathbb{R} \text{ such that } f(x_0) < 1/2 \text{ or equivalently } \inf_{x\in\mathbb{R}} f(x) < 1/2, \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

Indeed $\{f \in \Omega,\ f(x) < 1/2\}$ is a $\mathcal{F}_\sigma$ open set for all $x \in \mathbb{R}$.

Consider now $U = \{f \in \Omega,\ \inf_{x\in\mathbb{R}} f(x) < 1/2\} = \bigcup_{x\in\mathbb{R}}\{f \in \Omega,\ f(x) < 1/2\}$. We have $c^*(f) = \mathbb{0}$ for any $f \in U$. The set U is open but it is not a countable union of closed sets. Moreover, U is not included in a countable union of sets of the form $\{f \in \Omega, f(x) < 1/2\}$. Then, if we use the semi-σ-algebra of open sets $\mathcal{A}'$, $U \in \mathcal{A}'$ but $\mathbb{K}(U) = \mathbb{1} \neq \sup_{f\in U} c^*(f)$. Thus, $\mathbb{K}$ has no density in $\mathcal{A}'$. If this time we use the semi-σ-algebra of $\mathcal{F}_\sigma$ open sets $\mathcal{A}$, $\mathbb{K}$ has necessarily c^* as density on $\mathcal{A}$, but $\mathbb{K}^*$ has no density on $\mathcal{P}(\Omega)$ or even on $\mathcal{A}'$. Indeed, $\mathbb{K}^*$ is given on $\mathcal{P}(\Omega)$ by the same formula as $\mathbb{K}$, then $\mathbb{K}^*(U) = \mathbb{1} \neq \sup_{f\in U} c^*(f)$.

4. Idempotent integration

In [20], Maslov gives a construction of idempotent integrals over semirings $\mathbb{D}$ that are metric spaces with particular properties of the distance. In this context, he proves the following theorem concerning the integration of semi-measurable functions (the notations and assumptions will be made precise later).

Theorem 4.1 ([20, Ch. VIII, Th. 5.3]). *Consider an extension $\mathbb{K}'$ of a finite idempotent measure $\mathbb{K}$ to the least σ-algebra containing $\mathcal{A}$. The idempotent integrals with respect to $\mathbb{K}'$ and $\mathbb{K}^*$ of any* (*bounded*) *lower semi-measurable function taking its values in a separable subspace of $\mathbb{D}$ are equal.*

This result is a direct consequence of the construction of the integral. Since $\mathbb{K}^*$ has a density in many cases (see Section 3), Theorem 4.1 gives a justification to consider only idempotent measures with density. We generalize here the construction of the idempotent integral to locally continuous lattices and then prove the idempotent "Riesz representation theorem".

Theorem 4.1 was set when $\mathcal{A}$ is a Boolean algebra, but only the Boolean semi-algebra structure is needed. Moreover, $\mathbb{D}$ is supposed to be metrizable with a distance compatible with the semiring and lattice structures, and to have the following property: for any $a \prec b \in \mathbb{D}$, there exists $c \in \mathbb{D}$ such that $a \prec c \prec b$. Then, a lower semi-measurable function is a function f from Ω to $\mathbb{D}$ such that the sets $\Omega(a) = \{\omega \in \Omega, a \prec f(\omega)\}$ are elements of $\mathcal{G}$ for any $a \in \mathbb{D}$.

In order to generalize this result to any locally continuous lattice $\mathbb{D}$, we have to replace $\prec$ by $\ll$ (way below) in the definition of $\Omega(a)$. In this case, the property "for all $a \ll b \in \mathbb{D}$, there exists $c \in \mathbb{D}$ such that $a \ll c \ll b$" is a consequence of the continuity of the locally complete lattice $\mathbb{D}$ [15]. Then, the separability can be replaced by the existence of a countable basis to the lattice $\mathbb{D}$. Although the existence of a countable basis is equivalent, if $\mathbb{D}$ is a continuous lattice, to the property that $\mathbb{D}$ endowed with the Lawson topology is a compact metric space, the compatibility of the metric with the semiring and lattice structures is not ensured.

In the construction by Maslov of idempotent integrals, $\mathbb{D}$ was not necessarily an idempotent semiring but only an ordered semiring with the law $\oplus$ compatible with the order $\preceq$. This encompasses both classical and idempotent measure theories. Here, we treat idempotent measures with semi-algebras of sets (see Section 2) and semi-measurable functions, whereas classical probabilities or probabilities over symmetrizable ordered semirings (such as $(\mathbb{R}^+,+,\times)^n$) have to be treated with algebras and measurable functions. We thus restrict ourselves to idempotent measures and generalize the construction of idempotent integrals to general locally continuous lattices, using only lattice properties. The generalization of Theorem 4.1 will then be a consequence of this construction.

Remark 4.2. In a locally continuous lattice $\mathbb{D}$, the lower semi-continuity (l.s.c.) is equivalent to the continuity for the Scott topology, which is generated by the sets $\{x\in\mathbb{D},\ a\ll x\}$. Thus, semi-measurability is a natural generalization of semi-continuity in the same way as measurability is a generalization of continuity. For a general lattice $\mathbb{D}$, the set of l.s.c. functions from Ω to $\mathbb{D}$ (in the sense of Definition 1.11) is a sup-semilattice. It is a lattice if $\mathbb{D}$ is locally continuous. It is a $\mathbb{D}$-semimodule (a module over a semiring) if the $\otimes$ operation is distributive with respect to infinite sup.

If the continuity is defined in terms of the order topology (functions are continuous when they are both l.s.c. and u.s.c.), the set of continuous functions from Ω to $\mathbb{D}$ is a $\mathbb{D}$-semimodule if $\mathbb{D}$ is dually locally continuous and $\otimes$ is distributive with respect to infinite sup and filtered inf. It is a lattice if in addition $\mathbb{D}$ is locally continuous. A possible generalization of the classical integration in ordered symmetrizable semirings consists in defining measurable functions as functions f such that the sets $\{\omega\in\Omega,\ a\ll f(\omega)\}$ and $\{\omega\in\Omega,\ a\ll^{\mathrm{op}} f(\omega)\}$ are measurable, for instance Borel sets. But this requires both the local and dual local continuity of $\mathbb{D}$. Let us adopt the continuity notion of Definition 1.14, that is the continuity for the Lawson topology of $\mathbb{D}$, when $\mathbb{D}$ is locally continuous. The set of continuous functions from Ω to $\mathbb{D}$ is a semilattice when $\mathbb{D}$ is locally continuous. However, the sup-semilattice and then the semimodule property cannot be derived from the continuity or the dual continuity of $\mathbb{D}$. The Lawson continuity is then even less satisfactory than the order continuity, but since it is weaker, the normality condition of Definition 4.7 below is more easily fulfilled. In the sequel (for the Riesz representation theorem), we suppose that one of the two continuity notions is chosen once and for all and denote by $\mathcal{C}(\Omega,\mathbb{D})$ (resp. $\mathcal{C}_{\mathrm{b}}(\Omega,\mathbb{D})$) the set of continuous (resp. upper bounded continuous) functions from Ω to $\mathbb{D}$.

Proposition 4.3. *Let $\mathcal{A}$ be a Boolean semi-algebra of subsets of Ω and let $\mathcal{G}$ be the semi-σ-algebra generated by $\mathcal{A}$. We denote by $\mathcal{L}(\Omega,\mathcal{A})$ the set of (finite) $\mathbb{D}$-linear combinations of characteristic functions $\mathbb{1}_A$ of sets $A\in\mathcal{A}$ and by $\mathcal{I}(\Omega,\mathcal{A})$ the set of functions from Ω to $\mathbb{D}$ which are nondecreasing limits of elements of $\mathcal{L}(\Omega,\mathcal{A})$.*

For a general idempotent semiring $\mathbb{D}$, $\mathcal{L}(\Omega,\mathcal{A})$ is a $\mathbb{D}$-semi-algebra (an algebra over a semiring) and $\mathcal{I}(\Omega,\mathcal{A})=\mathcal{I}(\Omega,\mathcal{G})$ is stable by countable upper bounded (by any function) supremum. If the $\otimes$ law is distributive with respect to upper bounded countable sup, *$\mathcal{I}(\Omega,\mathcal{A})$ is a $\mathbb{D}$-semi-algebra. If in addition $\mathbb{D}$ is a locally continuous lattice, then $\mathcal{L}(\Omega,\mathcal{A})$ and $\mathcal{I}(\Omega,\mathcal{A})$ are lattices.*

Proof. By construction, $\mathcal{L}(\Omega, \mathcal{A})$ is a $\mathbb{D}$-semimodule thus a sup-semilattice and it is a semi-algebra since $\mathcal{A}$ is stable by finite intersection. Again by construction, $\mathcal{I}(\Omega, \mathcal{A})$ is stable by countable upper bounded supremum. It is equal to $\mathcal{I}(\Omega, \mathcal{G})$ since $a \otimes \mathbb{1}_{\bigcup_{n\in\mathbb{N}} A_n} = \bigoplus_{n\in\mathbb{N}} a \otimes \mathbb{1}_{A_n}$. The distributivity of $\otimes$ with respect to countable $\oplus$ implies that it is a $\mathbb{D}$-semi-algebra.

In order to prove that $\mathcal{L}(\Omega, \mathcal{A})$ and $\mathcal{I}(\Omega, \mathcal{A})$ are lattices, we need a formula of the form: $(\bigoplus_i \lambda_i \otimes \mathbb{1}_{A_i}) \wedge (\bigoplus_j \mu_j \otimes \mathbb{1}_{B_j}) = \bigoplus_{i,j} (\lambda_i \wedge \mu_j) \otimes \mathbb{1}_{A_i \cap B_j}$. This holds if $\mathbb{D}$ is locally continuous and the sums are directed and upper bounded. But any sum $\bigoplus_i \lambda_i \otimes \mathbb{1}_{A_i}$ may be replaced by the sum of all terms $(\bigoplus_{i\in I} \lambda_i) \otimes \mathbb{1}_{\bigcap_{i\in I} A_i}$ for I finite, whose values in any point form a directed set. $\square$

Proposition 4.4. *Let us denote by $\mathcal{S}(\Omega, \mathcal{G})$ the set of semi-measurable functions with respect to $\mathcal{G}$: $\mathcal{S}(\Omega, \mathcal{G}) = \{f : \Omega \mapsto \mathbb{D},\ \Omega_f(a) \in \mathcal{G}\ \forall a \in \mathbb{D}\}$, where $\Omega_f(a) = \{\omega \in \Omega,\ a \ll f(\omega)\}$. For any semi-measurable function f, we denote by $\mathcal{G}(f)$ the semi-σ-algebra generated by the sets $\Omega_f(a)$ for $a \in \mathbb{D}$.*

We have $\mathcal{I}(\Omega, \mathcal{G}) \subset \mathcal{S}(\Omega, \mathcal{G})$ and any function f of $\mathcal{I}(\Omega, \mathcal{G})$ is such that $\mathcal{G}(f)$ has a countable basis in $\mathcal{G}$ or $\mathcal{A}$, that is a countable subset $\mathcal{B}$ of $\mathcal{G}$ (not necessarily included in $\mathcal{G}(f)$) stable by finite intersection, such that the elements of $\mathcal{G}(f)$ are unions of elements of $\mathcal{B}$.

Let us suppose now that $\mathbb{D}$ is a locally continuous lattice and that $\otimes$ is distributive with respect to upper bounded infinite sup. *Then, $\mathcal{I}(\Omega, \mathcal{G})$ is exactly the set of functions $f \in \mathcal{S}(\Omega, \mathcal{G})$ such that $\mathcal{G}(f)$ has a countable basis. If $\mathbb{D}$ has a countable basis or $\mathcal{A}$ has a countable basis, then*

$$\mathcal{I}(\Omega, \mathcal{G}) = \mathcal{S}(\Omega, \mathcal{G}).$$

Proof. Consider a function $f = \bigoplus_i \lambda_i \otimes \mathbb{1}_{A_i} \in \mathcal{I}(\Omega, \mathcal{G})$ where the sum is countable and directed (as in previous proof) and the sets $A_i \in \mathcal{A}$ or $\mathcal{G}$. The set of A_i is stable by finite intersection and the set of λ_i by finite addition. For any $a \in \mathbb{D}$, $\Omega_f(a) = \bigcup_{i,\ a \ll \lambda_i} A_i \in \mathcal{G}$, thus $f \in \mathcal{S}(\Omega, \mathcal{G})$. In addition, $\Omega_f(a) \cap \Omega_f(b) = \Omega_f(a \oplus b)$, therefore $\mathcal{G}(f)$ is the set of countable unions of sets $\Omega_f(a)$ and thus is included in the set of unions of sets A_i which forms a countable basis of $\mathcal{G}(f)$.

Now, suppose that $\mathbb{D}$ is locally continuous and consider $f \in \mathcal{S}(\Omega, \mathcal{G})$ such that $\mathcal{G}(f)$ has a countable basis $\mathcal{B}$ in $\mathcal{G}$. Since $f(\omega) = \sup\{a \in \mathbb{D}, a \ll f(\omega)\}$ for any $\omega \in \Omega$, we obtain

$$f = \bigoplus_{a\in\mathbb{D}} a \otimes \mathbb{1}_{\Omega_f(a)} = \bigoplus_{a\in\mathbb{D}} a \otimes \left(\bigoplus_{B\in\mathcal{B},\ B\subset\Omega_f(a)} \mathbb{1}_B \right) = \bigoplus_{B\in\mathcal{B}} \lambda(B) \otimes \mathbb{1}_B$$

with $\lambda(B) = \sup\{a \in \mathbb{D},\ B \subset \Omega_f(a)\}$. Then, as $\mathcal{B}$ is countable, $f \in \mathcal{I}(\Omega, \mathcal{A})$. In the previous equalities, we have used the distributivity of the $\otimes$ law with respect to infinite $\oplus$.

Now, if $\mathcal{A}$ has a countable basis, for any $f \in \mathcal{S}(\Omega, \mathcal{G})$, $\mathcal{G}(f)$ has a countable basis, thus $\mathcal{I}(\Omega, \mathcal{G}) = \mathcal{S}(\Omega, \mathcal{G})$.

If this time $\mathbb{D}$ has a countable basis, the Scott topology has a countable basis and since $\mathcal{G}(f)$ is the inverse image of the Scott topology by the function f, $\mathcal{G}(f)$ also has a countable basis. $\square$

In general $\mathcal{S}(\Omega, \mathcal{G})$ is not a semi-algebra (it is not stable by addition) except if $\mathbb{D}$ has a countable basis or if $\mathcal{A}$ is stable by any union operation. But this last property implies that $\mathcal{A}$ is a topology and $\mathcal{S}(\Omega, \mathcal{G})$ is in fact the set of l.s.c. functions.

Remark 4.5. If $\mathbb{D}$ is a locally continuous lattice with countable basis, the distributivity of $\otimes$ with respect to infinite sup is equivalent to the distributivity with respect to countable sup.

Proposition 4.6. *Let $\mathbb{D}$ be a semiring such that $\otimes$ is distributive with respect to upper bounded infinite* sup *and let $\mathbb{K}$ be an idempotent $\mathbb{D}$-probability or a finite idempotent $\mathbb{D}$-measure on $(\Omega, \mathcal{A})$ with extension $\mathbb{K}^+$ to $\mathcal{G}$.*

If $\mathbb{D}$ is a locally continuous lattice or a locally dually continuous lattice, there exists a unique $\mathbb{D}$-linear form $\mathbb{V}$ on $\mathcal{I}(\Omega, \mathcal{A})$, continuous on converging nondecreasing sequences (i.e. such that $\mathbb{V}(f_n) \underset{n\to+\infty}{\nearrow} \mathbb{V}(f)$ if $f_n \underset{n\to+\infty}{\nearrow} f$), and extending $\mathbb{K}$ in the sense that $\mathbb{V}(\mathbb{1}_A) = \mathbb{K}(A)$ for any $A \in \mathcal{A}$.

In the two following cases, we have a general expression for $\mathbb{V}$:

- *If $\mathbb{D}$ is locally continuous, then*

$$\mathbb{V}(f) = \bigoplus_{a\in\mathbb{D}} a \otimes \mathbb{K}^+(\Omega_f(a)).$$

- *If $[\mathbb{0}, \mathbb{K}(\Omega)]$ is dually continuous and $\mathcal{A}$ has a countable basis, or more generally if $\mathbb{K}^+$ has a density c^*, then*

$$\mathbb{V}(f) = \bigoplus_{\omega\in\Omega} f(\omega) \otimes c^*(\omega).$$

Proof. Consider a $\mathbb{D}$-linear form $\mathbb{V}$ on $\mathcal{I}(\Omega, \mathcal{A})$, continuous on converging nondecreasing sequences and such that $\mathbb{V}(\mathbb{1}_A) = \mathbb{K}(A)$ for any A in $\mathcal{A}$. The continuity implies that $\mathbb{V}(\mathbb{1}_A) = \mathbb{K}^+(A)$ for any $A \in \mathcal{G}$. Now, if $f \in \mathcal{I}(\Omega, \mathcal{A})$, $f = \bigoplus_i \lambda_i \otimes \mathbb{1}_{A_i}$, where the sum is countable and $A_i \in \mathcal{A}$. Then, $\mathbb{V}(f) = \bigoplus_i \lambda_i \otimes \mathbb{K}(A_i)$. If this expression only depends on f, that is if

$$\bigoplus_{i\in I} \lambda_i \otimes \mathbb{1}_{A_i} = \bigoplus_{j\in J} \mu_j \otimes \mathbb{1}_{B_j} \Rightarrow \bigoplus_{i\in I} \lambda_i \otimes \mathbb{K}(A_i) = \bigoplus_{j\in J} \mu_j \otimes \mathbb{K}(B_j) \tag{7}$$

for any countable sets I and J and for A_i and B_j in $\mathcal{A}$, $\mathbb{V}$ may be defined in that way. Then, if $\otimes$ is distributive with respect to upper bounded countable sup, $\mathbb{V}$ satisfies the properties of the proposition. Before proving (7) for particular cases, let us note that it is equivalent to

$$\mu \otimes \mathbb{1}_B \preceq \bigoplus_{i\in I} \lambda_i \otimes \mathbb{1}_{A_i} \Rightarrow \mu \otimes \mathbb{K}(B) \preceq \bigoplus_{i\in I} \lambda_i \otimes \mathbb{K}(A_i) \tag{8}$$

for any countable set I. Moreover, we only need to prove (8) for "directed" sums (indeed, adding terms of the form $(\bigoplus_{i\in J} \lambda_j) \otimes \mathbb{1}_{\bigcap_{j\in J} A_j}$, with J finite, to the first expression does not change the second expression).

Let us suppose $\otimes$ distributive with respect to upper bounded infinite sup and first prove that (8) holds in a locally continuous lattice $\mathbb{D}$. In this case, for any $f \in \mathcal{I}(\Omega, \mathcal{A})$, $\mathcal{G}(f)$ has a countable basis in $\mathcal{A}$, denoted $\mathcal{B}$, and following the previous proof, we have

$$f = \bigoplus_{B\in\mathcal{B}} \lambda(B) \otimes \mathbb{1}_B$$

with $\lambda(B)=\sup\{a\in\mathbb{D},\ B\subset\Omega_f(a)\}$. Thus,

$$\begin{aligned}\mathbb{V}(f) &= \bigoplus_{B\in\mathcal{B}}\lambda(B)\otimes\mathbb{K}(B)\\ &= \bigoplus_{a\in\mathbb{D}} a\otimes\left(\bigoplus_{B\in\mathcal{B},\ B\subset\Omega_f(a)}\mathbb{K}(B)\right)\\ &= \bigoplus_{a\in\mathbb{D}} a\otimes\mathbb{K}^+(\Omega_f(a)).\end{aligned}$$

Suppose now that $\mu\otimes\mathbb{1}_B\preceq\bigoplus_{i\in I}\lambda_i\otimes\mathbb{1}_{A_i}$, that is $\mu\preceq\bigoplus_{i\in I,\ \omega\in A_i}\lambda_i$ for any $\omega\in B$, with I countable and the set of values of the sum directed. Thus, for any $a\ll\mu$, $B\subset\bigcup_{i\in I,\ a\preceq\lambda_i}A_i$, and

$$\begin{aligned}a\otimes\mathbb{K}(B) &\preceq a\otimes\left(\bigoplus_{i\in I,\ a\preceq\lambda_i}\mathbb{K}(A_i)\right)\\ &\preceq \bigoplus_{i\in I,\ a\preceq\lambda_i}\lambda_i\otimes\mathbb{K}(A_i)\\ &\preceq \bigoplus_{i\in I}\lambda_i\otimes\mathbb{K}(A_i).\end{aligned}$$

Taking the supremum over $a\ll\mu$ and using the infinite distributivity of $\otimes$, we obtain (8).

Let us prove now that (8) holds if $[\mathbb{0},\mathbb{K}(\Omega)]$ is a dually continuous lattice. Suppose that $\mu\otimes\mathbb{1}_B\preceq\bigoplus_{i\in I}\lambda_i\otimes\mathbb{1}_{A_i}$, with I countable and the set of values of the sum directed. Denote by $\mathcal{B}$ the set composed of sets A_i, B and $A_i\cap B$ and by $\mathcal{G}'$ the semi-σ-algebra generated by $\mathcal{B}$, that is the set of unions of elements of $\mathcal{B}$. Then, $\mathcal{B}$ is a countable basis of $\mathcal{G}'$ and by Corollary 3.12, $\mathbb{K}^+$ has a density c^* on $\mathcal{G}'$. Using the infinite distributivity of $\otimes$, we obtain

$$\begin{aligned}\mu\otimes\mathbb{K}(B) &= \bigoplus_{\omega\in B}\mu\otimes c^*(\omega)\\ &\preceq \bigoplus_{\omega\in B}\left(\bigoplus_{i\in I,\ \omega\in A_i}\lambda_i\right)\otimes c^*(\omega)\\ &\preceq \bigoplus_{i\in I}\lambda_i\otimes\left(\bigoplus_{\omega\in A_i\cap B}c^*(\omega)\right)\\ &\preceq \bigoplus_{i\in I}\lambda_i\otimes\mathbb{K}(A_i)\end{aligned}$$

which proves (8). We may prove along the same lines that

$$\mathbb{V}(f)=\bigoplus_{\omega\in\Omega}f(\omega)\otimes c^*(\omega),$$

where c^* is a density of $\mathbb{K}$ on the semi-σ-algebra generated by the A_i such that $f=\bigoplus_{i\in I}\lambda_i\otimes\mathbb{1}_{A_i}$ with I countable. But since c^* depends on the sets A_i, thus on f, this does not lead to a general expression for $\mathbb{V}(f)$, except if c^* is a density of $\mathbb{K}^+$ in the entire algebra $\mathcal{G}$. $\square$

A semi-measurable function f is said to be integrable if $\mathbb{V}(f) \in \mathbb{D}$. The linear form $\mathbb{V}$ coincides with the integral defined by Maslov. It will then be denoted by

$$\mathbb{V}(f) = \oint_{\Omega} f(\omega) \otimes \mathbb{K}(d\omega).$$

It can be defined for any Boolean semi-algebra or semi-σ-algebra $\mathcal{A}$. In particular, for any extension $\mathbb{K}'$ of $\mathbb{K}$ to a larger semi-σ-algebra $\mathcal{A}'$, we may define an integral $\mathbb{V}'$ on the set of semi-measurable functions with respect to $\mathcal{A}'$. By uniqueness, $\mathbb{V}'$ coincides with $\mathbb{V}$ on the set of semi-measurable functions with respect to $\mathcal{A}$. This is the result of Theorem 4.1. If $\mathcal{A}'$ is the σ-algebra generated by $\mathcal{A}$ and $\mathbb{D} = \mathbb{R}_{\max}$, semi-measurable functions coincide with classical measurable functions and there are at least as many integrals $\mathbb{V}'$ as extensions $\mathbb{K}'$ of $\mathbb{K}$. If $[\mathbb{0}, \mathbb{K}(\Omega)]$ is a dually continuous lattice, we can also consider the maximal continuous linear form $\mathbb{V}^*$ on $\mathcal{I}(\Omega, \mathcal{P}(\Omega))$, such that $\mathbb{V}^*(\mathbb{1}_A) = \mathbb{K}(A)$ for any $A \in \mathcal{A}$. By the linearity and the continuity of $\mathbb{V}^*$, $\mathbb{K}'(A) = \mathbb{V}^*(\mathbb{1}_A)$ is an idempotent $\mathbb{D}$-measure extending $\mathbb{K}$, and $\mathbb{V}^*$ is the integral associated with $\mathbb{K}'$. Thus, $\mathbb{V}^*$ is lower than the integral associated with $\mathbb{K}^*$ ($\mathbb{K}' \preceq \mathbb{K}^*$), and then coincides with it.

If $\mathbb{K}$ has a density c on $\mathcal{A}$, then for any $f \in \mathcal{I}(\Omega, \mathcal{A})$:

$$\mathbb{V}(f) = \bigoplus_{\omega \in \Omega} f(\omega) \otimes c(\omega). \tag{9}$$

In this case, we will say that $\mathbb{V}$ has a *density*. Theorem 4.1 or Propositions 4.4-4.6 together with the results of the previous section imply that for any l.s.c. function f, the integral of f with respect to a finite $\mathbb{R}_{\max}$-measure $\mathbb{K}$ defined on the open sets or Borel sets has the form:

$$\mathbb{V}(f) = \sup_{\omega \in \Omega} f(\omega) + c^*(\omega) \tag{10}$$

with c^* an u.s.c. function ($\mathbb{R}_{\max}$ is locally continuous with countable basis and locally dually continuous).

In [18, 19] Kolokoltsov and Maslov prove that any continuous (for the uniform convergence topology) linear form on the $\mathbb{D}$-semimodule $\mathcal{C}_K(\Omega)$ of continuous functions with compact support from Ω to $\mathbb{D} = \mathbb{R}_{\max}$ has the form (10). As from any bounded measure on $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is the set of open sets, we can construct an integral which is a continuous linear form on $\mathcal{C}_K(\Omega)$; the existence of a density to this measure may have been deduced from (10). Conversely, (10) can be deduced from the existence of the density of any idempotent measure by using the Riesz representation theorem [20]. However, even if some generalizations of (10) may be done (see Kolokoltsov [17] and Maslov and Kolokoltsov [21]), many restrictions on the semiring $\mathbb{D}$ and the topological space Ω are necessary in order to get (10) or the Riesz representation theorem, restrictions which are not needed to prove the existence of a density.

We give now a “probabilistic” version of the Riesz representation theorem. This approach allows us to consider general functional spaces, when the “integration” point of view adopted in [18, 19, 17] imposes to the topological space Ω to be locally compact. We thus consider idempotent probabilities (bounded measures can be handled identically) and linear forms on the set $\mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})$ of upper bounded continuous functions, where Ω needs only to be normal with respect to $\mathbb{D}$ [20]. Note that in general, $\mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})$ is not a $\mathbb{D}$-semimodule (see Remark 4.2). We adopt the following definition of normality.

Definition 4.7. We say that the topological space Ω is $\mathbb{D}$-*normal*, if for any disjoint closed sets F and G of Ω and for any $a \in \mathbb{D}$, there exists a continuous function f from Ω to $[\mathbb{0}, a] \subset \mathbb{D}$ such that f is equal to $\mathbb{0}$ on F and a on G.

A possible generalization of the classical definition of normality is given by the restriction of the previous condition to the case $a = \mathbb{1}$. However, the proof of the Riesz representation theorem requires the previous condition in its general form. On the other hand, we may deduce the general normality condition from the restricted normality, when the product $\otimes$ is continuous (for the good topology). If $\mathbb{D}$ is a connected by arcs topological space, in particular for $\mathbb{D} = \mathbb{R}_{\max}$, $\mathbb{R}_{\min}$ or $(\overline{\mathbb{R}}, \max, \min)$, the $\mathbb{D}$-normality follows from the classical normality. For instance, Ω may be any metric space.

Theorem 4.8 ("Riesz representation theorem")**.** *Suppose that $\mathbb{D}$ is a nontrivial* ($\neq \{\mathbb{0}\}$) *locally continuous lattice with countable basis, such that $\otimes$ is distributive with respect to upper bounded countable* sup *and let Ω be a $\mathbb{D}$-normal topological space.*

We suppose that $\mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})$ is a $\mathbb{D}$-semimodule and denote by $\mathcal{A}$ the minimal semi-σ-algebra of subsets of Ω leading (*upper bounded*) *continuous functions to be semi-measurable with respect to $\mathcal{A}$. Then, $\mathcal{A}$ is the set of $\mathcal{F}_\sigma$ open sets* (*where a $\mathcal{F}_\sigma$ set is defined as a countable union of closed sets*) *and the set $\mathcal{S}(\Omega, \mathbb{D})$ of semi-measurable functions with respect to $\mathcal{A}$ is equal to the set of functions from Ω to $\mathbb{D}$ which are nondecreasing limits of continuous functions.*

Let $\mathbb{V}$ be a linear form on $\mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})$, continuous on converging nondecreasing sequences and such that $\mathbb{V}(\mathbb{1}) = \mathbb{1}$. Then, $\mathbb{V}$ may be extended in a continuous linear form on $\mathcal{S}(\Omega, \mathbb{D})$. $\mathbb{V}$ is exactly the idempotent integral

$$\mathbb{V}(f) = \oint_\Omega f(\omega) \otimes \mathbb{K}(d\omega) \tag{11}$$

corresponding to the idempotent $\mathbb{D}$-probability $\mathbb{K}$ defined on $\mathcal{A}$ by

$$\mathbb{K}(A) = \mathbb{V}(\mathbb{1}_A) = \sup_{f \in \mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D}), f \preceq \mathbb{1}_A} \mathbb{V}(f).$$

An idempotent $\mathbb{D}$-probability $\mathbb{K}$ such that (11) *holds is unique. Thus,* (11) *sets up a bijective correspondence between continuous linear forms on $\mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})$ such that $\mathbb{V}(\mathbb{1}) = \mathbb{1}$ and idempotent $\mathbb{D}$-probabilities on* $(\Omega, \mathcal{A})$.

Moreover, if $\otimes$ is distributive with respect to infinite $\oplus$, then $\mathbb{V}$ has a density in the sense of (9) *if and only if $\mathbb{K}$ has a density.*

Proof. Let us prove that $\mathcal{A}$ is the set of $\mathcal{F}_\sigma$ open sets. By definition, $\mathcal{A}$ is the semi-σ-algebra generated by sets $\Omega_f(a)$, with f continuous (and upper bounded). Since $\mathbb{D}$ is locally continuous, any continuous function (for any of the two senses defined in Remark 4.2) is such that $\Omega_f(a)$ is open and $C_f(a) = \{\omega \in \Omega,\ a \preceq f(\omega)\}$ is closed for all $a \in \mathbb{D}$. If I is a countable basis of $\mathbb{D}$, then for any $x \ll y$ in $\mathbb{D}$, there exists $z \in I$ such that $x \ll z \preceq y$ and the converse is also true. Then, $\Omega_f(a) = \bigcup_{b \in I,\ a \ll b} C_f(b)$. Since I is countable, $\Omega_f(a)$ is a $\mathcal{F}_\sigma$ open set. The set of $\mathcal{F}_\sigma$ open sets is a semi-σ-algebra, then $\mathcal{A}$ is included in it.

Suppose now that U is a $\mathcal{F}_\sigma$ open set, i.e. $U = \bigcup_n F_n$ with F_n closed. From the $\mathbb{D}$-normality of Ω, there exist continuous functions f_n from Ω into $[\mathbb{0}, \mathbb{1}]$, such that $f_n = \mathbb{1}$ on F_n and $\mathbb{0}$ on U^c. Since $\mathbb{0} \neq \mathbb{1}$ ($\mathbb{D} \neq \{\mathbb{0}\}$), there exists $a \neq \mathbb{0}$ such that $a \ll \mathbb{1}$. Then, $F_n \subset \Omega_{f_n}(a) \subset U$ for any n, and $U = \bigcup_n \Omega_{f_n}(a) \in \mathcal{A}$.

By construction, the set $\mathcal{S}(\Omega, \mathbb{D}) = \mathcal{S}(\Omega, \mathcal{A}) = \mathcal{I}(\Omega, \mathcal{A})$ contains continuous functions and then nondecreasing limits of continuous functions. Conversely, if $f \in \mathcal{S}(\Omega, \mathbb{D})$ and $a \in \mathbb{D}$, then $\Omega_f(a)$ is an open set such that there exist closed sets $F_{n,a}$ with $\Omega_f(a) = \bigcup_n F_{n,a}$. Then, there exists continuous functions $f_{n,a}$ with values in $[\mathbb{0}, a]$, such that $f_{n,a} = a$ on $F_{n,a}$ and $\mathbb{0}$ on $\Omega_f(a)^c$. This implies $\bigoplus_n f_{n,a} = a \otimes \mathbb{1}_{\Omega_f(a)}$. If I is a countable basis, $f = \bigoplus_{a \in I} a \otimes \mathbb{1}_{\Omega_f(a)} = \bigoplus_{a \in I,\, n \in \mathbb{N}} f_{n,a}$, and then f is the nondecreasing limit of (upper bounded) continuous functions.

Suppose that $\mathbb{V}$ is a linear form on $\mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})$ continuous on converging nondecreasing sequences, and consider $f \in \mathcal{S}(\Omega, \mathbb{D})$. If $f_n \underset{n \to +\infty}{\nearrow} f$ with $f_n \in \mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})$ and $\mathbb{V}$ can be extended to $\mathcal{S}(\Omega, \mathbb{D})$, then $\mathbb{V}(f) = \lim_{n \to +\infty} \mathbb{V}(f_n)$. Since $\mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})$ is stable by finite $\oplus$ (by assumption) and $\wedge$ (by the continuity of $\mathbb{D}$) and $\mathbb{V}$ is continuous on nondecreasing sequences, this formula is independent of the sequence (f_n) and thus defines a $\mathbb{D}$-linear form on $\mathcal{S}(\Omega, \mathbb{D})$, which is continuous on converging nondecreasing sequences. As a consequence, $\mathbb{V}(f) = \sup_{g \preceq f,\, g \in \mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})} \mathbb{V}(g)$.

If $A \in \mathcal{A}$, $\mathbb{1}_A \in \mathcal{S}(\Omega, \mathbb{D})$, then $\mathbb{K}(A) = \mathbb{V}(\mathbb{1}_A)$ defines an idempotent $\mathbb{D}$-probability on $\mathcal{A}$. By uniqueness of the integral (see Proposition 4.6), $\mathbb{V}$ is exactly the integral associated with $\mathbb{K}$. Moreover, (11) implies $\mathbb{K}(A) = \mathbb{V}(\mathbb{1}_A)$ which implies the uniqueness of $\mathbb{K}$. By Proposition 4.6, $\mathbb{V}$ has a density if $\mathbb{K}$ has a density and $\otimes$ is distributive with respect to infinite $\oplus$. Conversely, if $\mathbb{V}$ has a density denoted by c, then clearly $\mathbb{K}$ has c as density: $\mathbb{K}(A) = \mathbb{V}(\mathbb{1}_A) = \bigoplus_{\omega \in \Omega} \mathbb{1}_A(\omega) \otimes c(\omega) = \bigoplus_{\omega \in A} c(\omega)$. □

If Ω is a metric space, every open set is a $\mathcal{F}_\sigma$ set. Then, in a Polish space, every continuous (on converging nondecreasing sequences) linear form on $\mathcal{C}_{\mathrm{b}}(\Omega, \mathbb{D})$ admits the representation (9) ((10) in $\mathbb{R}_{\max}$). On the other hand, by Theorem 4.8, the counter example of Section 3 leads to a counter example for linear forms. The following continuous linear form $\mathbb{V}$ on $\mathcal{C}_{\mathrm{b}}(\Omega = L^\infty(0,1), \mathbb{R}_{\max})$ does not have a representation of the form (10):

$$\begin{aligned} \mathbb{V}(f) &= \sup \left\{ a \in \mathbb{R},\ \nexists (\omega_n) \in \Omega^{\mathbb{N}},\ \{\omega,\ a < f(\omega)\} \subset \bigcup_n B(\omega_n, 1) \right\} \\ &= \inf_{(\omega_n) \in \Omega^{\mathbb{N}}} \left(\sup_{\omega,\ \|\omega - \omega_n\|_\infty > 1\ \forall n} f(\omega) \right). \end{aligned}$$

Moreover, $\mathbb{V}$ is continuous for the uniform convergence topology defined by the exponential distance $d(x, y) = |e^x - e^y|$; $d(\mathbb{V}(f), \mathbb{V}(g)) \leq \sup_{\omega \in \Omega} d(f(\omega), g(\omega))$.

Remark 4.9. The construction of integrals with respect to measures and the Riesz representation theorem are the necessary ingredients to define the notion of weak convergence of measures: $\mathbb{K}_n$ converges towards $\mathbb{K}$ if $\mathbb{K}_n(f) \to \mathbb{K}(f)$ for all bounded continuous functions, where $\mathbb{K}(f)$ denotes the integral of f with respect to the measure $\mathbb{K}$ [23, 4]. In classical probability theory, these ingredients together with some regularity properties of probabilities yield the equivalence of this weak convergence definition with another one which uses measures of sets only: $\liminf_n \mathbb{K}_n(U) \geq \mathbb{K}(U)$ for all open sets U (when Ω is a metric space; see Billingsley [8, Theorem 2.1]). This shows, in some sense, the continuity of Riesz correspondence (the bijection between measures and linear forms set up in the Riesz representation theorem). The $\mathbb{R}_{\min}$ version of this equivalence is proved in [4]. The large deviation principle, recalled in Definition 5.1 below, is the analogue of weak convergence [26, 25].

The large deviation version of [8, Theorem 2.1] is provided by Varadhan's integral lemma for one implication [33] and Bryc's theorem for the other [9, T.1.2] (see also [13] for both of them, and Jiang [16, Theorem 2.4]). For both $\mathbb{R}_{\min}$ and large deviation cases, Example 3.23 shows that if the sequence (and then the limit) is tight, the limit has necessarily a density (a rate function for large deviations), a property which is already proved in Bryc's theorem.

5. Application to large deviations

The purpose of large deviations is to find, for a given family of probabilities $(P_\varepsilon)_{\varepsilon>0}$ (resp. $(P_n)_{n\in\mathbb{N}}$) on $(\Omega,\mathcal{A})$, the asymptotic rate of convergence of P_ε when ε tends to 0 (resp. n tends to infinity). In practice, the limit is a Dirac measure at some point. For instance, if X_n are independent random variables with the same law, the law P_n of $\frac{X_1+\cdots+X_n}{n}$ tends to the Dirac measure at the mean point $\mathbb{E}(X_1)$. For almost all sets A, $P_\varepsilon(A)$ tends to 0 exponentially fast and for particular sets A, $-\varepsilon\log P_\varepsilon(A)$ has a limit which can be expressed as the minimum of a function I over the set A. In order to formalize this feature, Varadhan has introduced [33] the following concept:

Definition 5.1. Consider a probability space $(\Omega,\mathcal{A})$, where Ω is a complete separable metric space and $\mathcal{A}$ is the set of Borel sets of Ω, and a family $(P_\varepsilon)_{\varepsilon>0}$ of probabilities on $(\Omega,\mathcal{A})$. Then, (P_ε) obeys the *large deviation principle* (LDP) if there exists a lower semi-continuous (l.s.c.) *rate function* $I\colon\Omega\to[0,+\infty]$ such that

1. for each closed set $C\subset\Omega$
$$\limsup_{\varepsilon\to0}\varepsilon\log P_\varepsilon(C)\le-\inf_{\omega\in C}I(\omega),$$
2. for each open set $U\subset\Omega$
$$\liminf_{\varepsilon\to0}\varepsilon\log P_\varepsilon(U)\ge-\inf_{\omega\in U}I(\omega),$$
3. I is inf-compact, that is $\Omega_a=\{\omega\in\Omega, I(\omega)\le a\}$ is a compact set for any $a<+\infty$.

Since the functions $\mathbb{K}(A)=-\inf_{\omega\in A}I(\omega)$, with I as in point 3, are particular idempotent $\mathbb{R}_{\max}$-probabilities on $(\Omega,\mathcal{A})$, we will consider the following weak form of the previous definition.

Definition 5.2. Consider, for any $A\in\mathcal{A}$, the quantities
$$\mathbb{K}^\vee(A)\overset{\mathrm{def}}{=}\limsup_{\varepsilon\to0}\varepsilon\log P_\varepsilon(A),$$
$$\mathbb{K}^\wedge(A)\overset{\mathrm{def}}{=}\liminf_{\varepsilon\to0}\varepsilon\log P_\varepsilon(A).$$
We say that (P_ε) obeys the *idempotent large deviation principle* (idempotent LDP) if there exists an idempotent $\mathbb{R}_{\max}$-probability $\mathbb{K}$ on $(\Omega,\mathcal{A})$ such that

1. for each closed set $C\subset\Omega$, $\mathbb{K}^\vee(C)\le\mathbb{K}(C)$,
2. for each open set $U\subset\Omega$, $\mathbb{K}^\wedge(U)\ge\mathbb{K}(U)$.

We say that (P_ε) obeys the *tight idempotent large deviation principle* (tight idempotent LDP) if in addition the limit $\mathbb{K}$ is tight, that is there exists a sequence (Ω_n) of compact sets such that $\mathbb{K}(\Omega_n^c)\underset{n\to+\infty}{\to}\mathbb{0}=-\infty$.

The family (P_ε) obeys the LDP iff (P_ε) obeys the tight idempotent LDP with a measure $\mathbb{K}$ having an upper semi-continuous (u.s.c.) density $c = -I$ (condition 3 of Definition 5.1 is equivalent to the tightness of $\mathbb{K}$). But, if (P_ε) obeys the idempotent LDP with a measure $\mathbb{K}$, the maximal extension $\mathbb{K}^*$ of the restriction of $\mathbb{K}$ to open sets is also admissible ($\mathbb{K}^*(U) = \mathbb{K}(U) \leq \mathbb{K}^\wedge(U)$, $\mathbb{K}^*(C) \geq \mathbb{K}(C) \geq \mathbb{K}^\vee(C)$). Now, the theorems of Section 3 give sufficient conditions on Ω for $\mathbb{K}^*$ to have a density, and this density is necessarily an u.s.c. function. Conversely, if $\mathbb{K}$ has an u.s.c. density then $\mathbb{K} = \mathbb{K}^*$. In particular if Ω is a complete separable metric space, we have shown in Example 3.17 that $\mathbb{K}^*$ has necessarily a density, thus the classical and tight idempotent LDP are equivalent. We suppose now that Ω is a general topological space and search for conditions on (P_ε) in order to satisfy the (tight) idempotent LDP. For this we construct a measure $\mathbb{K}$ which is a good candidate for the LDP.

Remark 5.3. The maps $\mathbb{K}^\vee$ and $\mathbb{K}^\wedge$ are nondecreasing functions on $\mathcal{A}$ and $\mathbb{K}(\emptyset) = \mathbb{0}$, $\mathbb{K}(\Omega) = \mathbb{1}$ for $\mathbb{K} = \mathbb{K}^\vee$ or $\mathbb{K}^\wedge$. Moreover $\mathbb{K}^\vee(A \cup B) = \mathbb{K}^\vee(A) \oplus \mathbb{K}^\vee(B)$ for any A and B in $\mathcal{A}$, but this is false for $\mathbb{K}^\wedge$. However, this last property is not useful, as one can construct the maximal idempotent measure lower than a nondecreasing function $\mathbb{K}^\wedge$, but not the minimal measure greater than $\mathbb{K}^\vee$.

Proposition 5.4. *Let $\mathcal{U}$ denote the set of open sets of Ω. The maximal idempotent $\mathbb{R}_{\max}$-measure on $(\Omega,\mathcal{U})$ lower than $\mathbb{K}^\wedge$ is the following*:

$$\mathbb{K}(U) = \inf_{(U_n \in \mathcal{U})_n,\, \bigcup_n U_n = U} \left(\sup_n \mathbb{K}^\wedge(U_n) \right) \quad \forall U \in \mathcal{U}.$$

Proof. By additivity and continuity, any idempotent $\mathbb{R}_{\max}$-measure lower than $\mathbb{K}^\wedge$ is lower than $\mathbb{K}$. Let us prove that $\mathbb{K}$ is an idempotent measure. First, $\mathbb{K}(\emptyset) \leq \mathbb{K}^\wedge(\emptyset) = \mathbb{0}$, then $\mathbb{K}(\emptyset) = \mathbb{0}$ and as $\mathbb{K}^\wedge$ is nondecreasing, $\mathbb{K}$ is also nondecreasing.

Consider a (possibly finite) sequence of open sets (U_n) and $U = \bigcup_n U_n$. Since $\mathbb{K}$ is monotone, $\mathbb{K}(U) \geq \sup_n \mathbb{K}(U_n)$. On the other hand,

$$\begin{aligned}
\sup_n \mathbb{K}(U_n) &= \sup_n \left(\inf_{\substack{(U_{n,m} \in \mathcal{U})_m, \\ \bigcup_m U_{n,m} = U_n}} \left(\sup_m \mathbb{K}^\wedge(U_{n,m}) \right) \right) \\
&= \inf_{\substack{(U_{n,m} \in \mathcal{U})_{n,m}, \\ \bigcup_m U_{n,m} = U_n \ \forall n}} \left(\sup_{n,m} \mathbb{K}^\wedge(U_{n,m}) \right) \\
&\geq \inf_{\substack{(U_{n,m} \in \mathcal{U})_{n,m}, \\ \bigcup_{m,n} U_{n,m} = U}} \left(\sup_{n,m} \mathbb{K}^\wedge(U_{n,m}) \right) \\
&\geq \mathbb{K}(U).
\end{aligned}$$

Then, $\mathbb{K}(\bigcup_n U_n) = \sup_n \mathbb{K}(U_n)$ which implies both the additivity and continuity properties. □

Consider the maximal extension $\mathbb{K}^*$ of the measure $\mathbb{K}$ of Proposition 5.4 to the algebra of all subsets of Ω. We have

$$\mathbb{K}^*(A) = \inf_{(U_n \in \mathcal{U})_n,\, \bigcup_n U_n \supset A} \left(\sup_n \mathbb{K}^\wedge(U_n) \right).$$

It is the maximal measure on $\mathcal{A}$ (or $\mathcal{P}(\Omega)$) satisfying condition 2 of Definition 5.2 and it is a good candidate to satisfy the idempotent LDP. Indeed, suppose that $\mathbb{K}'$ satisfies the idempotent LDP, then $\mathbb{K}' \leq \mathbb{K}$ on open sets (by condition 2 and Proposition 5.4), thus $\mathbb{K}' \leq \mathbb{K}^*$ on $\mathcal{A}$ and $\mathbb{K}^*$ satisfies condition 1. Hence, $\mathbb{K}^*$ satisfies the idempotent LDP.

Remark 5.5. In a metric space Ω, the restriction to open sets of a measure $\mathbb{K}'$ satisfying the idempotent LDP is unique. Thus, it is equal to $\mathbb{K}^*$. Indeed, any open set U is the union of the open sets $U_n = \{\omega \in U, d(\omega, U^c) > 1/n\}$, which satisfy $\overline{U}_n \subset U$. For any measures $\mathbb{K}$ and $\mathbb{K}'$ satisfying the idempotent LDP, we have $\mathbb{K}(U) \geq \mathbb{K}(\overline{U}_n) \geq \mathbb{K}^\vee(\overline{U}_n) \geq \mathbb{K}^\wedge(U_n) \geq \mathbb{K}'(U_n)$, thus $\mathbb{K}(U) \geq \sup_n \mathbb{K}'(U_n) = \mathbb{K}'(U)$. By symmetry, we obtain the uniqueness.

Therefore, in a metric space, a measure $\mathbb{K}'$ satisfying the idempotent LDP is necessarily equal to $\mathbb{K}^*$ on open sets (the tightness of $\mathbb{K}'$ is then equivalent to that of $\mathbb{K}^*$) and $\mathbb{K}^*$ satisfies the idempotent LDP. As a consequence, the rate function I of Definition 5.1 is unique and equal to the opposite of the density of $\mathbb{K}^*$.

To summarize, we have the following result.

Theorem 5.6. 1. *P_ε obeys the idempotent LDP if and only if $\mathbb{K}^\vee(C) \leq \mathbb{K}^*(C)$ for any closed subset C of Ω, that is*

$$\mathbb{K}^\vee(C) \leq \inf_{(U_n \in \mathcal{U})_n,\, C \subset \bigcup_n U_n} \left(\sup_n \mathbb{K}^\wedge(U_n) \right). \tag{12}$$

2. *The following condition is sufficient in general and necessary in metric spaces for (P_ε) to obey the tight idempotent LDP*:
   Inequality (12) *holds and $\mathbb{K}^*$ is tight.*
3. *In a metric space, the classical LDP* (*Definition* 5.1) *and tight idempotent LDP* (*Definition* 5.2) *are equivalent. Indeed, if $\mathbb{K}^*$ is tight, then $\mathbb{K}^*$ has necessarily as u.s.c. density the function*:

$$c^*(\omega) = \inf_{U \in \mathcal{U},\, U \ni \omega} \mathbb{K}^\wedge(U).$$

Then, (12) *is equivalent to*

$$\mathbb{K}^\vee(C) \leq \sup_{\omega \in C} c^*(\omega). \tag{13}$$

Moreover, if the conditions of Definition 5.2 *hold, the rate function I is unique and equal to $-c^*$.*

Proof. We only have to prove point 3. Suppose that Ω is a metric space and that $\mathbb{K}^*$ is tight. From Example 3.20, $\mathbb{K}^*$ has necessarily $c^*(\omega) = \mathbb{K}^*(\{\omega\})$ as density, where

$$c^*(\omega) = \mathbb{K}^*(\{\omega\}) = \inf_{(U_n \in \mathcal{U})_n,\, \omega \in \bigcup_n U_n} \left(\sup_n \mathbb{K}^\wedge(U_n) \right) = \inf_{U \in \mathcal{U},\, U \ni \omega} \mathbb{K}^\wedge(U).$$

Then, (12) is equivalent to (13). □

Thus, in a metric space, the unique rate function can be calculated using open sets or even a basis of neighborhoods only. Then, conditions 1 and 3 of Definition 5.1 have to be verified.

Remark 5.7. When Ω is a metric space, inequality (13), and then the LDP, implies

$$c^*(\omega) = \bar{c}(\omega) \stackrel{\text{def}}{=} \inf_{U \in \mathcal{U}, U \ni \omega} \mathbb{K}^\vee(U). \tag{14}$$

Moreover, using the max-additivity of $\mathbb{K}^\vee$ and the technique of proof of Theorem 3.9, we can deduce, from (14), the inequality (13) for compact sets. Then, the *weak large deviation principle* (weak LDP) (where closed sets are replaced by compact sets and I is not necessarily inf-compact in Definition 5.1) is a consequence of (14). These results are proved for general topological spaces in [13, Theorems 4.1.11 and 4.1.18]. Condition (14) is analogous to one of the definitions of the epiconvergence of c_ε towards c^*, when $\varepsilon \log P_\varepsilon$ is replaced by the idempotent $\mathbb{R}_{\max}$-measure $\mathbb{K}_\varepsilon$ with density c_ε (see [5, Definition 1.9 and Remark 1.11]). Indeed, epiconvergence of l.s.c. functions, weak LDP and vague convergence of probabilities are specializations of the vague convergence of capacities defined in [26, 25] (see also [4]).

If Ω is not a metric space, but a normal space, open sets have to be replaced by $\mathcal{F}_\sigma$ open sets, in the previous study. Indeed, the LDP has to be compared with the weak convergence of probabilities defined with measures of sets (see Remark 4.9, and [26, 25, 16]). But as pointed out in Remark 4.9, Theorem 2.1 of [8] states the continuity of the Riesz correspondence. Generalizing it to a normal space Ω, we obtain that the weak convergence of P_n towards P is equivalent to i) $\liminf_n P_n(U) \geq P(U)$ for all $\mathcal{F}_\sigma$ open sets U, which is also equivalent to ii) $\limsup_n P_n(C) \leq P(C)$ for all $\mathcal{G}_\delta$ closed sets C (where a $\mathcal{G}_\delta$ set is a countable intersection of open sets). Moreover, P_n and P need only be defined on the σ-algebra of Baire sets (which is generated by $\mathcal{F}_\sigma$ open sets). Similarly, the LDP should be defined as follows.

Definition 5.8. Consider a family $(P_\varepsilon)_{\varepsilon>0}$ of probabilities on $(\Omega, \mathcal{A})$, where Ω is a normal topological space and $\mathcal{A}$ is the σ-algebra of its Baire sets. Let $\mathbb{K}^\vee$ and $\mathbb{K}^\wedge$ be as in Definition 5.2. We say that (P_ε) obeys the *idempotent normal large deviation principle* (idempotent normal LDP) if there exists an idempotent $\mathbb{R}_{\max}$-probability $\mathbb{K}$ on $(\Omega, \mathcal{A})$ such that

1. for each $\mathcal{G}_\delta$ closed set $C \subset \Omega$, $\mathbb{K}^\vee(C) \leq \mathbb{K}(C)$,
2. for each $\mathcal{F}_\sigma$ open set $U \subset \Omega$, $\mathbb{K}^\wedge(U) \geq \mathbb{K}(U)$.

The *normal LDP* and the *tight idempotent normal LDP* can be defined along the same lines. From the normality of Ω, conditions 1 of Definitions 5.2 and 5.8 are equivalent, when the probabilities P_ε are defined on all Borel sets. Using Example 3.23, we find that the tight idempotent normal LDP is equivalent to the normal LDP. Indeed, the restriction to $\mathcal{F}_\sigma$ open sets of the measure $\mathbb{K}$ appearing in the idempotent normal LDP is unique, and equal to the measure $\mathbb{K}$ of Proposition 5.4. If $\mathbb{K}$ is tight, $\mathbb{K}^*$ has c^* as density on all $\mathcal{F}_\sigma$ sets, then on $\mathcal{F}_\sigma$ open sets and on closed sets. Moreover, if the family (P_ε) is defined on all Borel sets, the measure $\overline{\mathbb{K}}$ with density c^* on Borel sets satisfies the tight idempotent LDP and $I = -c^*$ satisfies the LDP. This explains the existence of a rate function in Bryc's theorem [9, T.1.2], even in a nonmetric space. However, $\overline{\mathbb{K}}$ is not equal to the maximal extension $\mathbb{K}^*$ of $\mathbb{K}$, and $\mathbb{K}^*$ need not have a density (see Example 3.24). In general, as shown in the following example, the family P_ε is not defined on all Borel sets.

Example 5.9. Let $\mathbb{K}$ be the idempotent $\mathbb{R}_{\max}$-measure of Example 3.24, defined on the subsets of the compact topological space $\Omega = \mathcal{F}(\mathbb{R}, [0, 1]) : \mathbb{K}(A) = -\infty$

if $A \subset \{f \in \Omega,\ \inf_n f(x_n) < 1/2\}$ for some $(x_n) \in \mathbb{R}^{\mathbb{N}}$; $\mathbb{K}(A) = 0$ otherwise. Let P_0 be the probability on $[0,1]$ which is uniform on its support $[1/2,1]$. We denote by P the product probability on the Baire sets of Ω, with marginals P_0 : $P(f(x_1) \in A_1, \ldots, f(x_n) \in A_n) = P_0(A_1) \times \cdots \times P_0(A_n)$ for all distinct points $x_i \in \mathbb{R}$ and all Borel sets A_i of $[0,1]$. Let $P_\varepsilon \equiv P$. All $\mathcal{F}_\sigma$ open sets are such that $\varepsilon \log P_\varepsilon(A)$ tends to $\mathbb{K}(A)$. Then, $\limsup_\varepsilon \varepsilon \log P_\varepsilon(A) \leq \mathbb{K}(A)$ for all Baire sets A, and the family P_ε satisfies the tight idempotent normal LDP with limit $\mathbb{K}$. However, P_ε is not defined on Borel sets, since the open set $U = \{f \in \Omega,\ \inf_{x\in\mathbb{R}} f(x) < 1/2\}$ is such that $P^*(U) = 1$ and $P_*(U) = 0$. Therefore, the LDP for Borel sets cannot even be defined.

The present study shows that the tight idempotent and classical LDP are always equivalent, up to natural extensions of the definitions. The main ingredient is that any tight idempotent probability on the set of $\mathcal{F}_\sigma$ open sets has a density. Although classical probabilities over Polish spaces are always tight, idempotent probabilities are not, in general, and this condition has to be imposed. Compactness results may be proved as in classical probability theory using this tightness condition [30, 16, 4]. But cases where idempotent and classical LDP do not coincide may only be obtained when the tightness condition is relaxed.

References

1. M. Akian, *Theory of cost measures: Convergence of decision variables*, Rapport de Recherche 2611, INRIA, 1995.
2. ———, *On the continuity of the Cramer transform*, Rapport de Recherche 2841, INRIA, 1996.
3. M. Akian, J. P. Quadrat, and M. Viot, *Bellman processes*, 11th International Conference on Analysis and Optimization of Systems: Discrete Event Systems, Lecture Notes in Control and Information Sciences, vol. 199, Springer Verlag, 1994.
4. ———, *Duality between probability and optimization*, Idempotency (J. Gunawardena, ed.), Cambridge University Press, 1998. CMP 98:09
5. H. Attouch, *Variational convergence for functions and operators*, Pitman, 1984. MR **86f:**49002
6. F. Baccelli, G. Cohen, G. J. Olsder, and J. P. Quadrat, *Synchronization and linearity: An algebra for discrete events systems*, John Wiley & Sons, New York, 1992. MR **94b:**93001
7. F. Bellalouna, *Un point de vue linéaire sur la programmation dynamique. Détection de ruptures dans le cadre des problèmes de fiabilité*, Thèse, Université Paris-IX Dauphine, Paris, 1992.
8. P. Billingsley, *Convergence of probability measures*, John Wiley & Sons, New York, 1968. MR **38:**1718
9. W. Bryc, *Large deviations by the asymptotic value method*, Diffusion processes and related problems in analysis (Boston) (M. A. Pinsky, ed.), vol. 1, Birkhaüser, 1990. MR **93b:**60055
10. G. Choquet, *Theory of capacities*, Ann. Inst. Fourier **5** (1955), 131–295. MR **18:**295g
11. P. Del Moral, *Résolution particulaire des problèmes d'estimation et d'optimisation non-linéaires*, Thèse, Université Paul Sabatier, Toulouse, 1994.
12. P. Del Moral, T. Thuillet, G. Rigal, and G. Salut, *Optimal versus random processes: The non-linear case*, Rapport de recherche, LAAS, 1990.
13. A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, Jones and Barlett, Boston, MA, 1993. MR **95a:**60034
14. R. Engelking, *General topology*, Heldermann Verlag, Berlin, 1989. MR **91c:**54001
15. G. Gierz, K. H. Hoffman, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *A compendium of continuous lattices*, Springer Verlag, Berlin, 1980. MR **82h:**06005
16. T. Jiang, *The metric of large deviation convergence*, Preprint, September 1995.
17. V. N. Kolokoltsov, *Maslov's arithmetic in general topology*, Geometry, Topology and Applications. Moskov. Instrument. Inst. (1990), 64–68 (Russian).

18. V. N. Kolokoltsov and V. P. Maslov, *The general form of the endomorphisms in the space of continuous functions with values in a numerical commutative semiring (with the operation* $\oplus = \max$*)*, Soviet Math. Dokl. **36** (1988), no. 1, 55–59.
19. ———, *Idempotent analysis as a technique of the theory of control and optimal synthesis I*, Functional Anal. Appl. **23** (1989), no. 1, 1–11, English translation. MR **90i:**49003
20. V. Maslov, *Méthodes opératorielles*, Éditions MIR, Moscow, 1987. MR **91k:**47003
21. V. P. Maslov and V. N. Kolokoltsov, *Idempotent analysis and its applications to optimal control theory*, Nauka, Moscow, 1994 (Russian).
22. V. P. Maslov and S. N. Samborskii (eds.), *Idempotent analysis*, Advances In Soviet Mathematics, vol. 13, Amer. Math. Soc., Providence, 1992. MR **93h:**00018
23. ———, *Stationary Hamilton-Jacobi and Bellman equations (existence and uniqueness of solutions)*, Idempotent analysis, Advances In Soviet Mathematics, vol. 13, Amer. Math. Soc., Providence, 1992. MR **94d:**49048
24. T. Norberg and W. Vervaat, *Capacities on non-Hausdorff spaces*, Probability and lattices, Math. Centrum, Centrum Wisk. Inform., Amsterdam, 1997. CMP 97:17
25. G. L. O'Brien, *Sequences of capacities, with connections to large-deviation theory*, J. Theoretical Probab. **9** (1996), no. 1, 19–35. MR **97f:**60065
26. G. L. O'Brien and W. Vervaat, *Capacities, large deviations and loglog laws*, Stable processes and related topics (S. Cambanis, G. Samorodnitsky, and M.S. Taqqu, eds.), Progress in probability, vol. 25, Birkhaüser, 1991, pp. 43–83. MR **92k:**60007
27. E. Pap, *On nonadditive set functions*, Atti Sem. Mat. Fis. Univ. Modena **39** (1991), 345–360. MR **92g:**28004
28. ———, *Solution of nonlinear differential and difference equations*, EUFIT'93 (Aachen), Sept 7–10, 1993, pp. 498–503.
29. ———, *The Lebesgue decomposition of the null-additive fuzzy measures*, Zb.-Rad.-Prirod.-Mat.-Fak.-Ser.-Mat. **24** (1994), no. 1, 129–137.
30. A. Puhalskii, *Large deviations of semimartingales via convergence of the predictable characteristics*, Stochastic and Stochastics Reports **49** (1994), 27–85.
31. ———, *Large deviation analysis of the single server queue*, Queuing Systems **21** (1995), 5–66. MR **97b:**60045; Errata MR **97m:**60032
32. J. P. Quadrat, *Théorèmes asymptotiques en programmation dynamique*, Note CRAS **311** (1990), 745–748. MR **91k:**90213
33. S. R. S. Varadhan, *Large deviations and applications*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 46, SIAM, Philadelphia, Penn., 1984. MR **86h:**60067b

INRIA, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex, France
E-mail address: `marianne.akian@inria.fr`