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INSTABILITY OF RAPIDLY-OSCILLATING PERIODIC SOLUTIONS FOR DISCONTINUOUS DIFFERENTIAL DELAY EQUATIONS

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Abstract. We study the equation

$$(\star)\qquad \dot{x}(t) = -h(x(t-1)) + f(x(t)) \text{ for } t \geq 0,\ x_{|_{[-1,0]}} = x_0,$$

where h is an odd function defined by $h(y)$ is equal to a if $0 < y < c$, equal to b if $y \geq c$, $a > b > 0$ and $c > 0$ and f is an odd $\mathcal{C}^1$ function such that $\sup|f(x)| < b$. We first consider the equation $\dot{x}(t) = -h(x(t-1))$, corresponding to $f \equiv 0$. We find the admissible shapes of rapidly-oscillating symmetric periodic solutions and we show that these periodic solutions are all unstable. We then extend these results to our general equation $(\star)$.

1. Introduction

We study here the scalar delay-differential equation

$$\dot{x}(t) = -h(x(t-1)) + f(x(t)) \quad t \geq 0\,, \quad x_{|_{[-1,0]}} = x_0, \tag{1}$$

where $x_0 \in \mathcal{C}([-1,0])$, h is defined as follows:

$$h : \mathbb{R} \to \mathbb{R},\ \text{odd},\ h(y) = \begin{cases} a & \text{if } 0 < y < c \\ b & \text{if } y \geq c \end{cases} \text{ with } a > b > 0 \text{ and } c > 0, \tag{2}$$

$$f \text{ is an odd Lebesgue measurable function such that } \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < b. \tag{3}$$

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The motivation for considering such systems comes from an automotive control problem, namely the fuel-air ratio regulation problem for spark ignition engines, see [6].

Equation (1) is a particular case of a more general class of first order systems given by

$$\dot{x}(t) = g(x(t), x(t-1)), \tag{4}$$

which arises in a variety of models in the literature, see Diekmann et al [4].

Say that g satisfies a "negative feedback condition" if

$$\begin{aligned} g(x,y) < 0 \quad &\text{for} \quad x > 0 \text{ and } y > 0, \\ g(x,y) > 0 \quad &\text{for} \quad x < 0 \text{ and } y < 0, \\ yg(0,y) < 0 \quad &\text{for} \quad y \neq 0. \end{aligned} \tag{5}$$

If g satisfies some regularity conditions and in particular the negative feedback condition (5), then Equation (4) has a slowly-oscillating periodic solution, see Mallet-Paret and Nussbaum [12] and Mallet-Paret et al [13]. More general results are proved in [11, 12, 13].

If $g(x,y) = f(x) - \operatorname{sgn} y$, where f is $\mathcal{C}^1$ and $\sup |f(x)| < 1$, Condition (5) is satisfied. Equation (4) can be rewritten as

$$\dot{x}(t) = -\operatorname{sgn}(x(t-1)) + f(x(t)). \tag{6}$$

Although g is not regular in this case, a complete study of (6) has been realized by doing some explicit computations.

Define $V(t)$ as the number of zeros of x on $[t'-1, t')$ with t' the first time after t such that $x(t') = 0$. Fridman et al [7] have shown that if $V(0)$ is finite then $V(t)$ is even and nonincreasing. For any $n \in \mathbb{N}$, Equation (6) has a unique periodic solution x with $V(t) \equiv 2n$ [7]. The slowest one ($n = 0$) is the only orbitally stable periodic solution, see [7, 2].

In addition, Shustin [17], Nussbaum and Shustin [16] and Akian and Bliman [1] have proved that, for any solution x, V is finite after a finite time (there is no super-high-frequency). For related results, see [8, 9]. Thus, we have a complete description of the asymptotic behavior of (6) in which the slow cycle is the only stable part of the attractor.

Equation (1,2,3) is still a particular case of (4) such that g satisfies the negative feedback condition (5). In this paper, we try to generalize the results of [7, 2] to our equation (1,2,3). We characterize some of rapidly-oscillating 2-phase periodic solutions and we show that these solutions are all unstable.

2. Previous results

First, we recall some definitions:

Definition 2.1 (Rapidly-oscillating functions)**.** A continuous function x defined on $[t_0, +\infty)$ is called rapidly-oscillating (with respect to 1) if there exist $t, t' > t_0$, such that $t \neq t'$, $|t - t'| < 1$ and $x(t) = x(t') = 0$.

Definition 2.2 (2-phase periodic (2PP) functions)**.** A T-periodic ($T > 0$) continuous function x defined on $[t_0, +\infty)$ is called 2-phase periodic if there exists $t \geq t_0$ such that $x_{|_{(t,t+T)}}$ changes signs (strictly) exactly once.

Definition 2.3 (Symmetric periodic functions)**.** A T-periodic ($T > 0$) continuous function x defined on $[t_0, +\infty)$ is called symmetric if $x(t+\frac{T}{2}) = -x(t)$ for $t \geq t_0$.

The following result proves the existence of rapidly-oscillating periodic solutions of Equation (1,2,3):

Theorem 2.4 (Nussbaum [15])**.** *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a Lebesgue measurable function. Assume that for each $(x_0, y_0) \in \mathbb{R}^2$, there exist $R = R(x_0, y_0) > 0$ and $C = C(x_0, y_0) > 0$ such that for every $y \in B_R(y_0) = \{y \in \mathbb{R} : |y - y_0| < R\}$, the map $x \mapsto g(x, y)$ is Lipschitzian with Lipschitz constant C on $B_R(x_0)$. Assume that there are positive constants m and M such that $-M \leq g(x, y) \leq -m$ for all $(x, y) \in \mathbb{R}^2$ with $y > 0$ and $m \leq g(x, y) \leq M$ for all $(x, y) \in \mathbb{R}^2$ with $y < 0$. Then, for every integer $n \geq 0$, there exists a 2-phase periodic Lipschitzian function x_n which satisfies Equation (4) almost everywhere in $\mathbb{R}$ and has the property that, if $x_n(\tau) = 0$ for some τ then* $\operatorname{card}\{t \in (\tau - 1, \tau) : x_n(t) = 0\} = 2n$.

Example. Let us consider Equation (1) with $f \equiv 0$ and h given by (2). We seek some periodic solutions. If $x(t)$ denotes a 2-phase periodic solution of (1), then either (a) $|x(t)| \leq c$ for all t, or (b) $|x(t)| > c$ for some t. If case (a) holds, then $h(x(t-1)) = a\operatorname{sgn}(x(t-1))$ almost everywhere. Hence the results on the periodic solutions of (6) and their stability can be applied. We give an explicit form of the family of periodic solutions x_n of Equation (1) such that $\|x_n\|_\infty \leq c$: for $t \in [0, \frac{T_n}{2}]$, $x_n(t) = -\frac{aT_n}{4} + at$, and for $t \in [\frac{T_n}{2}, T_n]$, $x_n(t) = \frac{aT_n}{4} - a(t - \frac{T_n}{2})$ and the period T_n is equal to $\frac{4}{4n+1}$. For a given n, $x_n(t)$ solves (1) if and only if $\frac{a}{4n+1} \leq c$.

Ivanov and Losson [10] claim that first order differential delay equations with negative feedback can possess asymptotically stable rapidly oscillating solutions. They present an analytically tractable example for Equation (1) with h piecewise constant.

Here, we are interested in symmetric 2-phase periodic solutions of (1) which exceed c in magnitude (case(b) in the example). There are two changes of slopes in each monotone phase of x. We start out with $f \equiv 0$. The evolution of system (1) around each periodic solution can be described with a linear discrete dynamical system. We shall show that the fixed point of the discrete system associated to our problem is unstable. Then, we extend these results to the class of functions f verifying (3) for which the discrete system is nonlinear.

3. STUDY OF $\dot{x}(t) = -h(x(t-1))$

Let x be a solution of (1). We define the set Z of points where x vanishes and changes sign, and the cardinality V of the number of zeros with changes of sign: $Z = \{t \geq -1 : x(t) = 0 \text{ and } \forall \epsilon > 0 \, \exists \, t' \in (t, t+\epsilon], t'' \in [t-\epsilon, t) \,/\, x(t')x(t'') < 0\}$, $V(t) = \text{card}\,(Z \cap [t'-1, t'))$, where $t' = \inf\{s \geq t, s \in Z\}$. We write $V(t) = \infty$ if $Z \cap [t'-1, t')$ is not finite.

Theorem 3.1. *Let us consider the following equation*

$$\dot{x}(t) = -h(x(t-1)) \quad t \geq 0\,, \quad x_{|_{[-1,0]}} = x_0, \tag{7}$$

where h is given by (2)*, and $x_0 \in \mathcal{C}([-1,0])$ has finitely many zeros.*

For any x_0 with a finite number of zeros, there exists a unique function $x \in \mathcal{C}([-1,+\infty))$, absolutely continuous on $[0,+\infty)$, satisfying Equation (7) *almost everywhere. For such a solution, $V(t)$ is nonincreasing and even-valued. For any periodic solution of* (7)*, $V(t)$ is constant. Rapidly-oscillating symmetric 2-phase periodic solutions x, such that $\|x\|_\infty > c$, are characterized as in Cases* #2*,* #2′*,* #3*,* #3′*,* #4*,* #4′*,* #5 *and* #6*, defined below. These periodic solutions are all unstable.*

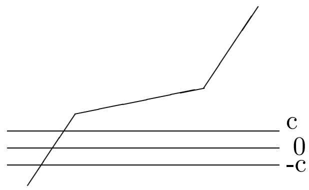
Proof. Existence and uniqueness of a solution of Equation (7) can be proved by induction. Since x_0 has a finite number of zeros, $-h(x(t-1))$ is a piecewise constant function on $[0,1]$, then a solution of (7) is easily determined.

The rest of the proof is done in the following subsections. □

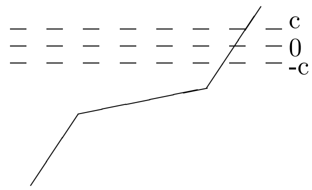
3.1. **Evolution of the number of zeros.**

The integer-valued Lyapunov function $V(t)$ has already been exploited for smooth function h by J. Mallet-Paret, see [11]. In [11, Theorem A,(i)], it has been proved that V is nonincreasing and even-valued for any solution of (4) if g satisfies the negative feedback condition (5). Indeed, the proof can be applied for non smooth g. Therefore, for any solution, V is constant after a finite time. Moreover, for any periodic solution, V is constant. This solution is rapidly oscillating if and only if $V > 0$.

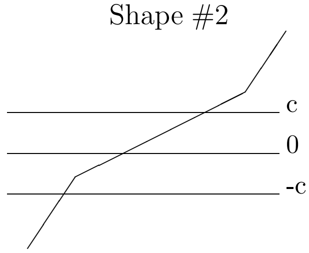
Shape #1



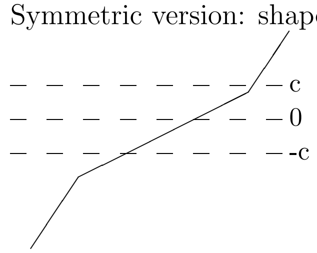
Symmetric version: shape #1′



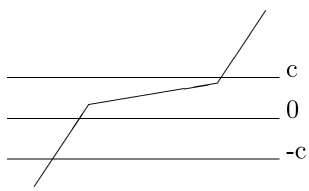
Shape #2



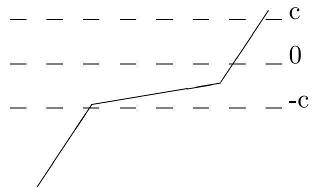
Symmetric version: shape #2′



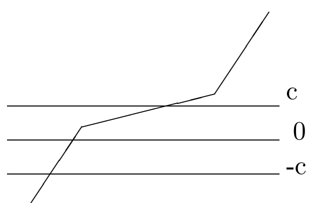
Shape #3



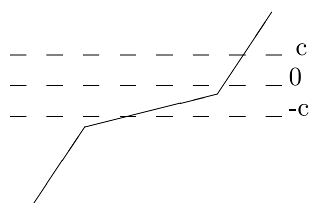
Symmetric version: shape #3′



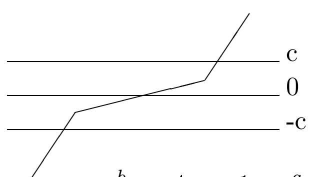
Shape #4



Symmetric version: shape #4′

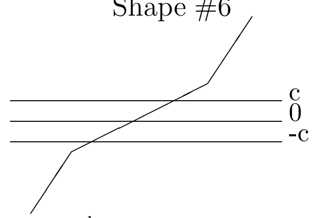


Shape #5



$\frac{b}{2c} < 4n+1 < \frac{a}{c}$

Shape #6



$\frac{b}{2c} > 4n+1$

Since we want to prove a result of local instability, we only consider perturbations of periodic solutions for which V does not change. From now on, our argument reduces to the study of discrete systems describing the evolution on each level $V(t) \equiv 2n$, $n \in \mathbb{N}\backslash\{0\}$.

3.2. **Shapes of rapidly-oscillating 2PP solutions.** The reader can check that any 2PP solution x of (7) increases strictly from its global minimum to its global maximum, decreases from its global maximum to its global minimum and then repeats. By exploiting the oddness of h, one obtains for each $n \geq 0$ a nonconstant periodic solution $x_n(t)$ of (7) as in Theorem 2.4 such that x_n has minimal period T_n and x_n is symmetric, i.e., $x_n(t + \frac{T_n}{2}) = -x_n(t)$ for all t. Consequently, T_n verifies $\frac{1}{n+1/2} < T_n < \frac{1}{n}$. We shall restrict our attention to such symmetric rapidly-oscillating (thus $n \geq 1$) periodic solutions of (7) such that $\|x_n\|_\infty > c$. In the following we draw all a priori possible shapes of these solutions from their minimum to their maximum (so in time interval of length $\frac{T_n}{2}$) . The key point is to determine where changes of slope are situated with respect to the straight lines $x = -c$, $x = 0$ and $x = c$. We only consider "non degenerate" solutions, that is solutions such that the changes of slope do not intersect the straight lines $x = -c$, $x = 0$ or $x = c$.

We show that Shapes #1 and #1′ are impossible. The existence of a symmetric 2PP solution of Equation (7) with a given other shape (shapes #2 to #6 or #2′ to #4′) depends on the constants a, b, c and on n.

3.3. **Evolution on level $V(t) \equiv 2n$, $n \in \mathbb{N}\backslash\{0\}$.** Let x be a solution of (7) such that $V(t) = 2n$ $\forall t \geq 0$ with $n \in \mathbb{N}\backslash\{0\}$. Let us consider an interval $I = [t-1, t)$ with $x(t) = 0$. As $V(t) = 2n$, the interval I "splits" in $2n+1$ intervals $(t-1, t_1)$, (t_1, t_2),…, (t_{2n}, t) and on each of these intervals, x is of constant sign. First of all, we consider $(t-1, t_1)$ on which x is positive for instance and $x(t-1) \neq 0$ and the interval (t_1, t_2) on which x is negative. We divide $(t-1, t_1)$ into three parts (possibly empty) : the first part is the time interval, with length a'_1 on which x is increasing and between 0 and c, the second part is the time interval, with length a'_2 on which x is above c and finally the third one is the time interval, with length a'_3, on which x is decreasing from c to 0. Concerning (t_1, t_2), we divide it into three parts: the first part is the time interval, with length a_1, on which $x(t)$ is decreasing from 0 to $-c$, the second part is the time interval, with length a_2, on which $x < -c$ and the third part is the time interval, with length a_3, on which x is increasing from $-c$ to 0. We define in a similar way $a_4, \ldots, a_{6n}$. From the definition of h, we see that the value of $h(x(.))|_I$ is uniquely determined by the value of $a = (a'_1, a'_2, a'_3, a_1, \dots, a_{6n})$.

Let us define the simplex $\sum_{6n+3} = \big\{\bar{a} = (a'_1, a'_2, a'_3, a_1, a_2, a_3, \dots, a_{6n-2}, a_{6n-1}, a_{6n}) \in \mathbb{R}^{6n+3} : a'_i \geq 0 \text{ for } 1 \leq i \leq 3,\ \sum_{i=1}^{3} a'_i > 0,\ a_i > 0 \text{ for } 1 \leq i \leq 6n \text{ and } \sum_{i=1}^{3} a'_i + \sum_{i=1}^{6n} a_i = 1\big\}$ and let Φ_n be the flow on Σ_{6n+3} associated with the evolution of (7) on the level $V \equiv 2n$, corresponding to the map $x_{|_{[t-1,t)}} \mapsto x_{|_{[t'-1,t')}}$, where t' is the first time $t' > t$ such that $x(t') = 0$. We have,

$$\begin{aligned} &\Phi_n : \Sigma_{6n+3} \longrightarrow \mathbb{R}^{6n+3}, \\ &\bar{a} = (a'_1, a'_2, a'_3, a_1, \dots, a_{6n}) \longmapsto \Phi_n(\bar{a}) = (a''_1, a''_2, a''_3, a_4, \dots, a_{6n+3}). \end{aligned} \tag{8}$$

In the following, we compute $\Phi_n(\bar{a})$, when $\bar{a}$ is near $\bar{a}^\star$, where $\bar{a}^\star$ corresponds to a "nondegenerate" symmetric periodic orbit of Equation (7). Thus, we can suppose that the shape of the solution x corresponding to $\bar{a}$ does not change with time and that $\bar{a}$ is such that $\Phi_n(\bar{a}) \in \Sigma_{6n+3}$.

If for instance x is negative on (t, t') and if $-\underline{x}$ denotes the minimum of x on $[t, t']$, then

$$\underline{x} = aa'_1 + ba'_2 + aa'_3 = a(a_1 - a''_1) + b(a_2 - a''_2) + a(a_3 - a''_3)\,. \tag{9}$$

With the additional conditions $0 \leq a''_i \leq a_i$ for $1 \leq i \leq 3$ and $(a''_i > 0 \Rightarrow a''_{i+1} = a_{i+1})$ for $i = 1, 2$, one determines a''_1, a''_2 and a''_3. The values of a_{6n+1}, a_{6n+2} and a_{6n+3} can then be computed. Their formula depends on the chosen shape. Indeed, for each shape, we shall obtain an affine function $\Phi_n(\bar{a})$ of $\bar{a}$. For non degenerate cases, this function has at most one fixed point $\bar{a}$, which corresponds to the unique (if it exists) symmetric periodic solution of Equation (7) with the corresponding shape. The fixed point $\bar{a}$ has to verify: $a''_1 = a'_1$, $a''_2 = a'_2$, $a''_3 = a'_3$, $a_i = a_{i+3}$ for $1 \leq i \leq 6n$, thus $a_i = a_{6n+i}$ for $1 \leq i \leq 3$. If T_n is the minimal period of the periodic solution, then $a_1 + a_2 + a_3 = \frac{T_n}{2}$ and $a'_1 + a'_2 + a'_3 = 1 - nT_n > 0$. Thus, $\frac{1}{n+1/2} < T_n < \frac{1}{n}$. Now, we determine more precisely the value of Φ_n corresponding to each shape. We compute the corresponding fixed point and if it exists, we prove that it is unstable for the dynamic Φ_n. This will imply the instability of the corresponding symmetric 2PP solution of Equation (7). A sufficient condition is that the tangent map of Φ_n at the fixed point has an eigenvalue of modulus strictly greater than 1, or that the tangent map of Φ_n^{-1} at the fixed point has an eigenvalue of modulus strictly less than 1. We shall thus compute the matrix associated to the tangent map of Φ_n and the characteristic polynomial of this matrix. Then, we prove that this polynomial has at least one root with modulus strictly greater than 1, using homotopy techniques. Since these characteristic polynomials are essentially of three types, we gather important results concerning their roots into Section 3.5.

3.4. Study of Φ_n for all cases of shape.

3.4.1. *Cases #1 and #1′.*

Lemma 3.2. *Φ_n has no fixed point or equivalently, there is no symmetric 2PP solution with shape #1 or #1′ .*

Proof. For Case #1, $a_{6n+2} \geq a'_2 + a'_3$ and $a''_2 = a_2$. Since any fixed point satisfies $a_{6n+2} = a_2$ and $a'_3 > 0$, we obtain a contradiction. So Φ_n has no fixed point. For Case #1′, we get $a_{6n+2} \geq a_1 + a_2$ and $a_1 > 0$, thus Φ_n has no fixed point. □

3.4.2. *Case #2.*

Proposition 3.3. *Any fixed point of Φ_n with shape #2 is unstable.*

Let us first compute the flow Φ_n near a fixed point with shape #2. It is given by (8), with $a'_1 = a''_1 = 0$ and

$$\begin{aligned} a''_2 &= -a'_2 - \frac{a}{b}a'_3 + \frac{a}{b}a_1 + a_2\,, \qquad a''_3 = a_3\,, \\ a_{6n+1} &= \frac{c}{b}\,, \qquad a_{6n+2} = a'_2(1+\frac{b}{a}) + 2a'_3 - \frac{c}{b} - \frac{c}{a}\,, \\ a_{6n+3} &= a_1(1-\frac{a}{b}) + a'_2(1-\frac{b}{a}) + a'_3(-1+\frac{a}{b}) + \frac{c}{a}. \end{aligned}$$

The fixed point is defined by the following equations:

$$\begin{aligned} &a'_2 = -a'_2 - \frac{a}{b}a'_3 + \frac{a}{b}a_1 + a_2\,, \quad a'_3 = a_3\,, \quad a_1 = \frac{c}{b}\,, \\ &a_2 = a'_2(1+\frac{b}{a}) + 2a'_3 - \frac{c}{b} - \frac{c}{a}\,, \quad 1 = 2n(a_1+a_2+a_3) + a'_2 + a'_3\,, \end{aligned}$$

which solution is given by $a'_2 = \frac{1}{d}\big((2n+1)\frac{c}{b}(-4+3\frac{a}{b}-\frac{b}{a})+2-\frac{a}{b}+\frac{c}{b}(3-2\frac{a}{b})\big)$, $a'_3 = \frac{1}{d}\big((2n+1)\frac{c}{b}(3\frac{b}{a}-\frac{a}{b})+1-\frac{b}{a}+\frac{c}{b}(1-2\frac{b}{a})\big)$, $a_1 = \frac{c}{b}$, $a_2 = \frac{1}{d}\big(2(2n+1)\frac{c}{b}(\frac{a}{b}-\frac{b}{a}-2)+3-\frac{a}{b}+2\frac{c}{b}(2-\frac{a}{b})\big)$, $a_3 = a'_3$, $d = (2n+1)(4-\frac{a}{b}-\frac{b}{a})-1$. We suppose that $d \neq 0$, so that the fixed point exists and is unique. To be compatible with the shape #2, the fixed point $\bar{a}$ has to satisfy $a'_2 > \frac{c}{b}$, $a'_3 > 0$ and $0 < a_2 - a'_2 < \frac{c}{b}$, which are equivalent to the conditions $\frac{a}{b} > 2$ with $(2n+1)(4-\frac{a}{b}-\frac{b}{a}) < 1$, $2n+1 > (\frac{b}{c}(1-\frac{b}{a})+1-2\frac{b}{a})/(\frac{a}{b}-3\frac{b}{a})$ and $(\frac{b}{c}+1)/(\frac{a}{b}+\frac{b}{a}) < 2n+1 < \frac{1}{2}(\frac{b}{2c}+1)$, or $(2n+1)(4-\frac{a}{b}-\frac{b}{a}) > 1$, $2n+1 < (\frac{b}{c}(1-\frac{b}{a})+1-2\frac{b}{a})/(\frac{a}{b}-3\frac{b}{a})$ and $(\frac{b}{c}+1)/(\frac{a}{b}+\frac{b}{a}) > 2n+1 > \frac{1}{2}(\frac{b}{2c}+1)$. These conditions can be satisfied for some constants a, b, c and some $n \geq 1$.

Let us now compute the tangent map of Φ_n. Any sufficiently small perturbation $\bar{a}$ of the fixed point (or any fixed point) of Φ_n satisfies

$$a_1' = 0 \quad \text{and} \quad a_{3k-2} = \frac{c}{b}, \text{ for } k = 1, \ldots, 2n\,. \tag{10}$$

We thus restrict Φ_n to the subspace Σ_{4n+2} of Σ_{6n+3} defined by (10) and we replace Φ_n by the map $\widetilde{\Phi}_n : (a_2', a_3', a_2, a_3, a_5, \ldots, a_{6n}) \mapsto (a_2'', a_3'', a_5, a_6, \ldots, a_{6n+3})$. Let us denote by $M \in \mathbb{R}^{(4n+2)\times(4n+2)}$, the matrix associated to the tangent map of $\widetilde{\Phi}_n$, its transposed matrix M^T is given by:

$$M^T = \begin{pmatrix} -1 & 0 & \cdots & \cdots & \cdots & 1+\frac{b}{a} & 1-\frac{b}{a} \\ -\frac{a}{b} & 0 & \cdots & \cdots & \cdots & 2 & -1+\frac{a}{b} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The following result will be useful to compute characteristic polynomials of tangent map matrices of Φ_n.

Lemma 3.4. *Let G and T be $k \times k$ real matrices, with k a positive integer. For any positive integer N, denote I_N the identity $N \times N$ matrix, and for any real e, denote $\mathcal{M}_{e,N} = \mathcal{G}_N + e\mathcal{T}_N$ with*

$$\mathcal{G}_N = \left(\begin{array}{c|c} G \;\big|\; 0 & 0 \\ \hline I_{N-k} & 0 \end{array}\right), \quad \mathcal{T}_N = \left(\begin{array}{c|c} 0 & T \\ \hline 0 & 0 \end{array}\right).$$

Then, for any positive integer m such that $N \geq k(m+1)$, the characteristic polynomial of $\mathcal{M}_{e,N}$ satisfies:

$$\det(\mathcal{M}_{e,N} - XI_N) = (-X)^{km} \det(\mathcal{M}_{\frac{e}{X^m}, N-km} - XI_{N-km})\,.$$

Proof. If $N - k \geq k$, we have $\mathcal{G}_N = \left(\begin{array}{c|c} \mathcal{G}_{N-k} & 0 \\ \hline 0 \;\big|\; I_k & 0 \end{array}\right)$, so

$$\det(\mathcal{M}_{e,N} - XI_N) = \det\left(\begin{array}{c|c} \mathcal{G}_{N-k} - XI_{N-k} & \begin{array}{c} eT \\ \hline 0 \end{array} \\ \hline 0 \quad\big|\; I_k & -XI_k \end{array}\right),$$

$$= \det(-XI_k) \times \det\left((\mathcal{G}_{N-k} - XI_{N-k}) + \frac{1}{X}\begin{pmatrix} eT \\ \hline 0 \end{pmatrix}\left(\begin{array}{c|c} 0 & I_k \end{array}\right)\right),$$

$$= (-X)^k \times \det\left((\mathcal{G}_{N-k} - XI_{N-k}) + \frac{1}{X} \left(\begin{array}{c|c} 0 & eT \\ \hline 0 & 0 \end{array} \right) \right),$$
$$= (-X)^k \times \det(\mathcal{M}_{\frac{e}{X},N-k} - XI_{N-k}) .$$

One concludes by induction on m. □

Using Lemma 3.4 with $N = 4n + 2$, $k = 2$ and $m = 2n$, we obtain:

Lemma 3.5. *The characteristic polynomial of M is equal to:*

$$R(X) = X^{4n+2} + X^{4n+1} - (2 + B)X^{2n+1} + B,$$

where $B = -2 + \frac{a}{b} + \frac{b}{a}$ is such that $B > 0$.

Proof of Proposition 3.3. The fixed point $\bar{a}$ of Φ_n found above is unstable if the tangent map of Φ_n at $\bar{a}$ has an eigenvalue with modulus strictly greater than 1, or equivalently if the characteristic polynomial R of M has one root z, such that $|z| > 1$.

a) If $B > 1$, then the product of all roots of R is greater than 1, so R has clearly one root z, such that $|z| > 1$.
b) If $0 < B \le 1$, we use Theorem 3.19 below, which implies that, for $n \ge 1$, there exists at least one root z such that $|z| > 1$. □

3.4.3. *Case #2′.*

Proposition 3.6. *Any fixed point of Φ_n with shape #2′ is unstable.*

The flow Φ_n near a fixed point with shape #2′ is given by (8), with $a'_1 = a''_1 = 0$ and

$$a''_2 = -a'_2 - \frac{a}{b}a'_3 + \frac{a}{b}a_1 + a_2 , \quad a''_3 = a_3 , \quad a_{6n+1} = a'_2(1 - \frac{b}{a}) + \frac{c}{a} ,$$
$$a_{6n+2} = a'_2(1 + \frac{b}{a}) + a'_3(1 + \frac{a}{b}) + a_1(1 - \frac{a}{b}) - \frac{c}{b} - \frac{c}{a} , \quad a_{6n+3} = \frac{c}{b} .$$

The fixed point is defined by the following equations:

$$a'_2 = \frac{1}{d}\big(2n\frac{c}{b}(\frac{b}{a} + \frac{a}{b}) + \frac{c}{b} - 1\big) \quad a'_3 = \frac{c}{b} , \quad a_1 = (1 - \frac{b}{a})a'_2 + \frac{c}{a} ,$$
$$a_2 = (3 - \frac{a}{b})a'_2 + \frac{c}{b}(\frac{a}{b} - 1) , \quad a_3 = \frac{c}{b} , \quad d = 2n(\frac{a}{b} + \frac{b}{a} - 4) - 1 .$$

We suppose that $d \neq 0$, so that the fixed point exists and is unique. To be compatible with the shape #2′, the fixed point $\bar{a}$ has to satisfy $0 < a'_2 < \frac{c}{b}$, $a_2 - a'_2 > \frac{c}{b}$ and $a_1 > 0$ which are equivalent to the conditions: $\frac{a}{b} > 2$ with $2n(\frac{a}{b} + \frac{b}{a} - 4) > 1$ and $(\frac{b}{c} - 1)/(\frac{a}{b} + \frac{b}{a}) < 2n < \frac{1}{2}\big(\frac{b}{2c} - 1\big)$, or $2n(\frac{a}{b} + \frac{b}{a} - 4) < 1$

and $(\frac{b}{c}-1)/(\frac{a}{b}+\frac{b}{a}) > 2n > \frac{1}{2}\big(\frac{b}{2c}-1\big)$. These conditions can be satisfied for some constants a, b, c and some $n \geq 1$.

Any sufficiently small perturbation $\bar{a}$ of a fixed point satisfies

$$a_1' = 0 \quad \text{and} \quad a_3' = a_{3k} = \frac{c}{b}, \ \text{for } k = 1, \ldots, 2n\,. \tag{11}$$

We thus restrict Φ_n to the subspace Σ_{4n+1} of Σ_{6n+3} defined by (11) and we replace Φ_n by the map $\widetilde{\Phi}_n : (a_2', a_1, a_2, a_4, \ldots, a_{6n-1}) \longmapsto (a_2'', a_4, a_5, \ldots, a_{6n+2})$.

The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+1)\times(4n+1)}$ associated to the tangent map of $\widetilde{\Phi}_n$ is given by:

$$M^T = \begin{pmatrix} -1 & 0 & \cdots & \cdots & \cdots & 1-\frac{b}{a} & 1+\frac{b}{a} \\ \frac{a}{b} & 0 & \cdots & \cdots & \cdots & 0 & 1-\frac{a}{b} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Using Lemma 3.4 with $N = 4n+1$, $k = 2$ and $m = 2n-1$, we obtain:

Lemma 3.7. *The characteristic polynomial of M is equal to:*

$$R(X) = -(X^{4n+1} + X^{4n} - (2+B)X^{2n} + B),$$

where $B = -2 + \frac{a}{b} + \frac{b}{a}$ is such that $B > 0$.

The proof of Proposition 3.6 follows by the same arguments as for Proposition 3.3, using Theorem 3.20 instead of Theorem 3.19.

3.4.4. *Case #3.*

Proposition 3.8. *Any fixed point of Φ_n with shape #3 is unstable.*

The flow Φ_n near a fixed point with shape #3 is given by (8) and

$$a_1'' = -a_1' - \frac{b}{a}a_2' + a_1 - a_3', \quad a_2'' = a_2\,, \quad a_3'' = a_3\,,$$
$$a_{6n+1} = a_2'(1-\frac{b}{a}) + \frac{c}{a}\,, \quad a_{6n+2} = 2(a_1' + \frac{b}{a}a_2' + a_3' - \frac{c}{a})\,, \quad a_{6n+3} = \frac{c}{a}\,.$$

The fixed point is unique and defined by the following equations:

$$a_1' = (\frac{1}{2} - \frac{b}{a})a_2\,, \quad a_2' = a_2\,, \quad a_3' = \frac{c}{a}\,,$$

$$a_1 = (1 - \frac{b}{a})a_2 + \frac{c}{a}\,, \quad a_2 = \frac{1-(4n+1)\frac{c}{a}}{(2n+1)(2-\frac{b}{a})-\frac{1}{2}}\,, \quad a_3 = \frac{c}{a}\,.$$

To be compatible with shape #3, the fixed point $\bar{a}$ has to satisfy $aa'_1 + ba'_2 < c < aa'_1 + ba'_2 + aa'_3$, $a'_1 > 0$ and $a_2 > 0$ which are equivalent to $\frac{a}{b} > 2$ and $(\frac{a}{2c}+1)/(3-\frac{b}{a}) < 2n+1 < \frac{1}{2}(\frac{a}{c}+1)$. Any sufficiently small perturbation $\bar{a}$ of the fixed point satisfies $a'_3 = a_{3k} = \frac{c}{a}$ for $k = 1, \ldots, 2n$. We thus restrict Φ_n to the subspace Σ_{4n+1} of Σ_{6n+3} defined by these conditions and we replace Φ_n by the map

$$(a'_1, a'_2, a_1, a_2, a_4, \ldots, a_{6n-1}) \xmapsto{\widetilde{\Phi}_n} (a''_1, a''_2, a_4, \ldots, a_{6n+2})\,. \tag{12}$$

The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+2)\times(4n+2)}$ associated to the tangent map of $\widetilde{\Phi}_n$ is given by:

$$M^T = \begin{pmatrix} -1 & 0 & \cdots & \cdots & \cdots & 0 & 2 \\ -\frac{b}{a} & 0 & \cdots & \cdots & 0 & 1-\frac{b}{a} & 2\frac{b}{a} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Using Lemma 3.4 with $N = 4n+2$, $k = 2$ and $m = 2n$, we obtain:

Lemma 3.9. *The characteristic polynomial of M is equal to:*

$$R(X) = X^{4n+2} + X^{4n+1} - (2+B)X^{2n+1} + B,$$

with $B = 2\frac{b}{a} - 2 < 0$.

Proof of Proposition 3.8. The proof of Proposition 3.8 follows by the same arguments as for Proposition 3.3, using Theorem 3.19. Indeed, since we know that $a > 2b$, then $|B| > 1$ and R has clearly one root z such that $|z| > 1$. This shows that the fixed point of Φ_n corresponding to shape #3 is unstable. □

3.4.5. *Case #3′.*

Proposition 3.10. *Any fixed point of Φ_n with shape #3′ is unstable.*

The flow Φ_n near a fixed point with shape #3′ is given by (8), with $a'_1 = a''_1 = a'_2 = a''_2 = 0$ and

$$a''_3 = -a'_3 + a_1 + \frac{b}{a}a_2 + a_3\,, \quad a_{6n+1} = \frac{c}{a}\,,$$

$$a_{6n+2} = 2(a_3' - \frac{c}{a}), \quad a_{6n+3} = a_2(1-\frac{b}{a}) + \frac{c}{a}.$$

The fixed point is unique and defined by the following equations:

$$a_3' = \frac{1}{2}a_2 + \frac{c}{a}, \quad a_1 = \frac{c}{a}, \quad a_2 = \frac{1-(4n+1)\frac{c}{a}}{2n(2-\frac{b}{a})+\frac{1}{2}}, \quad a_3 = a_2(1-\frac{b}{a}) + \frac{c}{a}.$$

To be compatible with the shape $\#3'$, the fixed point $\bar{a}$ has to satisfy $a_3'a > c > a_3'a - a_1 a$, $a_3' < a_3$ and $a_2 > 0$ which are equivalent to $\frac{a}{b} > 2$ and $(\frac{a}{2c}-1)/(3-\frac{b}{a}) < 2n < \frac{1}{2}(\frac{a}{c}-1)$. Any sufficiently small perturbation $\bar{a}$ of the fixed point satisfies $a_1' = a_2' = 0$ and $a_{3k-2} = \frac{c}{a}$ for $k = 1, \ldots, 2n$. We thus restrict Φ_n to the subspace Σ_{4n+1} of Σ_{6n+3} defined by these conditions and we replace Φ_n by the map

$$(a_3', a_2, a_3, a_5, \ldots, a_{6n}) \xmapsto{\widetilde{\Phi}_n} (a_3'', a_5, a_6, \ldots, a_{6n+3}). \tag{13}$$

The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+1)\times(4n+1)}$ associated to the tangent map of $\widetilde{\Phi}_n$ is given by:

$$M^T = \begin{pmatrix} -1 & 0 & \cdots & \cdots & \cdots & 2 & 0 \\ \frac{b}{a} & 0 & \cdots & \cdots & \cdots & 0 & 1-\frac{b}{a} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Using Lemma 3.4 with $N = 4n+1$, $k = 2$ and $m = 2n-1$, we obtain:

Lemma 3.11. *The characteristic polynomial of M is equal to:*

$$R(X) = -(X^{4n+1} + X^{4n} - (2+B)X^{2n} + B),$$

with $B = 2\frac{b}{a} - 2 < 0$.

The proof of Proposition 3.10 may follow by the same arguments as for Proposition 3.3, using Theorem 3.20, or by that of Proposition 3.8 ($a > 2b$ implies $|B| > 1$).

3.4.6. *Case #4.*

Proposition 3.12. *Any fixed point of Φ_n with shape #4 is unstable.*

The flow Φ_n near a fixed point with shape #4 is given by (8) and

$$a_1'' = -a_1' - \frac{b}{a}a_2' - a_3' + a_1, \quad a_2'' = a_2, \quad a_3'' = a_3, \quad a_{6n+1} = a_1'(1-\frac{a}{b}) + \frac{c}{b},$$
$$a_{6n+2} = a_2'(1+\frac{b}{a}) + a_1'(1+\frac{a}{b}) + 2a_3' - \frac{c}{a} - \frac{c}{b}, \quad a_{6n+3} = \frac{c}{a}.$$

The fixed point is unique and defined by the following equations :

$$\begin{aligned}
a_1' &= \frac{1}{d}\Big((2n+1)\frac{c}{a}(\frac{a}{b}+\frac{b}{a}) - \frac{b}{a} - \frac{c}{a}\Big), \quad a_2' = a_2, \quad a_3' = \frac{c}{a},\\
a_1 &= \frac{1}{d}\Big((2n+1)\frac{c}{a}(3\frac{a}{b}+\frac{b}{a}-2) - \frac{b}{a} - \frac{c}{a} + 1\Big),\\
a_2 &= \frac{1}{d}\Big(-4(2n+1)\frac{c}{b} + 1 + \frac{a}{b} + 2\frac{c}{b}\Big),\\
a_3 &= \frac{c}{a}, \quad d = (2n+1)(2+\frac{a}{b}-\frac{b}{a}) - 1 > 0.
\end{aligned}$$

To be compatible with the shape #4, the fixed point $\bar{a}$ has to satisfy $0 < a_1' < \frac{c}{a} < a_1' + a_2'\frac{b}{a}$ and $a_1 > 0$, which are equivalent to the conditions: $(\frac{b}{c}+1)/(\frac{a}{b}+\frac{b}{a}) < 2n+1 < (\frac{a}{2c}+1)/(3-\frac{b}{a})$. These conditions can be satisfied for some constants a, b, c and some $n \geq 1$, but they imply $\frac{a}{b} > 2$.

Since any sufficiently small perturbation $\bar{a}$ of the fixed point satisfies $a_3' = a_{3k} = \frac{c}{a}$ for $k = 1, \ldots, 2n$, we use the notations Σ_{4n+2} and $\widetilde{\Phi}_n$ defined for Case #3, see (12). The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+2)\times(4n+2)}$ associated to the tangent map of $\widetilde{\Phi}_n$, is given by:

$$M^T = \begin{pmatrix}
-1 & 0 & \cdots & \cdots & 0 & 1-\frac{a}{b} & 1+\frac{a}{b}\\
-\frac{b}{a} & 0 & \cdots & \cdots & \cdots & 0 & 1+\frac{b}{a}\\
1 & 0 & \cdots & \cdots & \cdots & 0 & 0\\
0 & 1 & 0 & \cdots & \cdots & \cdots & 0\\
\vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0\\
0 & \cdots & \cdots & 0 & 1 & 0 & 0
\end{pmatrix}.$$

Using Lemma 3.4 with $N = 4n+2$, $k = 2$ and $m = 2n$, we obtain:

Lemma 3.13. *The characteristic polynomial of M is equal to:*

$$R(X) = X^{4n+2} + X^{4n+1} - (2+B)X^{2n+1} + B\,, \quad \textit{with } B = \frac{b}{a} - \frac{a}{b} < 0.$$

The proof of Proposition 3.12 follows by the same arguments as for Proposition 3.3, using Theorem 3.19 or by that of Proposition 3.8 ($a > 2b$ implies $|B| > 1$).

3.4.7. *Case #4′.*

Proposition 3.14. *Any fixed point of Φ_n with shape #4′ is unstable.*

The flow Φ_n near a fixed point with shape #4′ is given by (8), with $a'_1 = a''_1 = a'_2 = a''_2 = 0$ and

$$\begin{aligned} a''_3 &= -a'_3 + a_1 + \frac{b}{a}a_2 + a_3\,, \quad a_{6n+1} = \frac{c}{a}\,,\\ a_{6n+2} &= a'_3(1+\frac{a}{b}) + a_1(1-\frac{a}{b}) - \frac{c}{a} - \frac{c}{b}\,,\\ a_{6n+3} &= a'_3(1-\frac{a}{b}) - a_1(1-\frac{a}{b}) + a_2(1-\frac{b}{a}) + \frac{c}{b}\,. \end{aligned}$$

The fixed point is unique and defined by the following equations:

$$\begin{aligned} a'_3 &= \frac{1}{d}\big(1 - 4n\frac{c}{a}(1-\frac{a}{b})\big)\,, \quad a_1 = \frac{c}{a}\,, \quad a_2 = \frac{1}{d}\big(1+\frac{a}{b} - 2\frac{c}{b} - 8n\frac{c}{b}\big)\,,\\ a_3 &= \frac{1}{d}\big(1 + \frac{c}{a} - \frac{b}{a} + 2n\frac{c}{a}(-2+3\frac{a}{b}+\frac{b}{a})\big)\,, \quad d = 2n(2+\frac{a}{b}-\frac{b}{a}) + 1\,. \end{aligned}$$

To be compatible with the shape #4′, the fixed point $\bar{a}$ has to satisfy $a'_3 a - a_1 a > c > a_3 a - a'_3 a > 0$ and $a_2 > 0$ which are equivalent to the conditions: $(\frac{b}{c} - 1)/(\frac{a}{b} + \frac{b}{a}) < 2n < (\frac{a}{2c} - 1)/(3 - \frac{b}{a})$. These conditions can be satisfied for some constants a, b, c and some $n \geq 1$, but they imply $\frac{a}{b} > 2$.

Since any sufficiently small perturbation $\bar{a}$ of the fixed point satisfies $a'_1 = a'_2 = 0$ and $a_{3k-2} = \frac{c}{a}$ for $k = 1, \ldots, 2n$, we use the notations Σ_{4n+1} and $\widetilde{\Phi}_n$ defined for Case #3′, see (13). The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+1)\times(4n+1)}$ associated to the tangent map of $\widetilde{\Phi}_n$, is given by:

$$M^T = \begin{pmatrix} -1 & 0 & \cdots & \cdots & \cdots & 1+\frac{a}{b} & 1-\frac{a}{b}\\ \frac{b}{a} & 0 & \cdots & \cdots & \cdots & 0 & 1-\frac{b}{a}\\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0\\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0\\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Using Lemma 3.4 with $N = 4n+1$, $k = 2$ and $m = 2n-1$, we obtain:

Lemma 3.15. *The characteristic polynomial of M is equal to:*

$$R(X) = -(X^{4n+1} + X^{4n} - (2+B)X^{2n} + B), \quad \textit{with } B = \frac{b}{a} - \frac{a}{b} < 0.$$

The proof of Proposition 3.14 may follow by the same arguments as for Proposition 3.3, using Theorem 3.20, or by that of Proposition 3.8 ($a > 2b$ implies $|B| > 1$).

3.4.8. *Case #5.*

Proposition 3.16. *Any fixed point of Φ_n with shape #5 is unstable.*

The flow Φ_n near a fixed point with shape #5 is given by (8), with $a'_1 = a''_1 = 0$ and

$$a''_2 = -a'_2 - \frac{a}{b}a'_3 + \frac{a}{b}a_1 + a_2,\quad a''_3 = a_3,\quad a_{6n+1} = a'_2(1-\frac{b}{a}) + \frac{c}{a},$$
$$a_{6n+2} = 2(\frac{b}{a}a'_2 + a'_3 - \frac{c}{a}),\ a_{6n+3} = \frac{c}{a} + (1-\frac{a}{b})a_1 + (1-\frac{b}{a})a'_2 + (\frac{a}{b}-1)a'_3.$$

If $\frac{a}{b} \neq 2$, the fixed point is unique and verifies:

$$a'_2 = \frac{a_2}{2},\quad a'_3 = a_3,\quad a_1 = a_3,\quad a_2 = \frac{1-\frac{c}{a}(4n+1)}{(1-\frac{b}{2a})(4n+1)},\quad a_3 = \frac{c}{a} + \frac{a_2}{2}(1-\frac{b}{a})\,.$$

To be compatible with the shape #5, the fixed point $\bar{a}$ has to satisfy $0 < a'_2 b < c < a'_2 b + a'_3 a$, $0 < (a_2 - a'_2)b < c$ and $a_3 > 0$ which are equivalent to the conditions: $\frac{b}{2c} < 4n+1 < \frac{a}{c}$. Since any sufficiently small perturbation $\bar{a}$ of a fixed point satisfies $a'_1 = 0$, we restrict Φ_n to the subspace Σ_{6n+2} of Σ_{6n+3} defined by this condition and we replace Φ_n by the map

$$(a'_2, a'_3, a_1, a_2, \dots, a_{6n}) \overset{\widetilde{\Phi}_n}{\longmapsto} (a''_2, a''_3, a_4, a_5, \dots, a_{6n+3})\,.$$

The transposed M^T of the matrix $M \in \mathbb{R}^{(6n+2)\times(6n+2)}$ associated to the tangent map of $\widetilde{\Phi}_n$ is given by:

$$M^T = \begin{pmatrix} -1 & 0 & \cdots & 0 & 1-\frac{b}{a} & 2\frac{b}{a} & 1-\frac{b}{a} \\ -\frac{a}{b} & 0 & \cdots & \cdots & 0 & 2 & \frac{a}{b}-1 \\ \frac{a}{b} & 0 & \cdots & \cdots & \cdots & 0 & 1-\frac{a}{b} \\ 1 & 0 & \cdots & \cdots & \cdots & \vdots & 0 \\ 0 & \ddots & 0 & \vdots & \vdots & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Using Lemma 3.4 with $N = 6n+2$, $k = 3$ and $m = 2n-1$, we get:

Lemma 3.17. *The characteristic polynomial of M is equal to:*

$$R(X) = X^{6n+2} + X^{6n+1} - (2+B)X^{4n+1} + B,$$

where $B = 2(\frac{a}{b} + \frac{b}{a} - 2)$ *is such that* $B > 0$.

The proof of Proposition 3.16 follows by the same arguments as for Proposition 3.3, using Theorem 3.21 instead of Theorem 3.19.

3.4.9. *Case #6.*

Proposition 3.18. *Any fixed point of* Φ_n *with shape #6 is unstable.*

The flow Φ_n near a fixed point with shape #6 is given by (8), with $a'_1 = a''_1 = 0$ and

$$a''_2 = -a'_2 - a'_3\frac{a}{b} + a_1\frac{a}{b} + a_2\,, \quad a''_3 = a_3\,,$$
$$a_{6n+2} = 2a'_2 + a'_3(1+\frac{a}{b}) + a_1(1-\frac{a}{b}) - 2\frac{c}{b}\,, \quad a_{6n+1} = \frac{c}{b}\,, \quad a_{6n+3} = \frac{c}{b}\,.$$

The fixed point is unique and defined by:

$$a'_2 = \frac{1}{2}a_2, \quad a'_3 = \frac{c}{b}, \quad a_1 = \frac{c}{b}, \quad a_2 = \frac{2}{4n+1} - \frac{2c}{b}, \quad a_3 = \frac{c}{b}$$

To be compatible with the shape #6, the fixed point $\bar{a}$ has to satisfy $c < a'_2 b < a_2 b - c$ which is equivalent to the condition: $\frac{b}{2c} > 4n+1$. Since any sufficiently small perturbation $\bar{a}$ of the fixed point satisfies $a'_1 = 0$ and $a'_3 = a_{3k-2} = a_{3k} = \frac{c}{a}$ for $k = 1, \ldots, 2n$, we restrict Φ_n to the subspace Σ_{2n+1} of Σ_{6n+3} defined by these conditions and we replace Φ_n by the map $\widetilde{\Phi}_n : (a'_2, a_2, a_5, \ldots, a_{6n-1}) \longmapsto (a''_2, a_5, \ldots, a_{6n+2})$.

The flow $\widetilde{\Phi}_n$ has an inverse $\widetilde{\Phi}_n^{-1}$ given by:

$$(a''_2, a_5, a_8, \ldots, a_{6n+2}) \overset{\widetilde{\Phi}_n^{-1}}{\longmapsto} (a'_2 = \frac{a_{6n+2}}{2}, a_2 = a''_2 + \frac{a_{6n+2}}{2}, a_5, \ldots, a_{6n-1})\,.$$

If $M \in \mathbb{R}^{(2n+1)\times(2n+1)}$ denotes the matrix associated to the tangent map of $\widetilde{\Phi}$, then M^{-1} is that associated to $\widetilde{\Phi}_n^{-1}$. It is equal to:

$$M^{-1} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{2} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{2} \\ 0 & 1 & \ddots & \cdots & \cdots & \ldots & 0 \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The matrix M^{-1} has nonnegative coefficients, and the sum of the columns are 1. Thus, $z = 1$ is an eigenvalue associated to an eigenvector with positive coefficients. According to Perron–Frobenius'theorem, see for instance [3], the other roots are of modulus strictly less than 1. Therefore, any fixed point of $\widetilde{\Phi}_n^{-1}$ is stable for the flow $\widetilde{\Phi}_n^{-1}$ and unstable for $\widetilde{\Phi}_n$. The argument here is related to that given in [7, 2] for the instability of rapidly-oscillating 2PP solutions of a non linear equation (1) with $h(y) = \operatorname{sgn} y$ (that is $a = b = 1$). One can also prove, by the same technique as in the previous sections, that the characteristic polynomial of the matrix of the tangent map of $\widetilde{\Phi}_n$ is equal to $R(X) = -(X^{2n+1} + X^{2n} - 2)$. This polynomial corresponds to the limit case $a = b$ or equivalently $B = 0$ in any of the previous polynomials, up to a product by a power of X. By Lemma 3.23, R has $2n$ roots with $|z| > 1$, and the other root is 1.

3.5. **Roots of the characteristic polynomials.** The following results are proved using homotopy techniques.

Theorem 3.19. *Assume that $-1 \leq B \leq 1$. The equation*

$$X^{4n+2} + X^{4n+1} - (2 + B)X^{2n+1} + B = 0$$

has precisely $2n$ solutions z (counting algebraic multiplicity) with $|z| > 1$.

Theorem 3.20. *Assume that $-1 \leq B \leq 1$. The equation*

$$X^{4n+1} + X^{4n} - (2 + B)X^{2n} + B = 0 \tag{14}$$

has precisely $2n$ solutions z (counting algebraic multiplicity) with $|z| > 1$.

Theorem 3.21. *Assume that $0 \leq B \leq 1$. The equation*

$$R(X) = X^{6n+2} + X^{6n+1} - (2 + B)X^{4n+1} + B = 0$$

has precisely $2n$ solutions z (counting algebraic multiplicity) with $|z| > 1$.

Let us denote by R the polynomial of Theorem 3.19 and consider the homotopy

$$f_t(z) = Q(z) - tP(z)\,, \tag{15}$$

with

$$\begin{aligned}
P(X) &= X^{2n} + \cdots + 1 = \frac{X^{2n+1} - 1}{X - 1}\,, && (16)\\
Q(X) &= X^{4n+1} + 2X^{4n} + \cdots + 2X^{2n+1} = X^{2n+1}\frac{S(X)}{X - 1}\,, && (17)\\
S(X) &= X^{2n+1} + X^{2n} - 2\,. && (18)
\end{aligned}$$

We have $R(X) = (X-1)f_B(X)$. So if we denote $D(0,1) = \{z \in \mathbb{C},\ |z| < 1\}$ and $\overline{D}(0,1)^c = \{z \in \mathbb{C},\ |z| > 1\}$ the complement of its closure, the roots of R in $D(0,1)$ or $\overline{D}(0,1)^c$ are exactly that of f_B (with same multiplicity).

Lemma 3.22. *Consider $n \geq 1$ and f_t defined by* (15,16,17,18)*. If $|z| = 1$ and $-1 < t \leq 1$, then $f_t(z) \neq 0$. Moreover, $f_{-1}(z) = 0 \Leftrightarrow z = -1$ and $z = -1$ is a simple root of f_{-1}.*

Proof. Consider z such that $|z| = 1$ and put $w = z^{2n+1}$. If $f_t(z) = 0$, we get

$$w^2 + \frac{1}{z}w^2 - (2+t)w + t = f_t(z)(z-1) = 0\,,$$

or

$$\frac{1}{z} = \frac{(2+t)w - t - w^2}{w^2}\,. \tag{19}$$

Taking modulus on both sides of (19) gives $1 = |w^2 - (2+t)w + t|^2$. Writing $w = e^{i\theta}$ and simplifying gives

$$(\cos\theta - 1)(4t\cos\theta - 2t - 2t^2 - 4) = 0\,. \tag{20}$$

Thus, for $t \neq 0$, we either have $\cos\theta = 1$ or $\cos\theta = \frac{1}{t} + \frac{t}{2} + \frac{1}{2} = g(t)$. Note that $g(1) = 2 > 1$, $g(-1) = -1$ and $g'(t) = -\frac{1}{t^2} + \frac{1}{2} < 0$ for $-1 \leq t \leq 1$, thus $g(t) > 1$ for $0 < t \leq 1$ and $g(t) < -1$ for $-1 < t < 0$. This implies that the equality $\cos\theta = g(t)$ is impossible for $-1 < t \leq 1$, $t \neq 0$. Then, for $-1 < t \leq 1$, $\cos\theta = 1$, so $w = 1$. But if $w = z^{2n+1} = 1$ and $f_t(z) = 0$, we find by (19), $z = 1$, which is impossible since $f_t(1) = 4n + 1 - t(2n+1) > 0$ for $n \geq 1$ and $t \leq 1$.

Suppose now that $t = -1$. Using (20), we obtain the two possibilities $\cos\theta = 1$ or $\cos\theta = -1$. the first one is impossible as before, whereas the second one implies $z = -1$, by using (19). Conversely, we have $f_{-1}(-1) = 0$ and $f'_{-1}(-1) = n + 1 > 0$, so that $z = -1$ is a simple root of f_{-1}. $\square$

Lemma 3.23. *The equation $S(z) = z^{2n+1} + z^{2n} - 2 = 0$ has no solutions z with $|z| < 1$, $2n$ solutions z with $|z| > 1$ (counting algebraic multiplicity) and 1 is the only root with $|z| = 1$.*

Proof. For $|z| \leq 1$, we have $|z^{2n+1} + z^{2n}| \leq 2$, with equality if and only if $|z| = 1$ and $|z+1| = 2$ or equivalently $z = 1$. So $z = 1$ is the only root of S such that $|z| \leq 1$. It satisfies $S'(1) = 4n + 1 > 0$ so that 1 is a simple root. Since S is a polynomial with degree $2n+1$, this implies that S has exactly $2n$ roots such that $|z| > 1$ (counting multiplicity). $\square$

Proof of Theorem 3.19 for $B \neq -1$. The function $f_t(z)$ defined in (15), (16), (17), (18) is continuous with respect to t and polynomial (thus analytic) with respect to z. By Lemma 3.23, S has no roots in $D(0,1)$. So the only root of $f_0 = Q$ in $D(0,1)$ is 0 with multiplicity $2n+1$. This shows that $\deg(f_0, D(0,1), 0) = 2n+1$. From Lemma 3.22, f_t has no roots on $\partial D(0,1)$ if $-1 < t \leq 1$. Applying Rouché's theorem to the homotopy f_t on $D(0,1)$, we get

$$\deg(f_t, D(0,1), 0) = \deg(f_0, D(0,1), 0) = 2n+1 \quad \text{for } -1 < t \leq 1 \,.$$

This implies that the equation $f_t(z) = 0$ has (counting algebraic multiplicity) $2n+1$ solutions z in $D(0,1)$. Since f_t is a polynomial with degree $4n+1$, $f_t(z) = 0$ has $2n$ solutions in $\overline{D}(0,1)^c$. $\square$

Proof of Theorem 3.19 for $B = -1$. By Lemma 3.22, f_t has no roots on $\partial D(0,1)$ when $-1 < t \leq 1$ and -1 is the only root of f_{-1} on $\partial D(0,1)$. Since $f'_{-1}(-1) \neq 0$, the implicit function theorem shows that there exists $\delta > 0$, $\rho > 0$ and a $\mathcal{C}^1$ function $Z : (-1-\delta, -1+\delta) \to \mathbb{C}$, such that $f_t(z) = 0$ and $|z+1| \leq \rho$ if and only if $z = Z(t)$. So $Z(-1) = -1$ and

$$Z'(-1) = -\frac{\frac{\partial f_t(z)}{\partial t}|_{t=-1,z=-1}}{\frac{\partial f_t(z)}{\partial z}|_{t=-1,z=-1}} = \frac{P(-1)}{f'_{-1}(-1)} = \frac{1}{n+1} > 0 \,,$$

which implies $|Z(t)| < 1$ for $-1 < t < -1+\delta$ (possibly by shrinking δ). Therefore, when $-1 < t < -1+\delta$, $f_t(z) = 0$ and $|z+1| \leq \rho$ imply $|z| < 1$. Denote $G_\rho = \{z \in \mathbb{C} \text{ such that } |z| < 1 \text{ or } |z+1| < \rho\}$. The properties of Z imply that for $-1 \leq t < -1+\delta$, f_t has no roots on ∂G_ρ, and that for $-1 < t < -1+\delta$, the number of roots of f_t in G_ρ is the same as in $D(0,1)$, that is $2n+1$. Applying Rouché's theorem to the homotopy f_t on G_ρ, we get

$$\deg(f_{-1}, G_\rho, 0) = \deg(f_t, G_\rho, 0) = 2n+1 \quad \text{for } -1 < t < -1+\delta \,,$$

so that f_{-1} has $2n+1$ roots in G_ρ and $2n$ roots outside the closure of G_ρ (f_t is a polynomial with degree $4n+1$). Since f_{-1} has no roots such that $0 < |z+1| \leq \rho$, it has exactly $2n$ roots in $\overline{D}(0,1)^c$. $\square$

Theorem 3.20 is proved by the same technique as for the case $B \neq 1$ of Theorem 3.19, changing P and Q into:

$$P(X) = X^{2n-1} + \cdots + 1 = \frac{X^{2n}-1}{X-1} \,, \tag{21}$$

$$Q(X) = X^{4n} + 2X^{4n-1} + \cdots + 2X^{2n} = X^{2n}\frac{S(X)}{X-1} \,. \tag{22}$$

Indeed, by the same technique as for Lemma 3.22, one proves:

Lemma 3.24. *Consider $n \geq 1$ and f_t defined by* (15), (21), (22), (18). *If $|z| = 1$ and $-1 \leq t \leq 1$, then $f_t(z) \neq 0$.*

Proof. Consider z such that $|z| = 1$ and $f_t(z) = 0$, and put $w = z^{2n}$. We obtain

$$z = \frac{(2+t)w - t - w^2}{w^2},$$

which leads, as in Lemma 3.22, to the only possibilities $w = z = 1$, or $w = z = -1$ and $t = -1$. Since $f_t(1) = 4n + 1 - 2nt > 0$ and $f_{-1}(-1) \neq 0$ (indeed $w = z^{2n} = 1$ for $z = -1$), we obtain the result of the lemma. □

The arguments for the proof of Theorem 3.21 are essentially the same as for Theorem 3.19, changing P and Q into:

$$P(X) \quad = \quad X^{4n} + \cdots + 1 = \frac{X^{4n+1} - 1}{X - 1}, \tag{23}$$

$$Q(X) \quad = \quad X^{6n+1} + 2X^{6n} + \cdots + 2X^{4n+1} = X^{4n+1}\frac{S(X)}{X-1}. \tag{24}$$

Lemma 3.25. *Consider $n \geq 1$, and P and Q defined by* (23), (24), (18). *If $|z| = 1$ and $0 \leq t < 1$, then $t|P(z)| < |Q(z)|$.*

Proof. Indeed

$$\left(|Q(z)|^2 - |P(z)|^2\right)|z - 1|^2 = |z^{2n} - 1|^2 |z^{2n+1} - 1|^2 \geq 0 \quad \text{if } |z| = 1\,, \tag{25}$$

and $Q(1) = 4n + 1 = P(1)$, so that

$$|P(z)| \leq |Q(z)| \quad \text{for } |z| = 1\,. \tag{26}$$

Moreover, $Q(z) = 0$ and $|z| = 1$ imply $S(z) = 0$. But by Lemma 3.23, the only root of S with $|z| = 1$ is $z = 1$ for which $Q(1) \neq 0$. Thus, $Q(z) \neq 0$ for all z such that $|z| = 1$, which with (26) leads to the result of the lemma. □

Proof of Theorem 3.21 for $B \neq 1$. Let f_t be defined by (15,23,24,18). From Lemma 3.25, one deduces by Rouché's theorem that f_B and f_0 have same number of roots in $D(0,1)$ (counting multiplicity). By Lemma 3.23, 0 is the only root of $f_0 = Q$ in $D(0,1)$ and it has multiplicity $4n + 1$. Since f_B is a polynomial with degree $6n + 1$ and has no roots on $\partial D(0,1)$ (due to Lemma 3.25), f_B has $2n$ roots in $\overline{D}(0,1)^c$. So does $R = (X - 1)f_B$. □

Proof of Theorem 3.21 for $B = 1$. We use the same technique as for the case $B = -1$ of Theorem 3.19. By Lemma 3.25, f_t has no roots on $\partial D(0,1)$ for $0 \leq t < 1$. For $t = 1$, $f_1(z) = 0$ implies, by (25), $z^{2n} = 1$ or $z^{2n+1} = 1$. Computing $f_1(z)$ for such a z, one obtains a contradiction, except for $z = 1$ which is indeed the only root of f_1 on $\partial D(0,1)$.

Since $f_1'(1) = 12n^2 + 6n + 1 > 0$, the implicit function theorem shows that there exists $\delta > 0$, $\rho > 0$ and a $\mathcal{C}^1$ function $Z : (1-\delta, 1+\delta) \to \mathbb{C}$, such that $f_t(z) = 0$ and $|z-1| \le \rho$ if and only if $z = Z(t)$. So $Z(1) = 1$ and

$$Z'(1) = -\frac{\frac{\partial f_t(z)}{\partial t}|_{t=1,z=1}}{\frac{\partial f_t(z)}{\partial z}|_{t=1,z=1}} = \frac{P(1)}{f_1'(1)} > 0\,,$$

which implies $|Z(t)| < 1$ for $1-\delta < t < 1$ (possibly by shrinking δ). Denote $G_\rho = \{z \in \mathbb{C} \text{ such that } |z| < 1 \text{ or } |z-1| < \rho\}$. We obtain that for $1-\delta < t \le 1$, f_t has no roots on ∂G_ρ, and that for $1-\delta < t < 1$, the number of roots of f_t in G_ρ is the same as in $D(0,1)$, that is $4n+1$. So by Rouché's theorem, the number of roots of f_1 in G_ρ is also $4n+1$, and then f_1 has $2n$ roots outside the closure of G_ρ. Since f_1 has no roots such that $0 < |z-1| \le \rho$, f_1 has exactly $2n$ roots in $\overline{D}(0,1)^c$. $\square$

4. Study of $\dot{x}(t) = -h(x(t-1)) + f(x(t))$.

The following result generalizes [1, Theorem 1].

Theorem 4.1. *Let f be a Lebesgue measurable function such that*

$$\operatorname{ess\,sup} |f(x)| < b$$

and let h be given by (2)*. Then, for any $x_0 \in \mathcal{C}([-1,0])$, such that*[1] meas $\{t \in [-1,0] : x_0(t) = 0\} = 0$*, there exists at least one function $x \in \mathcal{C}([-1,+\infty))$, absolutely continuous on $[0,+\infty)$, such that the composition $f \circ x$ is Lebesgue measurable and satisfying* (1) *almost everywhere. For any Lebesgue measurable function $\tilde{f}$ such that $\tilde{f}(x) = f(x)$ a.e. in $\mathbb{R}$, x satisfies $\tilde{f}(x(t)) = f(x(t))$ a.e. By definition, such a function x is called a solution of Equation* (1) *on* $[0,+\infty)$.

Sketch of Proof. We proceed by the same way as in [1]. We prove the existence of a solution of Equation (1) on $[0,1]$. The global existence is obtained by induction.

Let us denote $g(t) = -h(x(t-1))$ for $t \in [0,1]$. Since g is measurable on $[0,1]$ and takes its values in $\{-a,-b,a,b\}$ a.e., one can construct a sequence $(g_n)_{n\ge1}$ of piecewise constant functions $[0,1] \to \{-a,-b,b,a\}$ such that $\|g_n - g\|_{L^1} < \frac{C}{n}$, with $C > 0$. Indeed, let α_i, $1 \le i \le 4$, be the elements of $\{-a,-b,b,a\}$. Define $A_i = g^{-1}(\alpha_i)$ for $1 \le i \le 4$. Since A_1 is a measurable

[1] One can eliminate this condition by replacing the value of h in 0 ($h(0) = 0$ here due to the oddness of h) by any value in $\{-a,-b,a,b\}$, or by replacing $h(x(t-1))$ by $h(t, x(t-1))$, where $t \in (0,+\infty) \mapsto h(t,.)$ is Lebesgue measurable and $(h(t,x) \in \{-a,-b,a,b\}\ \forall x \in \mathbb{R})$, for a.e. $t \in (0,+\infty)$, see [1].

set, there exists a set O_1^n such that $\lambda(O_1^n \Delta A_1) < \frac{1}{n}$, where λ is the Lebesgue measure, and O_1^n is a finite union of open intervals. We set $A_2' = (A_1 \cup A_2) \backslash O_1^n$ and for $i = 3, 4$, $A_i' = A_i \backslash O_1^n$. We have $\lambda(A_i' \Delta A_i) < \frac{1}{n}$ for $i = 2, 3, 4$. In the same way, we modify successively A_i', $i = 2, 3, 4$ to find the associated sets O_i^n, $i = 2, 3, 4$, that are all finite unions of intervals and such that $\lambda(O_i^n \Delta A_i) < \frac{3}{n}$. Then, $g_n = \sum_{i=1}^4 \alpha_i 1_{O_i^n}$ is a piecewise constant function such that $\|g_n - g\|_{L^1} < \frac{3 \sum_{i=1}^4 |\alpha_i|}{n}$.

For any $n \geq 1$, the following equation

$$\dot{x}(t) = g_n(t) + f(x(t)) \,, \quad t \in [0, 1]$$

admits a unique solution x_n which is continuous and piecewise $\mathcal{C}^1$ in $[0, 1]$. To prove this result, let us define the increasing functions

$$\mathcal{F}_d(e) = \int_0^e \frac{\operatorname{sgn} d}{d + f(x)} dx, \quad e \in \mathbb{R}, \ d \in \{-a, -b, b, a\} \,. \tag{27}$$

The functions $\mathcal{F}_d$ are Lipschitz continuous and such that $\mathcal{F}_d'(x) = \frac{\operatorname{sgn} d}{d + f(x)}$ a.e. Moreover, for $d \in \{-a, -b, b, a\}$, we get

$$0 < \frac{1}{a + \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)|} \leq \mathcal{F}_d'(e) \leq \frac{1}{b - \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)|}, \quad \text{for a.e. } e \in \mathbb{R} \,.$$

Hence $\mathcal{F}_d$ is invertible with Lipschitz continuous inverse. We deduce, by the same way as in [1] that on any interval $[t_1, t_2]$ on which $g_n(t) \equiv d$, x_n is given by

$$x_n(t) = \mathcal{F}_d^{-1}(\mathcal{F}_d(x_n(t_1)) + (t - t_1) \operatorname{sgn} d). \tag{28}$$

The rest of the proof is the same as for [1, Theorem 1]: there exists a limit point x of the sequence x_n in $\mathcal{C}([0, 1])$, this limit point is such that $f \circ x$ is measurable, x is solution of (1) and $f \circ x = \tilde{f} \circ x$ a.e. when $f = \tilde{f}$ a.e. $\square$

Theorem 4.2. *Let us consider Equation* (1)*, where h is given by* (2)*, f satisfies* (3) *and $x_0 \in \mathcal{C}([-1, 0])$ has finitely many zeros. For any x_0 with a finite number of zeros, there exists a unique solution x of* (1)*. For such a solution, $V(t)$ is nonincreasing and even-valued. For any periodic solution of* (1)*, $V(t)$ is constant. Rapidly-oscillating symmetric* 2*-phase periodic solutions x, such that $\|x\|_\infty > c$, have qualitatively the same pictures as Shapes* #2, #2′, #3, #3′, #4, #4′, #5 *and* #6*, defined above except that straight lines are replaced by curves. These periodic solutions are all unstable.*

Proof. Existence and uniqueness of a solution of (1) follows by Theorem 4.1 and its proof, see (28). For the evolution of the number V of zeros, see Section 3.1.

For the rest of the proof, we use the notations $\bar{a}$, Σ_{6n+3} and Φ_n of Section 3.3. The properties stated in Section 3.3 still hold here, except that the flow Φ_n near a fixed point is no more affine, and that (a_1'', a_2'', a_3'') is determined by the equation:

$$\begin{aligned}
&\mathcal{F}_{-a}^{-1}(\mathcal{F}_{-a}\mathcal{F}_{-b}^{-1}(\mathcal{F}_{-b}\mathcal{F}_{-a}^{-1}(-a_1') - a_2') - a_3') = -\underline{x} \\
&\quad = \quad \mathcal{F}_a^{-1}(\mathcal{F}_a\mathcal{F}_b^{-1}(\mathcal{F}_b\mathcal{F}_a^{-1}(a_3'' - a_3) + a_2'' - a_2) + a_1'' - a_1)\,,
\end{aligned}$$

which replaces (9), and where the functions $\mathcal{F}_d$, for $d \in \{a, b, -a, -b\}$, are defined in (27). Since f is odd, we have $\mathcal{F}_d(e) = -\mathcal{F}_{-d}(-e)$. Thus, the flow Φ_n is independent of the sign of x on (t, t'), where t and t' are defined in Section 3.3.

In Section 4.1 below, we determine as in Section 3.4, the flow Φ_n corresponding to each shape (using the functions $\mathcal{F}_d$) and prove that any fixed point (if it exists) is unstable. We do not give the equations of the fixed point, since they cannot be solved explicitly as before. We only determine Φ_n, the matrix M associated to the tangent map of its restriction $\widetilde{\Phi}_n$ at some fixed point and the characteristic polynomial R of M. The restriction $\widetilde{\Phi}_n$ is defined in the same way as in the linear case and takes its values in a similar subspace of Σ_{6n+3}. As for the linear case, we shall prove that R has at least one root with modulus strictly greater than 1, by using homotopy techniques. We gather all results concerning roots of characteristic polynomials into Section 4.2. □

4.1. Study of Φ_n for all cases of generalized shape.

4.1.1. *Generalized cases #1 and #1′.*

Lemma 4.3. *Generalized shapes #1 and #1′ are impossible.*

Proof. We proceed as for Lemma 3.2 to show that Φ_n has no fixed point with shape #1 or #1′. □

4.1.2. *Generalized case #2.*

Proposition 4.4. *Any fixed point of Φ_n with generalized shape #2 is unstable.*

The flow Φ_n near a fixed point with generalized shape #2 is given by (8), with $a_1' = a_1'' = 0$ and

$$\begin{aligned}
&a_2'' = a_2 + \mathcal{A}(a_2', a_3', a_1)\,, \quad a_3'' = a_3\,, \quad a_{6n+1} = -\mathcal{F}_{-b}(-c)\,, \\
&a_{6n+2} = a_2' + a_3' + \mathcal{F}_{-b}(-c) + \mathcal{F}_a(-c) - \mathcal{B}(a_2', a_3')\,, \\
&a_{6n+3} = a_1 - \mathcal{F}_a(-c) + \mathcal{B}(a_2', a_3') - \mathcal{A}(a_2', a_3', a_1)\,,
\end{aligned}$$

where

$$\mathcal{B}(a_2', a_3') = \mathcal{F}_a \mathcal{F}_{-a}^{-1}(-a_3' + \mathcal{F}_{-a}\mathcal{F}_{-b}^{-1}(-a_2'))\,,$$
$$\mathcal{A}(a_2', a_3', a_1) = \mathcal{F}_b \mathcal{F}_a^{-1}(a_1 + \mathcal{B}(a_2', a_3'))\,.$$

Any point $\bar{a}$ in the image of the $2n$-th iterate $(\Phi_n)^{2n}$ of Φ_n, and so any sufficiently small perturbation $\bar{a}$ of a fixed point of Φ_n, satisfies $a_1' = 0$ and $a_{3k-2} = -\mathcal{F}_{-b}(-c)$ for $k = 1, \ldots, 2n$. We thus restrict Φ_n to the subspace Σ_{4n+2} of Σ_{6n+3} defined by these conditions and use the notation $\widetilde{\Phi}_n$ defined for Case #2, see Section 3.4.2. The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+2)\times(4n+2)}$ associated to the tangent map of $\widetilde{\Phi}_n$ at any point $\bar{a} \in \Sigma_{4n+2}$ is given by:

$$M^T = \begin{pmatrix} \mathcal{A}'_{a_2'} & 0 & \cdots & \cdots & 0 & 1-\mathcal{B}'_{a_2'} & \mathcal{B}'_{a_2'} - \mathcal{A}'_{a_2'} \\ \mathcal{A}'_{a_3'} & 0 & \cdots & \cdots & \cdots & 1-\mathcal{B}'_{a_3'} & \mathcal{B}'_{a_3'} - \mathcal{A}'_{a_3'} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \vdots & \vdots & \vdots & 0 \\ \vdots & 0 & \ddots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \end{pmatrix},$$

where for any function f, f_x' denotes the partial derivative of f with respect to x.

Lemma 4.5. *The characteristic polynomial of M is equal to:*

$$R(X) = X^{4n+2} + AX^{4n+1} - (1 + A + B)X^{2n+1} + B\,,$$

where $A = -\mathcal{A}'_{a_2'} > 0$ and $B = (\mathcal{A}'_{a_1} - 1)(\mathcal{B}'_{a_2'} - \mathcal{B}'_{a_3'}) > 0$.

Proof. We use the same arguments as for Lemma 3.5 and the property that $\mathcal{A}'_{a_i'} = \mathcal{A}'_{a_1}\mathcal{B}'_{a_i'}$ for $i = 2, 3$. In particular, $B = (\mathcal{A}'_{a_1} - 1)(\mathcal{B}'_{a_2'} - \mathcal{B}'_{a_3'}) = ((\mathcal{F}_b\mathcal{F}_a^{-1})'(x) - 1)\mathcal{B}'_{a_3'}((\mathcal{F}_{-a}\mathcal{F}_{-b}^{-1})'(-a_2') - 1)$ for some $x \in \mathbb{R}$. Since $\mathcal{B}'_{a_3'} < 0$ and

$$(\mathcal{F}_b\mathcal{F}_a^{-1})'(x) = \frac{\mathcal{F}_b'\mathcal{F}_a^{-1}(x)}{\mathcal{F}_a'\mathcal{F}_a^{-1}(x)} = \frac{a + f(\mathcal{F}_a^{-1}(x))}{b + f(\mathcal{F}_a^{-1}(x))} > 1$$

$$(\mathcal{F}_{-a}\mathcal{F}_{-b}^{-1})'(x) = \frac{b - f(\mathcal{F}_{-b}^{-1}(x))}{a - f(\mathcal{F}_{-b}^{-1}(x))} < 1$$

for any $x \in \mathbb{R}$ when $a > b$, we obtain $B > 0$. Indeed, as in Lemma 3.5, we also obtain that $B > 0$ when $a < b$ (and $\operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| < a$). $\square$

The proof of Proposition 4.4 follows by the same arguments as for Proposition 3.3, using Theorem 4.20 below instead of Theorem 3.19.

4.1.3. *Generalized case #2′.*

Proposition 4.6. *Any fixed point of Φ_n with generalized shape #2′ is unstable.*

The flow Φ_n near a fixed point with generalized shape #2′ is given by (8), with $a_1' = a_1'' = 0$ and

$$\begin{aligned}
&a_2'' = a_2 + \mathcal{A}(a_2', a_3', a_1)\,, \quad a_3'' = a_3\,, \quad a_{6n+1} = a_2' + \mathcal{B}(a_2') - \mathcal{F}_{-a}(-c)\,,\\
&a_{6n+2} = a_3' + a_1 - \mathcal{A}(a_2', a_3', a_1) - \mathcal{B}(a_2') + \mathcal{F}_{-a}(-c) + \mathcal{F}_b(-c)\,,\\
&a_{6n+3} = -\mathcal{F}_b(-c)\,,
\end{aligned}$$

where

$$\mathcal{B}(a_2') = \mathcal{F}_{-a}\mathcal{F}_{-b}^{-1}(-a_2'),\ \mathcal{A}(a_2', a_3', a_1) = \mathcal{F}_b\mathcal{F}_a^{-1}(a_1 + \mathcal{F}_a\mathcal{F}_{-a}^{-1}(-a_3' + \mathcal{B}(a_2'))).$$

Any point $\bar{a}$ in the image of $(\Phi_n)^{2n+1}$, and so any sufficiently small perturbation $\bar{a}$ of a fixed point of Φ_n, satisfies $a_1' = 0$ and $a_3' = a_{3k} = -\mathcal{F}_b(-c)$ for $k = 1, \dots, 2n$. We thus restrict Φ_n to the subspace Σ_{4n+1} of Σ_{6n+3} defined by these conditions and use the notation $\widetilde{\Phi}_n$ defined for Case #2′, see Section 3.4.3. The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+1)\times(4n+1)}$ associated to the tangent map of $\widetilde{\Phi}_n$ at any point $\bar{a} \in \Sigma_{4n+1}$ is given by:

$$M^T = \begin{pmatrix}
\mathcal{A}'_{a_2'} & 0 & \cdots & \cdots & 0 & 1 + \mathcal{B}'_{a_2'} & -\mathcal{B}'_{a_2'} - \mathcal{A}'_{a_2'}\\
\mathcal{A}'_{a_1} & 0 & \cdots & \cdots & \cdots & 0 & 1 - \mathcal{A}'_{a_1}\\
1 & 0 & \cdots & \cdots & \cdots & 0 & 0\\
0 & 1 & 0 & \vdots & \vdots & \vdots & 0\\
\vdots & 0 & \ddots & \vdots & \ddots & \vdots & \vdots\\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0\\
0 & \cdots & \cdots & 0 & 1 & 0 & 0
\end{pmatrix},$$

Lemma 4.7. *The characteristic polynomial of M is equal to:*

$$R(X) = -(X^{4n+1} + AX^{4n} - (1 + A + B)X^{2n} + B)\,,$$

where $A = -\mathcal{A}'_{a_2'} > 0$ and $B = (\mathcal{A}'_{a_1} - 1)(1 + \mathcal{B}'_{a_2'}) > 0$.

Here also we obtain $B > 0$ for any $a \neq b$, with $\operatorname{ess\,sup} |f| < \min(a, b)$.

The proof of Proposition 4.6 follows by the same arguments as for Proposition 3.3, using Theorem 4.21 below.

4.1.4. *Generalized case #3.*

Proposition 4.8. *Any fixed point of Φ_n with generalized shape #3 is unstable.*

The flow Φ_n near a fixed point with generalized shape #3 is given by (8) and

$$\begin{aligned}
a_1'' &= a_1 + \mathcal{A}(a_1', a_2', a_3')\,, \quad a_2'' = a_2\,, \quad a_3'' = a_3\,,\\
a_{6n+1} &= a_1' + a_2' + \mathcal{B}(a_1', a_2') - \mathcal{F}_{-a}(-c)\,,\\
a_{6n+2} &= a_3' + \mathcal{F}_{-a}(-c) - \mathcal{B}(a_1', a_2') - \mathcal{A}(a_1', a_2', a_3') + \mathcal{F}_a(-c)\,,\\
a_{6n+3} &= -\mathcal{F}_a(-c),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}(a_1', a_2') &= \mathcal{F}_{-a}\mathcal{F}_{-b}^{-1}(-a_2' + \mathcal{F}_{-b}\mathcal{F}_{-a}^{-1}(-a_1'))\,,\\
\mathcal{A}(a_1', a_2', a_3') &= \mathcal{F}_a\mathcal{F}_{-a}^{-1}(-a_3' + \mathcal{B}(a_1', a_2'))\,.
\end{aligned}$$

Any sufficiently small perturbation $\bar{a}$ of a fixed point of Φ_n satisfies $a_3' = a_{3k} = -\mathcal{F}_a(-c)$ for $k = 1, \ldots, 2n$. We thus restrict Φ_n to the subspace Σ_{4n+2} of Σ_{6n+3} defined by these conditions and use the notation $\widetilde{\Phi}_n$ defined for Case #3, see (12). The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+2)\times(4n+2)}$ associated to the tangent map of $\widetilde{\Phi}_n$ at any point $\bar{a} \in \Sigma_{4n+2}$ is given by:

$$M^T = \begin{pmatrix}
\mathcal{A}'_{a_1'} & 0 & \cdots & \cdots & 0 & 1 + \mathcal{B}'_{a_1'} & -\mathcal{B}'_{a_1'} - \mathcal{A}'_{a_1'}\\
\mathcal{A}'_{a_2'} & 0 & \cdots & \cdots & \cdots & 1 + \mathcal{B}'_{a_2'} & -\mathcal{B}'_{a_2'} - \mathcal{A}'_{a_2'}\\
1 & 0 & \cdots & \cdots & \cdots & 0 & 0\\
0 & 1 & 0 & \vdots & \vdots & \vdots & 0\\
\vdots & 0 & \ddots & \vdots & \ddots & \vdots & \vdots\\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0\\
0 & \cdots & \cdots & 0 & 1 & 0 & 0
\end{pmatrix}.$$

Lemma 4.9. *The characteristic polynomial of M is equal to:*

$$R(X) = X^{4n+2} + AX^{4n+1} - (1 + A + B)X^{2n+1} + B\,,$$

with $A = -\mathcal{A}'_{a_1'} > 0$ and $B = (1 - \mathcal{A}'_{a_3'})(\mathcal{B}'_{a_1'} - \mathcal{B}'_{a_2'}) > 0$.

Here we obtain $B > 0$ for $a > b$ and $B < 0$ for $a < b$ (with $\operatorname{ess\,sup} |f| < \min(a, b)$).

The proof of Proposition 4.8 follows by the same arguments as for Proposition 4.4, that are the arguments of Proposition 3.3, using Theorem 4.20 below.

4.1.5. *Generalized case #3′.*

Proposition 4.10. *Any fixed point of Φ_n with generalized shape #3′ is unstable.*

The flow Φ_n near a fixed point with generalized shape #3′ is given by (8), with $a_1' = a_1'' = a_2' = a_2'' = 0$ and

$$\begin{aligned} &a_3'' = a_3 + \mathcal{A}(a_3', a_1, a_2)\,, \quad a_{6n+1} = -\mathcal{F}_{-a}(-c)\,,\\ &a_{6n+2} = a_3' - \mathcal{B}(a_3') + \mathcal{F}_{-a}(-c) + \mathcal{F}_a(-c)\,,\\ &a_{6n+3} = a_1 + a_2 - \mathcal{A}(a_3', a_1, a_2) + \mathcal{B}(a_3') - \mathcal{F}_a(-c)\,, \end{aligned}$$

where

$$\mathcal{B}(a_3') = \mathcal{F}_a\mathcal{F}_{-a}^{-1}(-a_3'), \quad \mathcal{A}(a_3', a_1, a_2) = \mathcal{F}_a\mathcal{F}_b^{-1}(a_2 + \mathcal{F}_b\mathcal{F}_a^{-1}(a_1 + \mathcal{B}(a_3'))).$$

Any sufficiently small perturbation $\bar{a}$ of a fixed point of Φ_n satisfies $a_1' = a_2' = 0$ and $a_{3k-2} = -\mathcal{F}_{-a}(-c)$ for $k = 1, \ldots, 2n$. We thus restrict Φ_n to the subspace Σ_{4n+1} of Σ_{6n+3} defined by these conditions and use the notation $\widetilde{\Phi}_n$ defined for Case #3′, see (13). The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+1)\times(4n+1)}$ associated to the tangent map of $\widetilde{\Phi}_n$ at any point $\bar{a} \in \Sigma_{4n+1}$ is given by:

$$M^T = \begin{pmatrix} \mathcal{A}'_{a_3'} & 0 & \cdots & \cdots & 0 & 1-\mathcal{B}'_{a_3'} & \mathcal{B}'_{a_3'} - \mathcal{A}'_{a_3'} \\ \mathcal{A}'_{a_2} & 0 & \cdots & \cdots & \cdots & 0 & 1-\mathcal{A}'_{a_2} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \vdots & \vdots & \vdots & 0 \\ \vdots & 0 & \ddots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \end{pmatrix},$$

Lemma 4.11. *The characteristic polynomial of M is equal to:*

$$R(X) = -(X^{4n+1} + AX^{4n} - (1+A+B)X^{2n} + B)\,,$$

where $A = -\mathcal{A}'_{a_3'} > 0$ and $B = (\mathcal{A}'_{a_2} - 1)(1 - \mathcal{B}'_{a_3'}) < 0$.

Indeed, B has same sign as $b - a$ (when $\operatorname{ess\,sup} |f| < \min(a, b)$).

The proof of Proposition 4.10 follows by the same arguments as for Proposition 4.6, that are the arguments of Proposition 3.3 using Theorem 4.21 below.

4.1.6. *Generalized case #4.*

Proposition 4.12. *Any fixed point of Φ_n with generalized shape #4 is unstable.*

The flow Φ_n near a fixed point with generalized shape #4 is given by (8) and

$$\begin{aligned}
&a_1'' = a_1 + \mathcal{A}(a_1', a_2', a_3'),\ a_2'' = a_2,\ a_3'' = a_3, \quad a_{6n+1} = a_1' + \mathcal{B}(a_1') - \mathcal{F}_{-b}(-c),\\
&a_{6n+2} = a_2' + a_3' + \mathcal{F}_{-b}(-c) - \mathcal{B}(a_1') + \mathcal{F}_a(-c) - \mathcal{A}(a_1', a_2', a_3')\,,\\
&a_{6n+3} = -\mathcal{F}_a(-c),
\end{aligned}$$

where

$$\mathcal{B}(a_1') = \mathcal{F}_{-b}\mathcal{F}_{-a}^{-1}(-a_1'), \mathcal{A}(a_1', a_2', a_3') = \mathcal{F}_a\mathcal{F}_{-a}^{-1}(-a_3' + \mathcal{F}_{-a}\mathcal{F}_{-b}^{-1}(-a_2' + \mathcal{B}(a_1'))).$$

Any sufficiently small perturbation $\bar{a}$ of a fixed point of Φ_n satisfies $a_3' = a_{3k} = -\mathcal{F}_a(-c)$ for $k = 1, \ldots, 2n$. We thus restrict Φ_n to the subspace Σ_{4n+2} of Σ_{6n+3} defined by these conditions and use the notation $\widetilde{\Phi}_n$ defined for Case #3, see (12). The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+2)\times(4n+2)}$ associated to the tangent map of $\widetilde{\Phi}_n$ at any point $\bar{a} \in \Sigma_{4n+2}$ is given by:

$$M^T = \begin{pmatrix}
\mathcal{A}'_{a_1'} & 0 & \cdots & \cdots & 0 & 1 + \mathcal{B}'_{a_1'} & -\mathcal{B}'_{a_1'} - \mathcal{A}'_{a_1'}\\
\mathcal{A}'_{a_2'} & 0 & \cdots & \cdots & \cdots & 0 & 1 - \mathcal{A}'_{a_2'}\\
1 & 0 & \cdots & \cdots & \cdots & 0 & 0\\
0 & 1 & 0 & \vdots & \vdots & \vdots & 0\\
\vdots & 0 & \ddots & \vdots & \ddots & \vdots & \vdots\\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0\\
0 & \cdots & \cdots & 0 & 1 & 0 & 0
\end{pmatrix}.$$

Lemma 4.13. *The characteristic polynomial of M is equal to:*

$$R(X) = X^{4n+2} + AX^{4n+1} - (1 + A + B)X^{2n+1} + B\,,$$

with $A = -\mathcal{A}'_{a_1'} > 0$ and $B = (1 + \mathcal{B}'_{a_1'})(1 - \mathcal{A}'_{a_2'}) < 0$.

Indeed, B has same sign as $b - a$ (when $\operatorname{ess\,sup} |f| < \min(a, b)$).

The proof of Proposition 4.12 follows by the same arguments as for Proposition 4.4.

4.1.7. *Generalized case #4′.*

Proposition 4.14. *Any fixed point of Φ_n with generalized shape #4′ is unstable.*

The flow Φ_n near a fixed point with generalized shape #4′ is given by (8), with $a'_1 = a''_1 = a'_2 = a''_2 = 0$ and

$$\begin{aligned} a''_3 &= a_3 + \mathcal{A}(a'_3, a_1, a_2)\,, \quad a_{6n+1} = -\mathcal{F}_{-a}(-c)\,, \\ a_{6n+2} &= a'_3 + a_1 - \mathcal{B}(a'_3, a_1) + \mathcal{F}_{-a}(-c) + \mathcal{F}_b(-c)\,, \\ a_{6n+3} &= a_2 - \mathcal{A}(a'_3, a_1, a_2) + \mathcal{B}(a'_3, a_1) - \mathcal{F}_b(-c)\,, \end{aligned}$$

where

$$\mathcal{B}(a'_3, a_1) = \mathcal{F}_b\mathcal{F}_a^{-1}(a_1 + \mathcal{F}_a\mathcal{F}_{-a}^{-1}(-a'_3)), \mathcal{A}(a'_3, a_1, a_2) = \mathcal{F}_a\mathcal{F}_b^{-1}(a_2 + \mathcal{B}(a'_3, a_1)).$$

Any sufficiently small perturbation $\bar{a}$ of a fixed point of Φ_n satisfies $a'_1 = a'_2 = 0$ and $a_{3k-2} = -\mathcal{F}_{-a}(-c)$ for $k = 1, \dots, 2n$. We thus restrict Φ_n to the subspace Σ_{4n+1} of Σ_{6n+3} defined by these conditions and use the notation $\widetilde{\Phi}_n$ defined for Case #3′, see (13). The transposed M^T of the matrix $M \in \mathbb{R}^{(4n+1)\times(4n+1)}$ associated to the tangent map of $\widetilde{\Phi}_n$ at any point $\bar{a} \in \Sigma_{4n+1}$ is given by:

$$M^T = \begin{pmatrix} \mathcal{A}'_{a'_3} & 0 & \cdots & \cdots & 0 & 1 - \mathcal{B}'_{a'_3} & \mathcal{B}'_{a'_3} - \mathcal{A}'_{a'_3} \\ \mathcal{A}'_{a_2} & 0 & \cdots & \cdots & \cdots & 0 & 1 - \mathcal{A}'_{a_2} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \vdots & \vdots & \vdots & 0 \\ \vdots & 0 & \ddots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \end{pmatrix},$$

Lemma 4.15. *The characteristic polynomial of M is equal to:*

$$R(X) = -(X^{4n+1} + AX^{4n} - (1 + A + B)X^{2n} + B)\,,$$

where $A = -\mathcal{A}'_{a'_3} > 0$ and $B = (\mathcal{A}'_{a_2} - 1)(1 - \mathcal{B}'_{a'_3}) < 0$.

Indeed, B has same sign as $b - a$ (when $\operatorname{ess\,sup} |f| < \min(a, b)$).

The proof of Proposition 4.14 follows by the same arguments as for Proposition 4.6.

4.1.8. *Generalized case #5.*

Proposition 4.16. *Any fixed point of Φ_n with generalized shape #5 is unstable.*

The flow Φ_n near a fixed point with generalized shape #5 is given by (8), with $a_1' = a_1'' = 0$ and

$$\begin{aligned}
&a_2'' = a_2 + \mathcal{A}(a_2', a_3', a_1), \quad a_3'' = a_3, \quad a_{6n+1} = a_2' + \mathcal{B}(a_2') - \mathcal{F}_{-a}(-c),\\
&a_{6n+2} = a_3' - \mathcal{B}(a_2') - \mathcal{C}(a_2', a_3') + \mathcal{F}_{-a}(-c) + \mathcal{F}_a(-c),\\
&a_{6n+3} = a_1 + \mathcal{C}(a_2', a_3') - \mathcal{A}(a_2', a_3', a_1) - \mathcal{F}_a(-c).
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{B}(a_2') = \mathcal{F}_{-a}\mathcal{F}_{-b}^{-1}(-a_2'), \quad \mathcal{C}(a_2', a_3') = \mathcal{F}_a\mathcal{F}_{-a}^{-1}(-a_3' + \mathcal{B}(a_2')),\\
&\mathcal{A}(a_2', a_3', a_1) = \mathcal{F}_b\mathcal{F}_a^{-1}(a_1 + \mathcal{C}(a_2', a_3')).
\end{aligned}$$

Any sufficiently small perturbation $\bar{a}$ of a fixed point of Φ_n satisfies $a_1' = 0$. We thus restrict Φ_n to the subspace Σ_{6n+2} of Σ_{6n+3} defined by this condition and use the notation $\widetilde{\Phi}_n$ defined for Case #5. The transposed M^T of the matrix $M \in \mathbb{R}^{(6n+2)\times(6n+2)}$ associated to the tangent map of $\widetilde{\Phi}_n$ at any point $\bar{a} \in \Sigma_{6n+2}$ is given by:

$$M^T = \begin{pmatrix}
\mathcal{A}'_{a_2'} & 0 & \cdots & 0 & 1 + \mathcal{B}'_{a_2'} & -\mathcal{B}'_{a_2'} - \mathcal{C}'_{a_2'} & \mathcal{C}'_{a_2'} - \mathcal{A}'_{a_2'}\\
\mathcal{A}'_{a_3'} & 0 & \cdots & \cdots & 0 & 1 - \mathcal{C}'_{a_3'} & \mathcal{C}'_{a_3'} - \mathcal{A}'_{a_3'}\\
\mathcal{A}'_{a_1} & 0 & \cdots & \cdots & \cdots & 0 & 1 - \mathcal{A}'_{a_1}\\
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
0 & \ddots & 0 & \vdots & \vdots & \vdots & 0\\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots\\
0 & \cdots & 0 & 1 & 0 & \cdots & 0
\end{pmatrix}.$$

Lemma 4.17. *The characteristic polynomial of M is equal to:*

$$R(X) = X^{6n+2} + AX^{6n+1} - (1 + A + B)X^{4n+1} + B\,,$$

with $A = -\mathcal{A}'_{a_2'} > 0$ and $B = (1 + \mathcal{B}'_{a_2'})(1 - \mathcal{C}'_{a_3'})(\mathcal{A}'_{a_1} - 1) > 0$.

Here $B > 0$ for any $a \neq b$ (when $\operatorname{ess\,sup} |f| < \min(a, b)$).

The proof of Proposition 4.16 follows by the same arguments as for Proposition 3.3, using Theorem 4.22 below instead of Theorem 3.19.

4.1.9. *Generalized case #6.*

Proposition 4.18. *Any fixed point of Φ_n with generalized shape #6 is unstable.*

The flow Φ_n near a fixed point with generalized shape #6 is given by (8), with $a'_1 = a''_1 = 0$ and

$$a''_2 = a_2 + \mathcal{A}(a'_2, a'_3, a_1)\,, \quad a''_3 = a_3\,, \quad a_{6n+1} = -\mathcal{F}_{-b}(-c)\,,$$
$$a_{6n+2} = a'_2 + a' - 3 + a_1 - \mathcal{A}(a'_2, a'_3, a_1) + \mathcal{F}_{-b}(-c) + \mathcal{F}_b(-c) = a'_2 + \mathcal{A}(a'_2),$$
$$a_{6n+3} = -\mathcal{F}_b(-c)\,.$$

where

$$\mathcal{A}(a'_2, a'_3, a_1) = \mathcal{F}_b\mathcal{F}_a^{-1}(a_1 + \mathcal{F}_a\mathcal{F}_{-a}^{-1}(-a'_3 + \mathcal{F}_{-a}\mathcal{F}_{-b}^{-1}(-a'_2))).$$

Any sufficiently small perturbation $\bar{a}$ of a fixed point of Φ_n satisfies $a'_3 = a_{3k} = \mathcal{F}_{-b}(c)$ and $a_{3k-2} = \mathcal{F}_b(c)$ for $k = 1, \dots, 2n$. We thus restrict Φ_n to the subspace Σ_{2n+1} of Σ_{6n+3} defined by these conditions and use the notation $\widetilde{\Phi}_n$ defined for Case #6. The transposed M^T of the matrix $M \in \mathbb{R}^{(2n+1)\times(2n+1)}$ associated to the tangent map of $\widetilde{\Phi}_n$ at any point $\bar{a} \in \Sigma_{2n+1}$ is given by:

$$M^T = \begin{pmatrix} \mathcal{A}'_{a'_2} & 0 & \cdots & \cdots & \cdots & 0 & 1 - \mathcal{A}'_{a'_2} \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & \cdots & \cdots & \dots & 0 \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Lemma 4.19. *The characteristic polynomial of M is equal to:*

$$R(X) = -(X^{2n+1} + AX^{2n} - (1 + A))\,,$$

with $A = -\mathcal{A}'_{a'_2} > 0$.

The proof of Proposition 4.18 may follow by the same arguments as for Proposition 4.4, that is the arguments of Proposition 3.3, using Theorem 4.20 with $B = 0$ (that can be proved directly in a similar way as in Lemma 3.23). Indeed, since $A = -\mathcal{A}'_{a'_2} > 0$, then $1 + A > 1$, and R has clearly one root z such that $|z| > 1$. This shows that the fixed point of Φ_n corresponding to the generalized shape #6 is unstable.

4.2. **Roots of the characteristic polynomials.** The following results are proved using homotopy techniques.

Theorem 4.20. *Assume that* $-1 \leq B \leq 1$ *and* $A > 0$. *If* $B \neq 1$ *or* $A < 2 + \frac{1}{n}$, *then the equation*

$$X^{4n+2} + AX^{4n+1} - (1 + A + B)X^{2n+1} + B = 0$$

has precisely $2n$ *solutions* z (*counting algebraic multiplicity*) *such that* $|z| > 1$. *Otherwise it has* $2n - 1$ *such solutions.*

Theorem 4.21. *If* $-1 \leq B \leq 1$ *and* $A > 0$, *the equation*

$$X^{4n+1} + AX^{4n} - (1 + A + B)X^{2n} + B = 0$$

has precisely $2n$ *solutions* z (*counting algebraic multiplicity*) *such that* $|z| > 1$.

Theorem 4.22. *Assume that* $0 \leq B \leq 1$ *and* $A > 0$. *If* $A \geq 1 - \frac{4n+1}{2n}(1 - B)$, *the equation*

$$X^{6n+2} + AX^{6n+1} - (1 + A + B)X^{4n+1} + B = 0$$

has precisely $2n$ *solutions* z (*counting algebraic multiplicity*) *such that* $|z| > 1$. *Otherwise it has* $2n + 1$ *such solutions.*

Let us denote by R the polynomial of Theorem 4.20 and consider the homotopy

$$g_s(z) = f_B(z) + (s - 1)P(z)\,, \tag{29}$$

with

$$\begin{aligned} P(X) &= X^{4n} + \cdots + X^{2n+1} = \frac{X^{4n+1} - X^{2n+1}}{X - 1}\,, & (30)\\ f_B(X) &= X^{4n+1} + 2X^{4n} + \cdots + 2X^{2n+1} - BX^{2n} + \cdots - B \\ &= \frac{X^{4n+2} + X^{4n+1} - (2 + B)X^{2n+1} + B}{X - 1}\,. & (31) \end{aligned}$$

We have $R(X) = (X - 1)g_A(X)$ and $(X - 1)f_B(X)$ is the polynomial of Theorem 3.19.

Using the same technique as for Lemma 3.22, we obtain:

Lemma 4.23. *Consider* $n \geq 1$ *and* g_s *defined by* (29,30,31). *If* $|z| = 1$, $s > 0$ *and* $-1 < B \leq 1$, *then* $g_s(z) \neq 0$. *If* $|z| = 1$, $s > 0$ *and* $B = -1$, *then* $g_s(z) = 0 \Leftrightarrow z = -1$.

Proof. Consider z such that $|z| = 1$ and $g_s(z) = 0$, and put $w = z^{2n+1}$. We get

$$\frac{s}{z}w^2 = -w^2 + (1 + s + B)w - B. \tag{32}$$

Taking modulus on both sides gives $s^2 = |-w^2 + (1 + s + B)w - B|^2$, writing $w = e^{i\theta}$ gives $(1 - \cos\theta)(1 + B^2 + (1 + B)s - 2B\cos\theta) = 0$. Thus, we have either $\cos\theta = 1$ or $B \neq -1$ and $s = -\frac{(1+B^2-2B\cos\theta)}{1+B} \leq 0$ which is impossible, or $B = -1$ and $\cos\theta = -1$. The condition $\cos\theta = 1$ implies $w = 1$. Plugging $w = 1$ in (32), we obtain $z = 1$ $(s > 0)$, which is impossible since $g_s(1) = (2n + 1)(1 - B) + 2ns > 0$. So the only possibility is $B = -1$ and $\cos\theta = -1$, that is $w = -1$. With (32), this implies $z = -1$, which is a root of g_s when $B = -1$. $\square$

Proof of Theorem 4.20 for $B \neq -1$. The function $g_s(z)$ defined in (29), (30), (31) is continuous with respect to s and polynomial with constant degree $4n+1$ with respect to z. From Lemma 4.23, g_s has no roots on $\partial D(0, 1)$ for $s > 0$. So by Rouché's theorem, the number of roots of g_s in $D(0, 1)$ (resp. $\overline{D}(0, 1)^c$) is constant with respect to s. According to Theorem 3.19, $g_1(z) = 0$ has precisely $2n$ solutions in $\overline{D}(0, 1)^c$, so does $g_s(z) = 0$ for $s > 0$. $\square$

Proof of Theorem 4.20 for $B = -1$. When B is fixed equal to -1, $z = -1$ is a root of g_s for all $s > 0$ (see Lemma 4.23), so $g_s(z) = (z + 1)\tilde{g}_s(z)$ with

$$\tilde{g}_s(z) = \tilde{f}_B(z) + (s - 1)\widetilde{P}(z)\,, \tag{33}$$

and

$$\widetilde{P}(X) = \frac{P(X)}{X+1} = X^{4n-1} + X^{4n-3} + \cdots + X^{2n+3} + X^{2n+1}, \tag{34}$$

$$\begin{aligned}\tilde{f}_B(X) &= \frac{f_B(X)}{X+1} \\ &= X^{4n} + X^{4n-1} + \cdots + X^{2n+1} + X^{2n} + X^{2n-2} + \cdots + X^2 + 1.\end{aligned} \tag{35}$$

We have $\tilde{g}_s(-1) = g'_s(-1) = 2n + 1 - sn \neq 0$ for $s \neq \bar{s} = 2 + \frac{1}{n}$. So, with Lemma 4.23, we obtain that, for $s \neq \bar{s}$, $\tilde{g}_s$ has no roots on $\partial D(0, 1)$. It follows, by Rouché's theorem, that the number of roots of $\tilde{g}_s$ (and thus of g_s) in $D(0, 1)$ (resp. $\overline{D}(0, 1)^c$) is constant with respect to s in each of the intervals $(0, \bar{s})$ and $(\bar{s}, +\infty)$. According to Theorem 3.19, g_1 has precisely $2n$ roots in $\overline{D}(0, 1)^c$, so does g_s for $0 < s < \bar{s}$.

For $s = \bar{s}$, we have $\tilde{g}'_{\bar{s}}(-1) = \frac{g''_{\bar{s}}(-1)}{2} = n(2n + 1) > 0$. The implicit function theorem shows that there exists $\delta > 0$, $\rho > 0$ and a $\mathcal{C}^1$ function

$Z : (\bar{s}-\delta, \bar{s}+\delta) \to \mathbb{C}$, such that $\tilde{g}_s(z) = 0$ and $|z+1| \leq \rho$ if and only if $z = Z(s)$. So $Z(\bar{s}) = -1$ and

$$Z'(\bar{s}) = -\frac{\frac{\partial \tilde{g}_s(z)}{\partial s}|_{s=\bar{s},z=-1}}{\frac{\partial \tilde{g}_s(z)}{\partial z}|_{s=\bar{s},z=-1}} = -\frac{\widetilde{P}(-1)}{\tilde{g}'_{\bar{s}}(-1)} = \frac{1}{2n+1} > 0\,,$$

which implies $|Z(s)| < 1$ for $\bar{s} < s < \bar{s}+\delta$ and $|Z(s)| > 1$ for $\bar{s}-\delta < s < \bar{s}$ (possibly by shrinking δ). Since the root $Z(s)$ of $\tilde{g}_s$ goes from $\overline{D}(0,1)^c$ to $D(0,1)$ when s increases around $\bar{s}$, this implies that

$$\deg(\tilde{g}_s, D(0,1), 0) = 1 + \deg(\tilde{g}_{s'}, D(0,1), 0) \quad \text{for } \bar{s}-\delta < s' < \bar{s} < s < \bar{s}+\delta$$

(one may use the set $G_\rho = \{z \in \mathbb{C} \text{ such that } |z| < 1 \text{ or } |z+1| < \rho\}$ to prove this result). Since $\tilde{g}_A$ has $2n$ roots in $\overline{D}(0,1)^c$ when $0 < A < \bar{s}$, it follows that $\tilde{g}_A$ has $2n-1$ roots when $\bar{s} < A$ and also when $A = \bar{s}$. So does g_A and $R = (X-1)g_A(X)$. □

Theorem 4.21 is proved by the same technique as the case $B \neq 1$ of Theorem 4.20, changing P and Q into

$$P(X) = \frac{X^{4n} - X^{2n}}{X-1}\,, \tag{36}$$

$$f_B(X) = \frac{X^{4n+1} + X^{4n} - (2+B)X^{2n} + B}{X-1}\,. \tag{37}$$

Indeed, using the same technique as for Lemma 3.24, one gets:

Lemma 4.24. *Consider $n \geq 1$ and g_s defined by* (29,36,37). *If $|z| = 1$, $s > 0$ and $-1 \leq B \leq 1$, then $g_s(z) \neq 0$.*

Proof. Consider z such that $|z| = 1$ and $g_s(z) = 0$, and put $w = z^{2n}$. We obtain

$$\frac{z}{s}w^2 = -w^2 + (1 + \frac{1}{s} + \frac{B}{s})w - \frac{B}{s} \tag{38}$$

which leads, as (32), to the only possibilities $w = z = 1$ or $\frac{B}{s} = -1$ and $w = z = -1$. This last case is impossible since $w = z^{2n}$. The first one also since $g_s(1) = 1 + 2n(1 - B + s) > 0$. □

Theorem 4.22 is proved by the same technique as Theorem 4.20, changing P and Q into

$$P(X) = \frac{X^{6n+1} - X^{4n+1}}{X-1}\,, \tag{39}$$

$$f_B(X) = \frac{X^{6n+2} + X^{6n+1} - (2+B)X^{4n+1} + B}{X-1}\,. \tag{40}$$

Lemma 4.25. *Consider $n \geq 1$ and g_s defined by (29,39,40). If $|z| = 1$, $s > 0$ and $0 \leq B \leq 1$, then $g_s(z) = 0 \Rightarrow z = 1$. Moreover, $z = 1$ is a root of g_s if and only if $s = 1 - \frac{4n+1}{2n}(1-B)$. In this case, it is a simple root and $g_s'(1) > 0$.*

Proof. Consider z such that $|z| = 1$ and $g_s(z) = 0$, and put $w = z^{2n+1}$, $v = z^{2n}$. We get

$$w + sv - (1 + s + B) + B(wv)^{-1} = 0\,. \tag{41}$$

Set $w = e^{i\theta}$, $v = e^{i\theta'}$. Multiplying (41) by $Bv - 1$, and taking the real part, we obtain

$$(1-B^2)(1-\cos\theta)+((1-s)(s+B)+(s-B)^2)(1-\cos\theta')+2Bs(1-\cos\theta')^2 = 0\,.$$

Suppose first that $0 < s \leq 1$ and $0 \leq B \leq 1$. Since all the terms in the previous sum are non negative, and the sum is zero, all the terms are zero. This implies, in particular, $\cos\theta' = 1$ or $v = 1$. Plugging $v = 1$ in (41) and taking the real part, we obtain $(1+B)(\cos\theta - 1) = 0$, so $\cos\theta = 1$ or $w = 1$.

Equation (41) can be rewritten

$$v + \frac{1}{s}w - (1 + \frac{1}{s} + \frac{B}{s}) + \frac{B}{s}(wv)^{-1} = 0$$

which is nothing but (41) with w, v, s, B replaced respectively by $v, w, \frac{1}{s}, \frac{B}{s}$. Hence, the same conclusion $v = w = 1$ is obtained when $0 < \frac{1}{s} \leq 1$ and $0 \leq \frac{B}{s} \leq 1$, and in particular when $0 \leq B \leq 1 \leq s$. Therefore, one deduces $z = \frac{w}{v} = 1$ for any B and s such that $0 \leq B \leq 1$ and $s > 0$.

The last points of the lemma are obtained by simple computations: $g_s(1) = 2n(s-1) + (4n+1)(1-B)$ for all s and $2g_s'(1) = (g_s(z)(z-1))''(1) = 2n + 1 + B(4n+1)(6n+1) > 0$ for $s = 1 - \frac{4n+1}{2n}(1-B)$. □

Proof of Theorem 4.22. Let g_s be defined by (29,39,40) with B fixed such that $0 \leq B \leq 1$ and denote $\bar{s} = \max(0, 1 - \frac{4n+1}{2n}(1-B))$. From Lemma 4.25, g_s has no roots on $\partial D(0,1)$ for $s \neq \bar{s}$. It follows, by Rouché's theorem, that the number of roots of g_s in $D(0,1)$ (resp. $\overline{D}(0,1)^c$) is constant with respect to s in each of the intervals $(0, \bar{s})$ and $(\bar{s}, +\infty)$.

Suppose first that $\bar{s} = 0$. According to Theorem 3.21, g_1 has precisely $2n$ roots in $\overline{D}(0,1)^c$, so does g_s for all $s > 0$.

Suppose now that $\bar{s} > 0$. By Lemma 4.25, 1 is a simple root of $g_{\bar{s}}$, so one can apply the same technique as for the case $B = -1$ of Theorem 4.20. By the implicit function theorem, there exists $\delta > 0$, $\rho > 0$ and a $\mathcal{C}^1$ function $Z : (\bar{s} - \delta, \bar{s} + \delta) \to \mathbb{C}$, such that $g_s(z) = 0$ and $|z - 1| \leq \rho$ if and only if

$z = Z(s)$. Then, $Z(\bar{s}) = 1$ and

$$Z'(\bar{s}) = -\frac{\frac{\partial g_s(z)}{\partial s}|_{s=\bar{s},z=1}}{\frac{\partial g_s(z)}{\partial z}|_{s=\bar{s},z=1}} = -\frac{P(1)}{g'_{\bar{s}}(1)} = -\frac{2n}{g'_{\bar{s}}(1)} < 0\,,$$

which implies that the root $Z(s)$ of g_s goes from $D(0,1)$ to $\overline{D}(0,1)^c$ when s decreases around $\bar{s}$. It follows that the number of roots of g_s in $\overline{D}(0,1)^c$ is constant for $s \in [\bar{s}, +\infty)$, and thus equal to $2n$ by Theorem 3.21 (using $\bar{s} \leq 1$), and that it is constant equal to $2n+1$ for $s \in (0, \bar{s})$. The same result holds for $R = (X-1)g_A(X)$ and $s = A$. □

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References

[1] M. Akian and P.-A. Bliman, *On super-high-frequencies in discontinuous 1st-order delay-differential equations,* J. Diff. Equations, 162 (2000), 326–358.

[2] M. Akian and P.-A. Bliman, and M. Sorine, *Control of delay systems with relay,* IMA J. on Math. Control Information, to appear.

[3] A. Berman and R.J. Plemmons, "Nonnegative Matrices in the Mathematical Sciences," Academic press, 1979.

[4] O. Diekmann, S.A. Van Gils, S.M. Verduyn Lunel, and H.O. Walther, "Delay Equations: Functional-, Complex- and Nonlinear Analysis," Applied mathematical Sciences, 110, Springer Verlag, New York, 1995.

[5] J.G. Dix, *Infinite frequency solutions to discontinuous control systems with variable delay,* Israel J. of Math., 106 (1998), 165–176.

[6] G.F. Franklin, J.D. Powell, and A. Emami-Naeimi, "Feedback Control of Dynamic Systems," Addison-Westley Publishing Company, 1988.

[7] L. Fridman, E. Fridman, and E. Shustin, *Steady modes and sliding modes in the relay control systems with time delay,* Proc. of the 35th conference on decision and control, Kobe, Japan (1996), 4601–4606.

[8] L. Fridman, E. Fridman, and E. Shustin, *The phenomenon of the second order steady modes in the relay control systems with time delay,* Proc. of the 4th European control conference, Brussels, Belgium, 1997.

[9] U. an der Heiden and H.-O. Walther, *Existence of chaos in control systems with delayed feedback,* J. Diff. Equations, 47 (1983), 273–295.

[10] A.F. Ivanov and J. Losson, *Stable rapidly oscillating solutions in delay differential equations with negative feedback,* Cadsem Report 97-004, Deakin University, Australia 1997.

[11] J. Mallet-Paret, *Morse decomposition for delay-differential equations,* J. Diff. Equations, 72 (1988), 270–315.

[12] J. Mallet-Paret and R.D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state dependent time lags: II,* J. für die Reine und Angewandte Math., 477 (1996), 129–197.

[13] J. Mallet-Paret, R.D. Nussbaum, and P. Paraskevopoulos, *Periodic solutions for functional differential equations with multiple state dependent time lags,* Topological Meth. in Nonlinear Anal., 3, (1994), 101–162.

[14] R.D. Nussbaum, *Periodic solutions of nonlinear autonomous functional differential equations,,* Funct. Diff. Equations and approximations of fixed points, Lecture Notes in Mathematics, 730 (1979), 283–325, Springer-Verlag.

[15] R.D. Nussbaum, "Personal Communication", 1999.

[16] R.D. Nussbaum and E. Shustin, *Nonexpansive periodic operators in* l_1 *with application to super-high-frequency oscillations in a discontinuous dynamical system with time delay,* J. of Dynamics and Differential equations, 13 (2001), 381–424.

[17] E. Shustin, *Super-high-frequency oscillations in a discontinuous dynamic system with time delay,* Israel J. of Mathematics, 90 (1995), 199–219.