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DYNAMIC OPTIMIZATION OF LONG-TERM GROWTH RATE FOR A PORTFOLIO WITH TRANSACTION COSTS AND LOGARITHMIC UTILITY

Marianne Akian and Agnès Sulem

INRIA, Domaine de Voluceau, Le Chesnay Cedex, France

Michael I. Taksar*

Department of Applied Mathematics and Statistics, SUNY at Stony Brook

We study the optimal investment policy for an investor who has available one bank account and n risky assets modeled by log-normal diffusions. The objective is to maximize the long-run average growth of wealth for a logarithmic utility function in the presence of proportional transaction costs. This problem is formulated as an ergodic singular stochastic control problem and interpreted as the limit of a discounted control problem for vanishing discount factor. The variational inequalities for the discounted control problem and the limiting ergodic problem are established in the viscosity sense. The ergodic variational inequality is solved by using a numerical algorithm based on policy iterations and multigrid methods. A numerical example is displayed for two risky assets.

Key Words: singular ergodic stochastic control, viscosity solution, variational inequality, portfolio selection, transaction costs

1. INTRODUCTION

We consider a model of n risky assets (called *Stocks*) whose prices are governed by logarithmic Brownian motions and one risk-free asset (called *Bank*). Consider an investor who has an initial wealth invested in Stocks and Bank and who has the ability to transfer funds between the assets. When these transfers involve proportional transaction costs, this problem can be formulated as a singular stochastic control problem.

In one type of model, the objective is to maximize the cumulative expected utility of consumption over a planning horizon. Magill and Constantinides (1976) formulated the problem with one risky asset ($n = 1$) over an infinite horizon and conjectured that the no-transaction region is a cone in the two-dimensional space of position vectors. This fact proved in a discrete-time setting by Constantinides (1986) and in continuous time by Davis and Norman (1990), for HARA utility functions. Similar investment policy has been obtained by Framstad, Øksendal, and Sulem (2001) for a jump diffusion market. An analysis of the optimal strategy together with regularity results for the value function can be found for $n = 1$ in Fleming and Soner (1993, Sec. VIII.7) and Shreve and Soner (1994) and for any n in Akian, Menaldi, and Sulem (1996). Existence and uniqueness

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Address correspondence to Michael I. Taksar, Dept. of Applied Mathematics and Statistics, SUNY at Stony Brook, Stony Brook, NY 11794-3600; e-mail: taksar@ams.sunysb.edu.

of a solution for the corresponding variational inequality have been proved by Fleming and Soner and Akian et al. (1996). Fitzpatrick and Fleming (1991) studied the numerical approximation of the investment-consumption problem by a Markov chain control problem with convergence arguments based on viscosity solution techniques; in Akian et al. (1996) the variational inequality was solved numerically. A deterministic model was solved by Shreve, Soner, and Xu (1991) with a general utility function that is not necessarily of HARA type, and a stochastic model driven by a finite state Markov chain rather than a Brownian motion has been studied by Zariphopoulou (1992), who gives existence and uniqueness results by using viscosity solution techniques.

Another type of problem is to consider a model without consumption and to maximize a utility function of the growth of wealth over a finite time horizon. In Akian, Sulem, and Séquier (1995a) and Akian et al. (1995b), the associated variational inequality solved numerically and numerical results are presented dealing with the domestic allocation issue. Similar dynamic programming variational inequalities have been studied by Zhu (1991, 1992).

Finally, a third class of problem consists in maximizing a long-run average growth of wealth. A one-risky asset case is studied in Taksar, Klass, and Assaf (1988) with a logarithmic utility function. The optimal policy consists in keeping the ratio of funds within (a priori unknown) bounds with minimal effort. Recently, several papers treated other aspects of transaction cost problems. In Buckley and Korn (1997), Korn (1997), Øksendal and Sulem (1999), Chancelier, Øksendal, and Sulem (2001), and Morton and Pliska (1995) a fixed transaction cost is present and the solution of the problem requires the use of impulse control. Of a special interest is the latest work by Bielecki and Pliska (2000) in which risk-sensitive control techniques are used and the growth rate optimization becomes a special case of a more general scheme.

In the present paper, we are further developing the model of Taksar et al. (1988), for any n, combining some of the methods and numerical techniques of Akian et al (1996). The problem we are facing can be formulated as an ergodic singular control problem (see Sections 2 and 3) and the associated dynamic programming equation is an ergodic variational inequality (VI). The ergodic problem can be approximated by a discounted control problem over an infinite horizon (Sections 4 and 5). The value function of the discounted problem is characterized as the unique "constrained" viscosity solution of an elliptic variational inequality (Section 6). The existence follows from a weak Dynamic Principle and the uniqueness is derived by proving a comparison theorem, with special care for the boundary points (Theorem 6.7). The existence of a viscosity solution to the ergodic variational inequality is proved by studying the asymptotics of the viscosity solution of the elliptic variational inequality when the discount factor vanishes (Section 7). The uniqueness of the constant involved in this equation, namely the average growth rate, follows from the uniqueness of the viscosity solution of the discounted variational inequality. Concerning the uniqueness of the potential function, Ishii's techniques (Ishii 1985; Ishii and Lions 1990) do not apply, and the variational techniques used for example in Bensoussan (1988) are only valid for nondegenerate equations, which is not the case here. We shall thus only prove the uniqueness of the potential function within an additive constant for the one risky asset case, and give a formal argument in the general case (Section 8). Section 9 is devoted to numerical results: the variational inequality is solved by using a numerical algorithm based on policies iterations and multigrid methods, and the optimal transaction policy is provided for a two-risky-assets portfolio. Moreover, the explicit solution is computed in the case of one risky asset.

2. FORMULATION OF THE PROBLEM

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a given filtration $(\mathcal{F}_t)_{t\geq 0}$. We denote respectively by $S_0(t)$ and $S_i(t)$ the amount of money the investor has in Bank and in the ith risky asset at time t. In the absence of transaction, the process $S_0(t)$ grows deterministically at exponential rate r, while $(S_1(t), \dots, S_n(t))$ is governed by a logarithmic Brownian motion with drift $\alpha = (\alpha_1, \dots, \alpha_n)$ and symmetric positive definite diffusion matrix $a = (a_{ij})$. The control is described in terms of the set of nondecreasing functionals $\mathcal{L}_i(t)$ and $\mathcal{M}_i(t)$, $i = 1, \dots, n$, representing cumulative purchase and sale of the ith stock at time t. The usual requirement is that $\mathcal{L}(t) = (\mathcal{L}_1(t), \dots, \mathcal{L}_n(t))$ and $\mathcal{M}(t) = (\mathcal{M}_1(t), \dots, \mathcal{M}_n(t))$ are $\mathcal{F}_t$-adapted càdlàg processes, such that $\mathcal{L}(0^-) = \mathcal{M}(0^-) = 0$. The dynamics of the system under control is then given by

$$dS_0(t) = rS_0(t)\,dt + \sum_{i=1}^{n}[(1-\mu_i)d\mathcal{M}_i(t) - (1+\lambda_i)d\mathcal{L}_i(t)], \tag{2.1}$$

$$dS_i(t) = \alpha_i S_i(t)\,dt + S_i(t)dw_i(t) + d\mathcal{L}_i(t) - d\mathcal{M}_i(t), \qquad i = 1, \dots, n, \tag{2.2}$$

with initial condition

$$S_i(0^-) = x_i, \qquad i = 0, 1, \dots, n, \tag{2.3}$$

and we refer by $S(t) = (S_0(t), \dots, S_n(t))$ to the investor's position at time t. In (2.2), $(w_1(t), \dots, w_n(t))$ is a Brownian motion such that $Ew_i(t) = 0$ and $E\{w_i(t)w_j(t)\} = a_{ij}t$, $i, j = 1, \dots, n$; the coefficient λ_i represents the commission for the purchase of \$1 worth of the ith stock; and μ_i is the commission for the sale of \$1 worth of the ith stock. We suppose that $0 \leq \mu_i < 1$, $\lambda_i \geq 0$, and $\lambda_i + \mu_i > 0$. Equation (2.1) shows that all the proceeds from sales of stock go into Bank while all the purchases are financed by withdrawing cash from Bank. We define the net wealth $W(t)$ at time t as the total amount of cash available if all the risky assets are sold, that is $W(t) := \rho(S(t))$, where

$$\rho(x) = x_0 + \sum_{i=1}^{n}(1-\mu_i)x_i \quad \text{for } x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}. \tag{2.4}$$

We assume that there is no borrowing and no shortselling. Accordingly, we define the admissible region as $\mathcal{S} = \mathbb{R}_+^{n+1}$. A set of control functionals $(\mathcal{L}, \mathcal{M})$ is *admissible for* $x = (x_0, x_1, \dots, x_n) \in \mathcal{S}$ if (2.1–2.3) has a unique solution and $S(t) \in \mathcal{S}$ for all $t \geq 0$. With each admissible policy we associate a performance functional that depends continuously on the risk-aversion coefficient $\gamma \geq 0$:

$$J_x(\mathcal{L}, \mathcal{M}) = \begin{cases} \liminf\limits_{T\to+\infty} T^{-1}(1-\gamma)^{-1}\log E\big[W(T)^{1-\gamma}\big] & \text{when } \gamma \geq 0, \gamma \neq 1, \\ \liminf\limits_{T\to+\infty} T^{-1}E\big[\log W(T)\big] & \text{when } \gamma = 1. \end{cases} \tag{2.5}$$

When $W(t^-) = 0$, or equivalently $S(t^-) = 0$, the only admissible policy is to remain at zero. Then, $W(T) = 0$ for all $T \geq t$ and $J_x(\mathcal{L}, \mathcal{M}) = -\infty$. In particular, $J_0(\mathcal{L}, \mathcal{M}) = -\infty$ for the unique admissible policy $(\mathcal{L}, \mathcal{M})$ for 0. The objective is to find the optimal growth rate

$$\sup_{(\mathcal{L},\mathcal{M})} J_x(\mathcal{L}, \mathcal{M}) \quad \text{for } x \in \mathcal{S} \setminus \{0\}, \tag{2.6}$$

where the supremum is taken over all admissible policies $(\mathcal{L}, \mathcal{M})$.

In the present paper, we study the case $\gamma = 1$ only. The case $\gamma \neq 1$ is a multiplicative ergodic problem. The associated ergodic variational inequality contains an additional nonlinear quadratic term (which does not appear when $\gamma = 1$). When $\gamma < 1$, this equation can be interpreted as an ergodic variational inequality for a stochastic control problem with an additional control variable, whereas when $\gamma > 1$, it can be seen as an ergodic Isaac equation associated to a max-min problem. We refer to Whittle (1990), where multiplicative control problems related to robust control are studied. The case $\gamma < 1$ will be treated in a forthcoming paper.

3. AN ERGODIC STOCHASTIC CONTROL FORMULATION

We formulate problem (2.6) as a classical ergodic stochastic control problem. Set

$$s_i(t) := \frac{(1-\mu_i)S_i(t)}{W(t)}, \qquad i = 1, \dots, n,$$

and $s(t) := (s_1(t), \dots, s_n(t))$. From (2.1–2.2), we have

$$dW(t) = W(t)\Bigg(\Big(r + \sum_{i=1}^{n}(\alpha_i - r)s_i(t)\Big)dt + \sum_{i=1}^{n} s_i(t)dw_i(t) - \sum_{i=1}^{n}(\lambda_i + \mu_i)W(t)^{-1}d\mathcal{L}_i(t)\Bigg). \tag{3.1}$$

Let $\mathcal{L}_i^c$ and $\mathcal{M}_i^c$ denote the continuous parts of $\mathcal{L}_i$ and $\mathcal{M}_i$ respectively. Using Itô's formula for càdlàg processes (see Meyer 1976), we get

$$E\log W(T) = \log W(0^-) + E\Bigg[\int_0^T H(s(t))dt - \sum_{i=1}^{n}(\lambda_i + \mu_i)\int_0^T W(t)^{-1}d\mathcal{L}_i^c(t) + \sum_{0\le t\le T}(\log W(t) - \log W(t^-))\Bigg], \tag{3.2}$$

where

$$H(y) = r + \sum_{i=1}^{n}(\alpha_i - r)y_i - \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}y_iy_j \quad \text{for } y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

From (3.1), we see that the discontinuities in the wealth process happen only at the times of jumps of the control processes associated with the purchase of stocks. More precisely,

$$W(t) = W(t^-) - \sum_{i=1}^{n}(\lambda_i + \mu_i)(\mathcal{L}_i(t) - \mathcal{L}_i(t^-))$$

and

$$\begin{aligned}\log W(t) - \log W(t^-) &= \int_0^1 \frac{W(t) - W(t^-)}{W(t^-) + u(W(t) - W(t^-))}du \\ &= -\sum_{i=1}^{n}(\lambda_i + \mu_i)\int_0^1 \frac{\mathcal{L}_i(t) - \mathcal{L}_i(t^-)}{(1-u)W(t^-) + uW(t)}du.\end{aligned}$$

The variation of $\log W(t)$ at the times of discontinuities of W is thus the same as the one resulting from a continuous purchase of the same amount of all stocks at a constant rate, provided that the diffusion part of the process is frozen. In the sequel, we use the following notation. Let $\xi(t)$ be an n-dimensional càdlàg process with bounded variation and let $X(t)$ be a càdlàg process. For any continuous function f with values in $\mathbb{R}$ or in a space of linear operators over $\mathbb{R}^n$, we write

$$(3.3)\quad \int_{t^-}^{t'} f(s,X(s))\circ d\xi(s) \equiv \int_t^{t'} f(s,X(s))\,d\xi^c(s) + \sum_{t\le s\le t'} \int_0^1 f\big(s,(1-u)X(s^-)+uX(s)\big)(\xi(s)-\xi(s^-))\,du,$$

where ξ^c denotes the continuous part of ξ. This notation allows us to "simplify" Itô's formula for the càdlàg process $X(t) \in \mathbb{R}^n$, satisfying

$$dX(t) = g(t, X(t))\,dt + \sigma(t, X(t))\,dw(t) + \sum_{i=1}^k a_i(t)d\xi^i(t),$$

where $(\xi^i)_{i=1,\dots,k}$ are càdlàg processes with bounded variation with values in $\mathbb{R}^n$, $a_i(t)$ are continuous functions, and w is a normalized m-dimensional Brownian motion. Namely, for any function ϕ that is $\mathcal{C}^2$ in $\mathbb{R}^n$, Itô's formula reads

$$d\phi(X(t)) = D\phi(X(t))\big(g(t, X(t))\,dt + \sigma(t, X(t))\,dW(t)\big) + \sum_{i=1}^k a_i(t)D\phi(X(t))\circ d\xi^i(t) + \tfrac{1}{2}\,\mathrm{tr}\,(\sigma\sigma^T(t, X(t))D^2\phi(X(t)))\,dt.$$

Equations (3.2) and (2.5) (with $\gamma = 1$) can be rewritten as

$$(3.4)\quad E\log W(T) = \log\rho(x)+E\left[\int_0^T H(s(t))dt-\sum_{i=1}^n(\lambda_i+\mu_i)\int_{0^-}^T W(t)^{-1}\circ d\mathcal{L}_i(t)\right],$$

and

$$(3.5)\quad J_x(\mathcal{L},\mathcal{M}) = \liminf_{T\to+\infty} T^{-1}E\left[\int_0^T H(s(t))\,dt - \sum_{i=1}^n(\lambda_i+\mu_i)\int_{0^-}^T W(t)^{-1}\circ d\mathcal{L}_i(t)\right].$$

The original problem is thus equivalent to an *ergodic problem for a singular stochastic control model* with dynamics given by (2.1–2.3) and performance functional given by (3.5).

REMARK 3.1. From (3.5), it follows immediately that J_x only depends on the trajectory of the n-dimensional state $s(t)$ ($\mathcal{L}(t)$ has to be adjusted proportionally to the initial wealth $W(0^-) = \rho(x)$). The dimension of the state can then be reduced from $n+1$ to n (see Sec. 5).

4. A DISCOUNTED CONTROL PROBLEM APPROXIMATION

Using the formulation of (2.6) as an ergodic control problem with criterion (3.5), we can write it as the limit of a discounted control problem. Consider the discounted version of (3.4):

$$(4.1)\quad J_x^\delta(\mathcal{L},\mathcal{M}) = \log\rho(x) + E\left[\int_0^{+\infty} e^{-\delta t}H(s(t))\,dt - \sum_{i=1}^n(\lambda_i+\mu_i)\int_{0^-}^{\infty} e^{-\delta t}W(t)^{-1}\circ d\mathcal{L}_i(t)\right].$$

Here $x \in \mathcal{S}$ is the initial position, $\mathcal{L}, \mathcal{M}$ are admissible control functionals, and $\delta > 0$ is the discount rate. Since H is upper bounded ((a_{ij}) is a positive definite matrix) and $\mathcal{L}_i$

are nondecreasing processes, the integrals in (4.1) and $J_x^\delta(\mathcal{L},\mathcal{M})$ are well defined with values in $\mathbb{R}\cup\{-\infty\}$. Set

$$V^\delta(x) = \sup_{(\mathcal{L},\mathcal{M})} J_x^\delta(\mathcal{L},\mathcal{M}). \tag{4.2}$$

In Section 7, we shall prove that $\delta V^\delta(x)$ tends to the optimal growth rate defined in (2.6) as δ goes to 0. Definition (4.2) implies $J_x^\delta(\mathcal{L},\mathcal{M}) \le \log\rho(x) + \kappa/\delta$, where

$$\kappa = \max_{z\in\mathbb{R}^n} H(z). \tag{4.3}$$

Hence,

$$V^\delta(x) \le \log\rho(x) + \kappa/\delta < +\infty \quad \text{in } \mathcal{S}, \tag{4.4}$$

and $V^\delta(0) = -\infty$. On the other hand, one can easily see that for $x \in \mathcal{S}\setminus\{0\}$ and some particular $(\mathcal{L},\mathcal{M})$, $J_x^\delta(\mathcal{L},\mathcal{M}) > -\infty$ (see the proof of Proposition 5.1 below). Thus, the value function V^δ is well defined in $\mathcal{S}\setminus\{0\}$. We now state a result that will be useful to prove properties of V^δ.

LEMMA 4.1. *Let $(\mathcal{L},\mathcal{M})$ be an admissible policy for $x \in \mathcal{S}$ and let W be the corresponding wealth process. Then*

$$J_x^\delta(\mathcal{L},\mathcal{M}) = \liminf_{T\to+\infty} \delta \int_0^T e^{-\delta t} E \log W(t)\, dt. \tag{4.5}$$

Proof. Equation (4.5) holds for the unique admissible policy $(\mathcal{L},\mathcal{M})$ for $x = 0$. Suppose $x \neq 0$. Applying Itô's formula to $e^{-\delta t}\log W(t)$, we obtain

$$\begin{aligned} E\big[e^{-\delta T}\log W(T)\big] = \log\rho(x) + E\bigg[&\int_0^T e^{-\delta t} H(s(t))\, dt \\ &- \sum_{i=1}^n (\lambda_i+\mu_i)\int_{0^-}^T e^{-\delta t} W(t)^{-1}\circ d\mathcal{L}_i(t) - \delta\int_0^T e^{-\delta t}\log W(t)\, dt\bigg]. \end{aligned}$$

Then

$$J_x^\delta(\mathcal{L},\mathcal{M}) = \lim_{T\to+\infty}\bigg[e^{-\delta T} E\log W(T) + \delta\int_0^T e^{-\delta t} E\log W(t)\, dt\bigg]. \tag{4.6}$$

In view of (3.4) and (4.3), $E\log W(T) \le \log\rho(x) + \kappa T$. Then,

$$\limsup_{T\to+\infty} e^{-\delta T} E\log W(T) \le 0.$$

If the above limit vanishes then equation (4.5) holds. Otherwise, the right-hand sides of (4.5) and (4.6) are both equal to $-\infty$, and equation (4.5) remains true. □

PROPOSITION 4.2. *For all $\delta > 0$, the function V^δ is concave in $\mathcal{S}$.*

Proof. For all $x \in \mathcal{S}$ and admissible policies $(\mathcal{L}, \mathcal{M})$ for x, denote by $S_{x,\mathcal{L},\mathcal{M}}(t)$ the solution of (2.1–2.3) and by $W_{x,\mathcal{L},\mathcal{M}}(t) = \rho(S_{x,\mathcal{L},\mathcal{M}}(t))$ the corresponding wealth process. Let $x^1, x^2 \in \mathcal{S}$ be two initial positions and $(\mathcal{L}_1, \mathcal{M}_1)$, $(\mathcal{L}_2, \mathcal{M}_2)$ two sets of admissible controls for x^1 and x^2 respectively. Let $0 < \theta < 1$ and set

$$x^3 = \theta x^1 + (1-\theta)x^2, \qquad (\mathcal{L}^3, \mathcal{M}^3) = \theta(\mathcal{L}^1, \mathcal{M}^1) + (1-\theta)(\mathcal{L}^2, \mathcal{M}^2).$$

From the linearity of equations (2.1), (2.2), and (2.4), one gets

$$S_{x^3,\mathcal{L}^3,\mathcal{M}^3}(t) = \theta S_{x^1,\mathcal{L}^1,\mathcal{M}^1}(t) + (1-\theta)S_{x^2,\mathcal{L}^2,\mathcal{M}^2}(t),$$
$$W_{x^3,\mathcal{L}^3,\mathcal{M}^3}(t) = \theta W_{x^1,\mathcal{L}^1,\mathcal{M}^1}(t) + (1-\theta)W_{x^2,\mathcal{L}^2,\mathcal{M}^2}(t).$$

Since $\mathcal{S}$ is convex, the latter shows that $(\mathcal{L}^3, \mathcal{M}^3)$ is an admissible set of controls for the initial position x^3. In view of Lemma 4.1, the concavity of the logarithm, and the superadditivity of the lim inf operation, we obtain

$$V^\delta(x^3) \geq J^\delta_{x^3}(\mathcal{L}^3, \mathcal{M}^3) \geq \theta J^\delta_{x^1}(\mathcal{L}^1, \mathcal{M}^1) + (1-\theta)J^\delta_{x^2}(\mathcal{L}^2, \mathcal{M}^2).$$

Taking the supremum over $(\mathcal{L}^1, \mathcal{M}^1)$ and $(\mathcal{L}^2, \mathcal{M}^2)$, we get the statement of the proposition. □

PROPOSITION 4.3. *The function V^δ is nondecreasing with respect to each of its arguments.*

Proof. Let $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n) \in \mathcal{S}$ be such that $x \leq y$ (i.e., $x_i \leq y_i$, $i = 0, \ldots, n$). Let $(\mathcal{L}, \mathcal{M})$ be a policy admissible for x. Itô's formula implies that $S_{x,\mathcal{L},\mathcal{M}}(t) \leq S_{y,\mathcal{L},\mathcal{M}}(t)$ for all $t \geq 0$. Then, $(\mathcal{L}, \mathcal{M})$ is admissible for y and $W_{x,\mathcal{L},\mathcal{M}}(t) \leq W_{y,\mathcal{L},\mathcal{M}}(t)$ for all $t \geq 0$. Lemma 4.1 leads to

$$J^\delta_x(\mathcal{L}, \mathcal{M}) \leq J^\delta_y(\mathcal{L}, \mathcal{M}) \leq V^\delta(y).$$

Taking the supremum over $(\mathcal{L}, \mathcal{M})$ yields the statement of the proposition. □

PROPOSITION 4.4. *If $x \in \mathcal{S} \setminus \{0\}$, then for any $\rho > 0$*

$$V^\delta(\rho x) = \log \rho + V^\delta(x).$$

Proof. Let $(\mathcal{L}, \mathcal{M})$ be any admissible control for the initial position x. The linearity of equations (2.1–2.2) implies that $(\rho\mathcal{L}, \rho\mathcal{M})$ is an admissible control for ρx and $W_{\rho x,\rho\mathcal{L},\rho\mathcal{M}}(t) = \rho W_{x,\mathcal{L},\mathcal{M}}(t)$ for all $t \geq 0$. Therefore, by Lemma 4.1,

$$V^\delta(\rho x) \geq J^\delta_{\rho x}(\rho\mathcal{L}, \rho\mathcal{M}) = \log \rho + J^\delta_x(\mathcal{L}, \mathcal{M}).$$

Taking the supremum over $(\mathcal{L}, \mathcal{M})$, we get $V^\delta(\rho x) \geq \log \rho + V^\delta(x)$. Similarly, $V^\delta(x) \geq \log(\rho^{-1}) + V^\delta(\rho x)$, and the statement of the proposition follows. □

5. REDUCTION IN DIMENSION

Let $\Delta = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n, y_i \geq 0, i = 1, \ldots, n, \sum_{i=1}^n y_i \leq 1\}$. For $y \in \Delta$, set

$$\mathcal{V}^\delta(y) = V^\delta\left(1 - \sum_{i=1}^n y_i, \frac{y_1}{1-\mu_1}, \ldots, \frac{y_n}{1-\mu_n}\right). \tag{5.1}$$

The function $\mathcal{V}^\delta$ is equal, up to a change of coordinates, to the restriction of V^δ to the subset of $\mathcal{S} : \{x \in \mathcal{S}, \rho(x) = 1\}$.

PROPOSITION 5.1. *The function $\mathcal{V}^\delta$ is concave, nondecreasing with respect to each of its arguments, and bounded Lipschitz continuous in Δ. Moreover, there exists a norm $\|\cdot\|$ in $\mathbb{R}^n$ independent of δ such that*

$$|\mathcal{V}^\delta(y') - \mathcal{V}^\delta(y)| \leq \|y' - y\| \qquad \forall y, y' \in \Delta.$$

Proof. (i) $\mathcal{V}^\delta$ *is concave*. The concavity of $\mathcal{V}^\delta$ is a consequence of the convexity of Δ, the concavity of V^δ, and the affine property of

$$y \mapsto x = \left(1 - \sum_{i=1}^n y_i, \frac{y_1}{1-\mu_1}, \ldots, \frac{y_n}{1-\mu_n}\right). \tag{5.2}$$

(ii) $\mathcal{V}^\delta$ *is nondecreasing with respect to each of its arguments* y_i, $i = 1, \ldots, n$. Let $x \in \mathcal{S}$ and $m = (m_1, \ldots, m_n)$ be such that $x_i - m_i \geq 0$ for $i = 1, \ldots, n$. An initial sale of stock m brings the investor from the position x to the position $x' = (x_0 + \sum_{i=1}^n (1-\mu_i)m_i, x_1 - m_1, \ldots, x_n - m_n)$ without changing the total wealth: $\rho(x') = \rho(x)$. For any admissible policy $(\mathcal{L}', \mathcal{M}')$ for the initial position x', the policy $(\mathcal{L}, \mathcal{M})$ defined for $t \geq 0$ by $\mathcal{L}(t) = \mathcal{L}'(t)$, $\mathcal{M}(t) = \mathcal{M}'(t) + m$, is such that $S_{x,\mathcal{L},\mathcal{M}}(t) = S_{x',\mathcal{L}',\mathcal{M}'}(t)$ for $t \geq 0$. Then, $(\mathcal{L}, \mathcal{M})$ is admissible for x and

$$V^\delta(x) \geq J_x^\delta(\mathcal{L}, \mathcal{M}) = J_{x'}^\delta(\mathcal{L}', \mathcal{M}').$$

Taking the supremum over $(\mathcal{L}', \mathcal{M}')$, we get

$$V^\delta(x) \geq V^\delta(x'). \tag{5.3}$$

Applying (5.3) to the position x given by (5.2), we obtain

$$\begin{aligned} \mathcal{V}^\delta(y) &\geq V^\delta\left(1 - \sum_{i=1}^n (y_i - (1-\mu_i)m_i), \frac{y_1}{1-\mu_1} - m_1, \ldots, \frac{y_n}{1-\mu_n} - m_n\right) \\ &= \mathcal{V}^\delta(y - q), \end{aligned} \tag{5.4}$$

where $q_i = (1-\mu_i)m_i$. Since $\mu_i < 1$ and m_i is any positive real such that $y_i - q_i \geq 0$, the nondecreasing property of $\mathcal{V}^\delta$ follows.

(iii) $\mathcal{V}^\delta$ *is bounded in* Δ. In view of (4.4) and (5.1), we see that $\mathcal{V}^\delta$ is upper bounded by κ/δ in Δ. Since $\mathcal{V}^\delta$ is nondecreasing, $\mathcal{V}^\delta(y) \geq \mathcal{V}^\delta(0)$ for all y in Δ and the function $\mathcal{V}^\delta$ is lower bounded iff $\mathcal{V}^\delta(0) = V^\delta(1, 0, \ldots, 0) > -\infty$. Consider the special policy $(\mathcal{L}, \mathcal{M})$ consisting of no transaction. Starting from the initial position $x = (1, 0, \ldots, 0)$

and applying this policy, the position at time t is $S_0(t) = e^{rt}$ and $S_i(t) = 0$. This policy is thus admissible for x and using (4.1, 4.2), we obtain

$$\mathcal{V}^\delta(0) \geq J_x^\delta(\mathcal{L}, \mathcal{M}) = E\int_0^{+\infty} e^{-\delta t} H(0)\,dt = \frac{r}{\delta} > -\infty.$$

Therefore, $\mathcal{V}^\delta$ is bounded on Δ:

$$\frac{r}{\delta} \leq \mathcal{V}^\delta(y) \leq \frac{\kappa}{\delta} \qquad \forall y \in \Delta. \tag{5.5}$$

(iv) $\mathcal{V}^\delta$ *is Lipschitz continuous*. Since $\mathcal{V}^\delta$ is concave and bounded (thus finite), $\mathcal{V}^\delta$ is locally Lipschitz continuous in the interior of Δ. The continuity at the boundary may also be proved by using the concavity of $\mathcal{V}^\delta$ and the properties of Δ, but we shall prove the Lipschitz continuity of $\mathcal{V}^\delta$ in Δ directly. Proceeding as in point (ii), we consider an initial purchase $l = (l_1, \ldots, l_n)$ of stock. We get

$$V^\delta(x) \geq V^\delta(x_0 - \sum_{i=1}^n (1+\lambda_i)l_i, x_1 + l_1, \ldots, x_n + l_n) \tag{5.6}$$

for all x in $\mathcal{S}$ and $l_i \geq 0$ such that $x_0 - \sum_{i=1}^n (1+\lambda_i)l_i \geq 0$. Combining (5.4) and (5.6), we obtain

$$\begin{aligned}\mathcal{V}^\delta(y) &\geq V^\delta\Bigg(1 - \sum_{i=1}^n \big(y_i + (1+\lambda_i)l_i - (1-\mu_i)m_i\big),\\ &\qquad \frac{y_1}{1-\mu_1} + l_1 - m_1, \ldots, \frac{y_n}{1-\mu_n} + l_n - m_n\Bigg)\\ &\geq V^\delta\Bigg(1 - \sum_{i=1}^n \big(y_i + (1+\nu_i)p_i - q_i\big), \frac{y_1 + p_1 - q_1}{1-\mu_1}, \ldots, \frac{y_n + p_n - q_n}{1-\mu_n}\Bigg)\end{aligned} \tag{5.7}$$

with

$$\nu_i = \frac{\lambda_i + \mu_i}{1 - \mu_i}, \tag{5.8}$$

$p_i = (1-\mu_i)l_i$, $q_i = (1-\mu_i)m_i$, such that[1]

$$p_i, q_i \geq 0, \quad y_i + p_i - q_i \geq 0, \quad \sum_{i=1}^n (y_i + (1+\nu_i)p_i - q_i) \leq 1. \tag{5.9}$$

Taking into account (5.7) and Proposition 4.4, we get

$$\mathcal{V}^\delta(y) \geq \mathcal{V}^\delta(y') + \log\rho,$$

with

$$\begin{aligned}\rho &= 1 - \nu \cdot p > 0,\\ y_i' &= \frac{y_i + p_i - q_i}{\rho}, \quad i = 1, \ldots, n,\end{aligned} \tag{5.10}$$

[1] Inequality (5.7) is first proved when $p_i q_i = 0$, if we do not purchase and sell the same stock. Then, from the nondecreasing property of V^δ, it also holds for all p_i, q_i satisfying (5.9).

where $\nu \cdot p$ denotes the scalar product of $\nu = (\nu_1, \dots, \nu_n)$ and $p = (p_1, \dots, p_n)$ in $\mathbb{R}^n$. Thus,

$$\mathcal{V}^\delta(y') - \mathcal{V}^\delta(y) \le -\log(1 - \nu \cdot p).$$

In order to finish the proof of the proposition, it is sufficient to prove that for all y, $y' \in \Delta$, there exist $p, q \ge 0$ satisfying (5.10) and such that $\nu \cdot p$ tends to 0 when $y' - y$ goes to 0. Equation (5.10) is equivalent to

$$(y - y')^+ - (y - y')^- = q - (p + \nu \cdot p \;\; y'),$$

with $z^\pm = (z_1^\pm, \dots, z_n^\pm)$ for $z \in \mathbb{R}^n$ and $z^+ = \max(z, 0), z^- = \max(-z, 0)$ for $z \in \mathbb{R}$. Hence, (5.10) holds for $p = (y - y')^-$ and $q = (y - y')^+ + \nu \cdot (y - y')^- \; y'$. For all $y, y' \in \Delta$ such that $\sum_{i=1}^n \nu_i |y_i - y_i'| < 1$, we have

$$\mathcal{V}^\delta(y') - \mathcal{V}^\delta(y) \le -\log\left(1 - \sum_{i=1}^n \nu_i (y_i - y_i')^-\right) \le -\log\left(1 - \sum_{i=1}^n \nu_i |y_i - y_i'|\right).$$

By symmetry, we get

$$|\mathcal{V}^\delta(y) - \mathcal{V}^\delta(y')| \le f(y - y') \qquad \forall y, y' \in \Delta,$$

with $f(z) = -\log((1 - \sum_{i=1}^n \nu_i |z_i|)^+)$. This property implies the continuity of $\mathcal{V}^\delta$ up to the boundary. Consider the following norm of $\mathbb{R}^n$ independent of δ: $\|z\| = 2\sum_{i=1}^n \nu_i |z_i|$. We have $f(z) \le \|z\|$ when $\|z\| \le 1$. Moreover,

$$|\mathcal{V}^\delta(y) - \mathcal{V}^\delta(y')| \le 2C\|y - y'\|,$$

for $\|y - y'\| \ge 1$ if C is a bound of $\mathcal{V}^\delta$. Hence, $\mathcal{V}^\delta$ is Lipschitz continuous in Δ. However, a bound of $\mathcal{V}^\delta$ depends on δ. Since Δ is compact, it can be covered by a finite number N of open balls of diameter 1 for $\|\cdot\|$. Then $|\mathcal{V}^\delta(y) - \mathcal{V}^\delta(y')| \le 1$ for all y and y' in the same ball and $|\mathcal{V}^\delta(y) - \mathcal{V}^\delta(y')| \le N$ for all $y, y' \in \Delta$. Therefore, $\mathcal{V}^\delta$ is Lipschitz continuous with constant N for the norm $\|\cdot\|$. Moreover, since $\|\cdot\|$ and Δ are independent of δ, so is N. Then, the norm $N\|\cdot\|$ satisfies the statement of the proposition. □

Consider the change of variables:

$$\begin{cases} \rho = \rho(x) = x_0 + \sum_{i=1}^n (1 - \mu_i) x_i \\ y = Y(x) = (Y_1(x), \dots, Y_n(x)) \quad \text{with } Y_i(x) = \frac{(1-\mu_i)x_i}{\rho}. \end{cases} \tag{5.11}$$

Using Proposition 4.4 and equation (5.1), we get

$$V^\delta(x) = \log \rho + \mathcal{V}^\delta(y). \tag{5.12}$$

COROLLARY 5.2. *The function V^δ is finite and continuous in $\mathcal{S} \setminus \{0\}$ and tends to $-\infty$ at* 0.

Proof. This follows from the fact that the function $\mathcal{V}^\delta$ is bounded and continuous and the fact that $\log \rho(x)$ tends to $-\infty$ when x goes to 0. □

6. VARIATIONAL INEQUALITIES FOR THE DISCOUNTED PROBLEM

6.1. The n-Dimensional Controlled Process

The change of variables (5.11) transforms the state variable $S(t)$ with initial condition $S(0^-) = x$ into the process $(W(t), s(t))$ with initial condition $W(0^-) = \rho$ and $s(0^-) = y$. From (4.1, 4.2, 5.12), we see that $\mathcal{V}^\delta$ only depends on the process $s(t)$. Consequently, we should be able to obtain the variational inequality satisfied by $\mathcal{V}^\delta$ by using the dynamic equation of the process $s(t)$. For $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, set

$$\sigma_{ij}(y) = y_i(\delta_{ij} - y_j), \tag{6.1}$$

$$\beta_i(y) = y_i \sum_{k=1}^{n} \left(-\sum_{l=1}^{n} a_{kl} y_l + \alpha_k - r \right)(\delta_{ki} - y_k), \tag{6.2}$$

where δ_{ik} is the Kronecker index, that is equal to 1 when $i = k$ and to 0 otherwise. Formally, $s(t)$ satisfies

$$\begin{aligned} ds_i(t) &= \beta_i(s(t))\,dt + \sum_{j=1}^{n} \sigma_{ij}(s(t))\,dw_j(t) + d\mathcal{P}_i(t) - d\mathcal{Q}_i(t) \\ &\quad + s_i(t) \sum_{j=1}^{n} \nu_j d\mathcal{P}_j(t), \qquad i = 1, \ldots, n; \quad s(0^-) = y, \end{aligned} \tag{6.3}$$

where $\mathcal{P}(t) = (\mathcal{P}_1(t), \ldots, \mathcal{P}_n(t))$ and $\mathcal{Q}(t) = (\mathcal{Q}_1(t), \ldots, \mathcal{Q}_n(t))$ are càdlàg processes such that $\mathcal{P}(0^-) = \mathcal{Q}(0^-) = 0$ and

$$d\mathcal{P}_i(t) = (1 - \mu_i) W(t)^{-1} \circ d\mathcal{L}_i(t), \qquad d\mathcal{Q}_i(t) = (1 - \mu_i) W(t)^{-1} \circ d\mathcal{M}_i(t),$$

and ν_j is defined by (5.8). Equations (4.1, 4.2, 5.12) imply that the function $\mathcal{V}^\delta$ satisfies

$$\mathcal{V}^\delta(y) = \sup_{(\mathcal{P}, \mathcal{Q})} \mathcal{J}_y^\delta(\mathcal{P}, \mathcal{Q}), \quad \text{with}$$

$$\mathcal{J}_y^\delta(\mathcal{P}, \mathcal{Q}) = E\left[\int_0^{+\infty} e^{-\delta t} H(s(t))\,dt - \sum_{i=1}^{n} \nu_i \int_0^{+\infty} e^{-\delta t} d\mathcal{P}_i(t) \right],$$

where the supremum is taken over all càdlàg processes $(\mathcal{P}, \mathcal{Q})$ such that $s(t)$ remains in Δ for all $t \geq 0$. Similarly, using (3.5), the ergodic problem (2.6) is reduced to

$$\sup_{(\mathcal{P}, \mathcal{Q})} \mathcal{J}_y(\mathcal{P}, \mathcal{Q}), \quad \text{with}$$

$$\mathcal{J}_y(\mathcal{P}, \mathcal{Q}) = \liminf_{T \to \infty} T^{-1} E\left[\int_0^{T} H(s(t))\,dt - \sum_{i=1}^{n} \nu_i \mathcal{P}_i(T) \right].$$

However, in (6.3), the product $s_i(t) d\mathcal{P}_j(t)$ is ill-posed when $\mathcal{P}_j$ is discontinuous. In order to define in a rigorous way the product $s_i(t) d\mathcal{P}_j(t)$, one needs to reintroduce the process $W(t)$ which is precisely the quantity that we want to get rid of. For this reason, we first state the variational inequality satisfied by V^δ (Theorem 6.4). We then obtain the VI satisfied by $\mathcal{V}^\delta$ by performing the change of variables and reducing the dimension of the problem directly on the associated VI (see Corollary 6.6).

6.2. Viscosity Solutions

Let us first recall the definition of viscosity solutions. Consider a nonlinear elliptic second-order partial differential equation of the form

$$F(D^2 v(x), Dv(x), v(x), x) = 0 \quad \text{in } \mathcal{D}, \tag{6.4}$$

where F is a given function in $S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathcal{D}$ (not necessarily continuous), S^N is the space of symmetric $N \times N$ matrices, $\mathcal{D}$ is a subset of $\mathbb{R}^N$, and F satisfies the ellipticity condition:

$$F(A, p, v, x) \geq F(B, p, v, x) \tag{6.5}$$
$$\text{if } A \geq B,\ A, B \in S^N,\ p \in \mathbb{R}^N,\ v \in \mathbb{R},\ x \in \mathcal{D}.$$

Let C be any subset of $\mathbb{R}^N$. The upper semi-continuous and the lower semi-continuous envelopes of a function $z : C \to \mathbb{R}$ are respectively defined as $z^*(x) = \limsup_{\substack{y\to x\\ y\in C}} z(y)$, and $z_*(x) = \liminf_{\substack{y\to x\\ y\in C}} z(y)$. Let $USC(C)$ (respectively, $LSC(C)$) denote the set of upper (respectively, lower) semicontinuous functions $v : C \to \mathbb{R}$, that is the functions v such that $v = v^*$ (respectively, $v = v_*$). Following Ishii (1985), Barles and Souganidis (1991), and Barles (1994), we define the notion of viscosity solutions as follows.

DEFINITION 6.1. Let $\mathcal{D}$ be a locally compact subset of $\mathbb{R}^N$. A function $v \in USC(\mathcal{D})$ is a viscosity subsolution of (6.4) if for all $\mathcal{C}^2$-function ϕ in a neighborhood of $\mathcal{D}$, if $x \in \mathcal{D}$ is a local maximum point of $v - \phi$, one has

$$F^*(D^2\phi(x), D\phi(x), v(x), x) \geq 0.$$

A function $v \in LSC(\mathcal{D})$ is a viscosity supersolution of (6.4) if for all $\mathcal{C}^2$-function ϕ in a neighborhood of $\mathcal{D}$, if $x \in \mathcal{D}$ is a local minimum point of $v - \phi$, one has

$$F_*(D^2\phi(x), D\phi(x), v(x), x) \leq 0. \tag{6.6}$$

A continuous function v on $\mathcal{D}$ is a viscosity solution of (6.4) if it is both a sub- and a supersolution of (6.4).

An equivalent definition of viscosity solutions which is useful for proving uniqueness results is the following (see Crandall, Ishii, and Lions 1992, Sec. 2):

DEFINITION 6.2. A function $v \in USC(\mathcal{D})$ is a viscosity subsolution of (6.4) if

$$F^*(X, p, v(x), x) \geq 0 \qquad \forall (p, X) \in \overline{J}_{\mathcal{D}}^{2,+} v(x), \forall x \in \mathcal{D}.$$

A function $v \in LSC(\mathcal{D})$ is a viscosity supersolution of (6.4) if

$$F_*(X, p, v(x), x) \leq 0 \qquad \forall (p, X) \in \overline{J}_{\mathcal{D}}^{2,-} v(x), \forall x \in \mathcal{D}.$$

The second-order "superjets" and their "closures" on a subset $\mathcal{D}$ of $\mathbb{R}^n$ are defined by

$$J_{\mathcal{D}}^{2,+} v(x) = \left\{ (p, X) \in \mathbb{R}^N \times S^N,\ \limsup_{\substack{y\to x\\ y\in\mathcal{D}}} \left[v(y) - v(x) - p\cdot(y-x) - \frac{1}{2} X(y-x)\cdot(y-x) \right] |y-x|^{-2} \leq 0 \right\},$$

$$\overline{J}_{\mathcal{D}}^{2,+} v(x) = \big\{ (p, X) \in \mathbb{R}^N \times S^N,\ \exists (x_n, p_n, X_n) \in \mathcal{D} \times \mathbb{R}^N \times S^N,\ (p_n, X_n) \in J_{\mathcal{D}}^{2,+} v(x_n) \text{ and } (x_n, v(x_n), p_n, X_n) \to (x, v(x), p, X) \text{ when } n \to \infty \big\},$$

and $J_{\mathcal{D}}^{2,-} v = -J_{\mathcal{D}}^{2,+}(-v)$, $\overline{J}_{\mathcal{D}}^{2,-} v = -\overline{J}_{\mathcal{D}}^{2,+}(-v)$.

REMARK 6.3. In Barles and Souganidis (1991) and Barles (1994), the set $\mathcal{D}$ is supposed to be either an open set or the closure of an open set. Since the notion of viscosity solution is a "local" property, conditions on $\mathcal{D}$ need only to be local. Here, we assume that $\mathcal{D}$ is locally compact so that uniqueness and stability properties of viscosity solutions can be proved for "good" functions F (see Crandall et al. 1992). The set $\mathcal{S}\backslash\{0\}$ is locally compact. Indeed, a function is a viscosity (sub-, super-) solution of (6.4) in $\mathcal{S}\backslash\{0\}$ if and only if it is a viscosity (sub-, super-) solution of (6.4) in any subset of $\mathcal{S}\backslash\{0\}$ of the form $\{x \in \mathcal{S}, \varepsilon \leq \rho(x)\}$, with $\varepsilon > 0$.

6.3. The Variational Inequalities for V^δ and $\mathcal{V}^\delta$

THEOREM 6.4. *Set $\overline{V}^\delta(x) = V^\delta(x) - \log\rho(x) = \mathcal{V}^\delta(Y(x))$ for $x \in \mathcal{S}\backslash\{0\}$. The function $\overline{V}^\delta$ is the unique bounded continuous function in $\mathcal{S}\backslash\{0\}$ which is a viscosity solution of the variational inequality*

$$F(D^2V(x), DV(x), \delta V(x), x) = 0 \quad \text{in } \mathcal{S}\backslash\{0\}, \tag{6.7}$$

where

$$F(D^2V, DV, v, x) = \max\bigl\{AV - v + H(Y(x)), F_0(DV, x)\bigr\}, \tag{6.8}$$

$$AV = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}x_ix_j\frac{\partial^2 V}{\partial x_i\partial x_j} + \sum_{i=1}^{n}\alpha_i x_i\frac{\partial V}{\partial x_i} + rx_0\frac{\partial V}{\partial x_0},$$

$$\begin{aligned} F_0(DV, x) &= \sup_{(l,m)}\sum_{i=1}^{n}\biggl[\bigl((1-\mu_i)m_i - (1+\lambda_i)l_i\bigr)\frac{\partial V}{\partial x_0} \\ &\qquad + (l_i - m_i)\frac{\partial V}{\partial x_i} - l_i\frac{\lambda_i + \mu_i}{\rho(x)}\biggr] \\ &= \sup_{(l,m)}\sum_{i=1}^{n}\biggl[l_i\biggl(L_iV - \frac{\lambda_i+\mu_i}{\rho(x)}\biggr) + m_iM_iV\biggr], \end{aligned} \tag{6.9}$$

$$L_iV = -(1+\lambda_i)\frac{\partial V}{\partial x_0} + \frac{\partial V}{\partial x_i},$$

$$M_iV = (1-\mu_i)\frac{\partial V}{\partial x_0} - \frac{\partial V}{\partial x_i},$$

and the supremum in (6.9) *is taken over all $l = (l_1, \ldots, l_n)$ and $m = (m_1, \ldots, m_n)$ such that $l_i, m_i \geq 0$, $\sum_{i=1}^{n}(l_i + m_i) = 1$, $l_i - m_i \geq 0$ if $x_i = 0$, and $\sum_{i=1}^{n}((1-\mu_i)m_i - (1+\lambda_i)l_i) \geq 0$ if $x_0 = 0$.*

The proof of Theorem 6.4 is postponed to the end of the section.

REMARK 6.5. Equation (6.7) is satisfied up to the boundary of $\mathcal{S}\backslash\{0\}$ since the process $S(t)$ is allowed to go everywhere in $\mathcal{S}\backslash\{0\}$, and the optimal performance V^δ depends on the entire trajectory of S. At any time, all the controls are allowed, except those that bring the investor's position outside $\mathcal{S}\backslash\{0\}$. This restriction on the controls is expressed in the optimization variables (l, m) defining F_0. The quantities l_i and m_i represent the proportions of purchase and sale of stock i with respect to the total amount of transactions (at each time t for the position $S(t^-)$). The functional F satisfies $F_* = F$,

$$F^*(D^2V, DV, v, x) = \max\left\{AV - v + H(Y(x)), \max_{1\le i\le n}\left(L_iV - \frac{\lambda_i + \mu_i}{\rho(x)}\right), \max_{1\le i\le n} M_iV\right\},$$

and $F = F_* = F^*$ in the interior of $\mathcal{S}$ (since all the controls are allowed). Since $F_* \geq \tilde{F} = \tilde{F}_*$ with

$$\tilde{F}(D^2V, DV, v, x) = \max\left\{AV-v+H(Y(x)), \max_{1\le i\le n_{x_0\neq 0}} \left(L_iV-\frac{\lambda_i + \mu_i}{\rho(x)}\right), \max_{1\le i\le n_{x_i\neq 0}} M_iV\right\},$$

and $\tilde{F}^* = F^*$, a viscosity solution of (6.7) is necessarily a viscosity solution of

$$\tilde{F}(D^2V(x), DV(x), \delta V(x), x) = 0 \quad \text{in } \mathcal{S}\backslash\{0\}. \tag{6.10}$$

In addition, a viscosity solution of (6.10) is necessarily a solution of $F^*(D^2V(x), DV(x), \delta V(x), x) \geq 0$ in $\mathcal{S}\backslash\{0\}$, and a solution of $F^*(D^2V(x), DV(x), \delta V(x), x) \leq 0$ in the interior int $(\mathcal{S})$ of $\mathcal{S}$. These two conditions define the notion of "constrained" viscosity solution of $F^*(D^2V(x), DV(x), \delta V(x), x) = 0$ in $\mathcal{S}\backslash\{0\}$, as introduced in Soner (1986a, 1986b) (see also Crandall et al. 1992 and Tourain and Zariphopoulou 1997). Conversely, a constrained viscosity solution of the previous equation is necessarily a viscosity solution of (6.7). Indeed, consider a solution of $F^*(D^2V(x), DV(x), \delta V(x), x) \leq 0$ in int $(\mathcal{S})$. Adding $\varepsilon \sum_{i\in I} \frac{1}{x_i}$ to V, with $I \subset \{0, \ldots, n\}$, and passing to the limit when ε goes to 0, one obtains (6.6) for any $x \in \partial\mathcal{S}$ with $I = \{i \in \{0, \ldots, n\}, x_i = 0\}$. Hence, the three formulations are equivalent. We choose formulation (6.7) for its control interpretation.

COROLLARY 6.6. *The function $\mathcal{V}^\delta$ is the unique viscosity solution of the variational inequality*

$$\mathcal{F}(D^2V(y), DV(y), \delta V(y), y) = 0 \quad \text{in } \Delta, \tag{6.11}$$

where

$$\mathcal{F}(D^2V, DV, v, y) = \max\big(BV - v + H(y), \mathcal{F}_0(DV, y)\big), \tag{6.12}$$

$$b(y) = \sigma(y)a\sigma(y)^T,$$

$$BV = \frac{1}{2}\sum_{i,j=1}^n b_{ij}(y)\frac{\partial^2 V}{\partial y_i \partial y_j} + \sum_{i=1}^n \beta_i(y)\frac{\partial V}{\partial y_i}, \tag{6.13}$$

$$\mathcal{F}_0(DV, y) = \sup_{(p,q)} \sum_{i=1}^n (p_i(P_iV - \nu_i) + q_iQ_iV), \tag{6.14}$$

$$P_iV = \nu_i \sum_{j=1}^n y_j \frac{\partial V}{\partial y_j} + \frac{\partial V}{\partial y_i}, \tag{6.15}$$

$$Q_iV = -\frac{\partial V}{\partial y_i}, \tag{6.16}$$

σ and β are defined in (6.1, 6.2), and the supremum in (6.14) is taken over all $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$ such that $p_i, q_i \geq 0$, $\sum_{i=1}^n (p_i + q_i) = 1$, $p_i - q_i \geq 0$ if $y_i = 0$, and $\sum_{i=1}^n (q_i - (1 + \nu_i)p_i) \geq 0$ if $\sum_{i=1}^n y_i = 1$.

Proof. Since $\phi : x \in \mathcal{S}\backslash\{0\} \to (\rho(x), Y(x)) \in (0, +\infty) \times \Delta$ is a $\mathcal{C}^\infty$ diffeomorphism, Theorem 6.4 implies that the function $\overline{V}^\delta \circ \phi^{-1}$ is the unique bounded continuous viscosity solution of $F(D^2(V \circ \phi)(\phi^{-1}(\rho, y)), D(V \circ \phi)(\phi^{-1}(\rho, y)), \delta V(\rho, y), \phi^{-1}(\rho, y)) = 0$ in $(0, +\infty) \times \Delta$. By Proposition 4.4

$$\overline{V}^\delta \circ \phi^{-1}(\rho, y) = \overline{V}^\delta \circ \phi^{-1}(1, y) = \mathcal{V}^\delta(y) \quad \forall(\rho, y) \in (0, +\infty) \times \Delta;$$

thus $\mathcal{V}^\delta$ is the unique bounded continuous viscosity solution of (6.11). Since Δ is compact, any continuous function on Δ is bounded, so the boundness condition on $\mathcal{V}^\delta$ may be removed. □

Proof of Theorem 6.4. The fact that $\overline{V}^\delta$ is a viscosity solution of (6.7) follows from the weak dynamic programming principle, which can be proved by using the uniform continuity of $\overline{V}^\delta$:

$$(6.17) \qquad \overline{V}^\delta(x) = \sup_{(\mathcal{L},\mathcal{M})} E\bigg[\int_0^\theta e^{-\delta t} H(s(t))\, dt - \sum_{i=1}^n (\lambda_i + \mu_i) \times \int_{0^-}^{\theta^-} e^{-\delta t} W(t)^{-1} \circ d\mathcal{L}_i(t) + e^{-\delta\theta}\overline{V}^\delta(S(\theta^-))\bigg]$$

for any stopping time θ. The proof is standard and we omit it here. We refer to Lions (1983a, 1983b) where the proof is given in the case of regular stochastic control problems, and to Fleming and Soner (1993, Chap. VIII), Zariphopoulou (1994), and Tourain and Zariphopoulou (1997) where the proof is given for the same type of singular stochastic control problem for $n = 1$. The uniqueness results from the following comparison result which proof is obtained by adapting the Ishii technique (e.g., see Ishii and Lions 1990 and Crandall et al. 1992) as in Zhu (1991, App. A), Zariphopoulou (1992), and Akian et al. (1996), and by using some ideas of Barles and Perthame (1990) and Barles (1994). □

THEOREM 6.7. *If V is an upper bounded viscosity subsolution and V' is a lower bounded viscosity supersolution of* (6.7), *then $V \le V'$ in* int $(\mathcal{S})$. *Moreover, if*

$$(6.18) \qquad V'(x) = \liminf_{\substack{y\to x \\ y\in \mathrm{int}(\mathcal{S})}} V'(y) \qquad \forall x \in \partial\mathcal{S}\backslash\{0\},$$

then $V \le V'$ in $\mathcal{S}\backslash\{0\}$.

Proof. First, we redefine V' in $\partial\mathcal{S}\backslash\{0\}$ by (6.18). A priori V' is only a supersolution of (6.7) in int $(\mathcal{S})$, but using Remark 6.5, it is also a supersolution in $\mathcal{S}\backslash\{0\}$. The result of the theorem reduces to $V \le V'$ in $\mathcal{S}\backslash\{0\}$ for this new function V'.

Second, we construct a perturbation of V', which is a strict supersolution of (6.7) and tends to $+\infty$ when x goes to infinity or 0. Consider the function $f(x) = (c \cdot x)^k/k - \log \rho(x)$ for $k > 0$ and $c = (c_0, \ldots, c_n)$ with $c_0 = 1, c_i = 1 + (\lambda_i - \mu_i)/2,\ i = 1, \ldots, n.$

By assumption (see Sec. 2), $c_i > 1 - \mu_i > 0$, $i = 1, \dots, n$, then $0 < \rho(x) \le c \cdot x$ for all $x \in \mathcal{S} \setminus \{0\}$. The function f is $\mathcal{C}^\infty$ in $\mathcal{S} \setminus \{0\}$ and satisfies:

$$L_i f - \frac{\lambda_i + \mu_i}{\rho(x)} = M_i f = -(c \cdot x)^{k-1} \frac{(\lambda_i + \mu_i)}{2} < 0,$$

$$Af + H(Y(x)) = (c \cdot x)^k \left[\frac{k-1}{2} \sum_{i,j=1}^n a_{ij} y_i y_j + \sum_{i,j=1}^n (\alpha_i - r) y_i + r \right],$$

with $y_i = c_i x_i / (c \cdot x)$, $i = 1, \dots, n$. If $k \le \frac{1}{2}$, then $Af + H(Y(x)) \le K(c \cdot x)^k$ for some constant K independent of k. Moreover,

$$f(x) = \frac{(c \cdot x)^k}{k} \left(1 - \frac{k \log \rho(x)}{(c \cdot x)^k} \right) \ge \frac{(c \cdot x)^k}{k} \left(1 - \frac{k \log (c \cdot x)}{(c \cdot x)^k} \right) \ge \frac{(c \cdot x)^k}{k} \left(1 - \frac{1}{e} \right) \ge 0.$$

Hence, $Af + H(Y(x)) \le kK' f$ for some constant K' and

$$Af - \delta f + H(Y(x)) \le (kK' - \delta) f < 0$$

for k small enough. For such k, f is a strict supersolution of (6.7) in any open subset G of $\mathcal{S} \setminus \{0\}$ (open with respect to $\mathcal{S} \setminus \{0\}$) with compact closure; that is, f is a supersolution of

$$F(D^2 V(x), DV(x), \delta V(x), x) = -\eta < 0 \quad \text{in } G.$$

Since F_* is convex with respect to $(D^2 V, DV, v)$, for any supersolution v of (6.7), and $\varepsilon \in (0, 1]$, $(1 - \varepsilon) v + \varepsilon f$ is a strict supersolution of (6.7) in G.

Let us finally prove the theorem by contradiction. Suppose that the supremum of $V - V'$ in $\mathcal{S} \setminus \{0\}$ is positive. Choose $\varepsilon > 0$ such that the supremum of $w = V - V''$ is positive, where $V'' = (1 - \varepsilon) V' + \varepsilon f$. Since w is upper semicontinuous and tends to $-\infty$ when x goes to 0 or to infinity, the set $\arg\max w(x)$ is nonempty and compact in $\mathcal{S} \setminus \{0\}$. It is included in some open subset G of $\mathcal{S} \setminus \{0\}$, with compact closure. Hence, in order to get a contradiction, it is sufficient to prove $V \le V''$ in G. We are then reduced to prove a comparison result for a strict supersolution V'' and a subsolution V of (6.7) in an open subset G of $\mathcal{S} \setminus \{0\}$ with compact closure $\overline{G}$, assuming that the supremum of $V - V''$ is attained in G only. This is proved by using Ishii's technique adapted as in Barles and Perthame (1990) or Barles (1994, Thm. 4.6) for the boundary conditions. Note that F is continuous in the interior of $\mathcal{S}$. Then one can prove that, when it is positive, the maximum of w is attained at the boundary of $\mathcal{S}$, using the standard Ishii technique, that is considering the maximum points of $w_k(x, y) = V(x) - V''(y) - \frac{k}{2} |x - y|^2$. We omit this proof here since it is included in the proof below concerning boundary points. Suppose that $\hat{x} \in \arg\max w(x) \cap \partial \mathcal{S}$ and denote $m = w(\hat{x})$ and $I = \{i \in \{0, \dots, n\}, \hat{x}_i = 0\}$. From (6.18), there exists a sequence (z^k) in $\operatorname{int}(\mathcal{S}) \cap \operatorname{int}(G)$ converging to $\hat{x}$, such that $V''(z^k)$ tends to $V''(\hat{x})$ when k goes to infinity. Let $\varepsilon_k = |z^k - \hat{x}|$, where $|\cdot|$ denotes the Euclidian norm, and set

$$w_k(x, y) = V(x) - V''(y) - \varphi(x, y) \quad \text{with}$$

$$\varphi(x, y) = \frac{|x - y|^2}{2\varepsilon_k} + \frac{1}{4} \sum_{i \in I} \left(\frac{y_i - x_i}{z_i^k} - 1 \right)^4 + \frac{1}{4} |x - \hat{x}|^4.$$

The function w_k is upper semicontinuous in $\mathcal{S} \setminus \{0\} \times \mathcal{S} \setminus \{0\}$ and $\overline{G}$ is compact. Thus, there exists $(x^k, y^k) \in \overline{G} \times \overline{G}$ such that $w_k(x^k, y^k) = m_k = \sup_{(x,y) \in \overline{G} \times \overline{G}} w_k(x, y)$.

Moreover, there exists a subsequence of (x^k, y^k), also denoted (x^k, y^k), converging to $(x, y) \in \overline{G} \times \overline{G}$. Since

$$m_k \geq w_k(\hat{x}, z^k) = V(\hat{x}) - V''(z^k) - \frac{\varepsilon_k}{2} \underset{k\to\infty}{\to} V(\hat{x}) - V''(\hat{x}) = m,$$

and V and $-V''$ are upperbounded, $\frac{|x^k - y^k|^2}{\varepsilon_k}$ is bounded and $x = y$. Moreover,

$$0 \leq \limsup_{k\to\infty} \varphi(x^k, y^k) = \limsup_{k\to\infty} (V(x^k) - V''(y^k) - m_k) \leq V(x) - V''(x) - m \leq 0;$$

thus, $x = \hat{x}$, $\frac{|x^k - y^k|^2}{\varepsilon_k} \to 0$ and $\frac{y_i^k - x_i^k}{z_i^k} \to 1$ for all $i \in I$, when k goes to infinity. In particular, $y_i^k > x_i^k \geq 0$ for $i \in I$ and k large enough. Since when $j \notin I$, y_j^k tends to $\hat{x}_j \neq 0$ as k goes to infinity, $y^k \in \operatorname{int}(\mathcal{S})$ for k large enough. Moreover, since $\hat{x} \in G$ and G is open with respect to $\mathcal{S}$, $x^k, y^k \in G$ for k large enough. Applying Crandall et al. (1992, Thm. 3.2) we obtain that, for any sequence (ε'_k) of positive numbers, there exist $X^k, Y^k \in S^{n+1}$ such that $(p^k, X^k) \in \overline{J}_G^{2,+} V(x^k)$, $(q^k, Y^k) \in \overline{J}_G^{2,-} V''(y^k)$, and

$$\begin{pmatrix} X^k & 0 \\ 0 & -Y^k \end{pmatrix} \leq D^2\varphi(x^k, y^k) + \varepsilon'_k (D^2\varphi(x^k, y^k))^2, \tag{6.19}$$

with

$$q^k = -D_y\varphi(x^k, y^k) = \frac{x^k - y^k}{\varepsilon_k} - \Big(\frac{1_{i\in I}}{z_i^k}\big(\frac{y_i^k - x_i^k}{z_i^k} - 1\big)^3\Big),$$

$$p^k = D_x\varphi(x^k, y^k) = q^k + \big((x_i^k - \hat{x}_i)^3\big),$$

and where 1_A denotes the indicator function. After computation, we obtain

$$D^2\varphi(x, y) = \begin{pmatrix} D_k(x, y) + E_k(x) & -D_k(x, y) \\ -D_k(x, y) & D_k(x, y) \end{pmatrix},$$

where $D_k(x, y)$ and $E_k(x)$ are the $(n + 1) \times (n + 1)$ diagonal matrices with diagonal entries

$$(D_k(x, y))_{ii} = \frac{1}{\varepsilon_k} + 3\frac{1_{i\in I}}{(z_i^k)^2}\left(\frac{y_i - x_i}{z_i^k} - 1\right)^2, \qquad (E_k(x))_{ii} = 3(x_i - \hat{x}_i)^2.$$

Then, $(D^2\varphi(x^k, y^k))^2$ is bounded in norm by $C(\min_{i\in I} z_i^k)^{-4}$ (since $\varepsilon_k \geq z_i^k$) and choosing ε'_k small enough, we obtain that $\varepsilon'_k(D^2\varphi(x^k, y^k))^2$ tends to 0 when k goes to infinity. Here and below C denotes a positive constant. Using Definition 6.2 of viscosity solutions and the fact that $y^k \in \operatorname{int}(\mathcal{S})$, we get

$$F^*(X^k, p^k, \delta V(x^k), x^k) \geq 0,$$

$$F^*(Y^k, q^k, \delta V''(y^k), y^k) \leq -\eta\varepsilon < 0.$$

Since $F_0^*(p, x)$ is Lipschitz continuous with respect to $p \in \mathbb{R}^{n+1}$ and $x \in G$, we get

$$\begin{aligned} F_0^*(p^k, x^k) &\leq F_0^*(q^k, y^k) + C(|x^k - \hat{x}|^3 + |x^k - y^k|) \\ &\leq -\eta\varepsilon + C(|x^k - \hat{x}|^3 + |x^k - y^k|) \to -\eta\varepsilon \quad \text{when } k \to \infty. \end{aligned}$$

Therefore, for k large enough, the maximum in the definition of $F^*(X^k, p^k, \delta V(x^k), x^k)$ is attained in the diffusion part and

$$\begin{aligned} 0 < \eta\varepsilon \leq \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x_i^k x_j^k X_{ij}^k - y_i^k y_j^k Y_{ij}^k) + \sum_{i=0}^{n} \alpha_i x_i^k (x_i^k - \hat{x}_i)^3 \\ + \sum_{i=0}^{n} \alpha_i (x_i^k - y_i^k) q_i^k - \delta(V(x^k) - V''(y^k)) + H(x^k) - H(y^k), \end{aligned}$$

where $\alpha_0 = r$. Since $V(x^k) - V''(y^k) \geq m_k > 0$ for k large enough, $x^k, y^k \to \hat{x}$, $(|x^k - y^k|^2)/(\varepsilon_k) \to 0$, and $(y_i^k - x_i^k)/z_i^k \to 1$, all the terms of the right-hand side of the above inequality, except perhaps the first one, have a limit lower than 0 when k goes to infinity. For the first one, using a square root of the matrix a and applying (6.19), we obtain

$$\sum_{i,j=1}^{n} a_{ij}(x_i^k x_j^k X_{ij}^k - y_i^k y_j^k Y_{ij}^k) \le C\sum_{i=1}^{n}(D_k(x^k, y^k))_{ii}(x_i^k - y_i^k)^2 + o(1)$$

$$= C\left(\frac{|x^k - y^k|^2}{\varepsilon_k} + 3\sum_{i\in I}\left(\frac{x_i^k - y_i^k}{z_i^k}\right)^2\left(\frac{y_i^k - x_i^k}{z_i^k} - 1\right)^2\right) + o(1),$$

which tends to 0 when k goes to infinity. This leads to a contradiction and finishes the proof of the comparison result. □

7. THE VARIATIONAL INEQUALITY FOR THE ERGODIC PROBLEM

In this section we derive the equation for the limiting ergodic control problem and show that the so-called "potential function" for the ergodic problem can be obtained from the functions $\mathcal{V}^\delta$ when $\delta \to 0$.

PROPOSITION 7.1. *Let* $\mathcal{W}^\delta(y) = \mathcal{V}^\delta(y) - \mathcal{V}^\delta(0)$. *For any sequence* δ_n *tending to* 0, *there exists a subsequence also denoted* δ_n, *a constant* π, *and a continuous function* $\mathcal{V}$ *on* Δ *such that*

$$\lim_{n\to\infty} \delta_n \mathcal{V}^{\delta_n}(y) = \pi \quad \textit{uniformly in } \Delta, \tag{7.1}$$

$$\lim_{n\to\infty} \mathcal{W}^{\delta_n}(y) = \mathcal{V}(y) \quad \textit{uniformly in } \Delta. \tag{7.2}$$

The function $\mathcal{V}$ *is concave. It is a viscosity solution of the variational inequality*

$$\mathcal{F}(D^2\mathcal{V}(y), D\mathcal{V}(y), \pi, y) = 0 \quad \textit{in } \Delta, \tag{7.3}$$

where $\mathcal{F}$ *is defined in (6.12).*

Proof. In view of (5.5), the function $\delta\mathcal{V}^\delta$ is bounded in Δ. Therefore, for any sequence δ_n converging to 0, there exists a subsequence also denoted δ_n and a constant π such that $\delta_n \mathcal{V}^{\delta_n}(0) \to \pi$. By virtue of Proposition 5.1, $\delta|\mathcal{V}^\delta(y) - \mathcal{V}^\delta(0)| \le \delta\|y\|$, for all $y \in \Delta$. Since Δ is compact, the above inequality implies that $\delta_n\mathcal{V}^{\delta_n}(y)$ converges to π uniformly in Δ and (7.1) follows. Using Proposition 5.1, we conclude to the equicontinuity of the family of functions $\mathcal{W}^\delta$. Therefore, by Ascoli's Theorem and since $\mathcal{W}^\delta(0) = 0$, there exists a subsequence of δ_n, also denoted δ_n, and a continuous function $\mathcal{V}$ in Δ such that (7.2) holds. Substituting $\mathcal{W}^\delta + \mathcal{V}^\delta(0)$ to $\mathcal{V}^\delta$ in (6.11), we see that $\mathcal{W}^\delta$ satisfies (in the viscosity sense) the following equation:

$$\mathcal{F}(D^2\mathcal{W}^\delta(y), D\mathcal{W}^\delta(y), \delta\mathcal{V}^\delta(y), y) = 0 \quad \text{in } \Delta.$$

By virtue of (7.1), $\delta_n\mathcal{V}^{\delta_n}(y)$ tends to π uniformly in $y \in \Delta$ as n goes to infinity. Thus,

$$\liminf_{A'\to A,\, p'\to p, n\to+\infty, y'\to y} \mathcal{F}_*(A', p', \delta_n\mathcal{V}^{\delta_n}(y'), y') \ge \mathcal{F}_*(A, p, \pi, y)$$

and

$$\limsup_{A'\to A,\, p'\to p, n\to+\infty, y'\to y} \mathcal{F}^*(A', p', \delta_n\mathcal{V}^{\delta_n}(y'), y') \le \mathcal{F}^*(A, p, \pi, y).$$

Since $\mathcal{W}^{\delta_n}$ converges to $\mathcal{V}$ uniformly in Δ, $\mathcal{V}$ is a viscosity solution of (7.3) with π as in (7.1) (see Lions 1983b for continuous $\mathcal{F}$ and see Barles and Perthame 1987 and Barles 1994 for discontinuous $\mathcal{F}$). The concavity of $\mathcal{V}^\delta$ implies the concavity of $\mathcal{W}^\delta$ and $\mathcal{V}$. □

The following theorem relates the value π in (7.1) to the optimal growth rate.

THEOREM 7.2. *For any viscosity solution $\mathcal{V}$ of (7.3), we have*

$$\pi = \sup_{(\mathcal{L},\mathcal{M})} J_x(\mathcal{L},\mathcal{M}) \qquad \forall x \in \mathcal{S}\backslash\{0\}.$$

COROLLARY 7.3. *The function $\delta\mathcal{V}^\delta$ converges uniformly in Δ toward the optimal growth rate defined in* (2.6) *which is constant.*

Proof. By virtue of Proposition 7.1, the function $\mathcal{V}$ given by (7.2) is a viscosity solution of equation (7.3) with π given by (7.1). Thus, Theorem 7.2 implies that any limit π of a converging sequence $\delta_n\mathcal{V}^{\delta_n}(y)$ with $\delta_n \to 0$ is equal to the optimal growth rate defined in (2.6). Hence, this optimal rate is a constant π and $\delta\mathcal{V}^\delta(y) \to \pi$ when $\delta \to 0$. Moreover, by the same arguments as in the proof of Proposition 7.1, this convergence is uniform in $y \in \Delta$. □

In order to prove Theorem 7.2, we need the following auxiliary result.

PROPOSITION 7.4. *Let $\mathcal{V}$ be a viscosity solution of (7.3). For all* $0 \le t \le T$*, we have*

$$\sup_{(\mathcal{L},\mathcal{M})} E(\log W(T) + \mathcal{V}(s(T)) - \pi T) = E(\log W(t^-) + \mathcal{V}(s(t^-)) - \pi t), \tag{7.4}$$

where the supremum is taken over all admissible policies $(\mathcal{L},\mathcal{M})$ *for the initial position* $S(t^-)$ *at time* t.

Proof. Let T be fixed and let $\overline{V} = \mathcal{V}\circ Y$ with Y defined in (5.11). For a process S satisfying (2.1–2.2) and $S(t^-) = x$, we consider the following performance functional:

$$\begin{aligned} J_{t,x}^T(\mathcal{L},\mathcal{M}) &= E\big[\log W(T) - \log W(t^-) + \overline{V}(S(T)) - \pi(T-t)\big] \\ &= E\left[\int_t^T (H(s(u)) - \pi)\,du - \sum_{i=1}^n (\lambda_i + \mu_i)\int_{t^-}^T W(u)^{-1}\circ d\mathcal{L}_i(u) + \overline{V}(S(T))\right], \end{aligned}$$

with optimum overall admissible policies $(\mathcal{L},\mathcal{M})$:

$$V_T(t,x) = \sup_{(\mathcal{L},\mathcal{M})} J_{t,x}^T(\mathcal{L},\mathcal{M}).$$

Equation (7.4) is equivalent to the equality $V_T(t,x) = \overline{V}(x)$ for all t. Since $\overline{V}$ is bounded ($\mathcal{V}$ is continuous and Δ is compact), V_T is a bounded viscosity solution of the following parabolic variational inequality

$$\begin{cases} F\left(D^2 V_T(t,x), DV_T(t,x), \pi - \dfrac{\partial V_T}{\partial t}(t,x), x\right) = 0 & \text{in } (0,T)\times(\mathcal{S}\backslash\{0\}), \\ \mathcal{V}_T(T,x) = \overline{V}(x), \end{cases} \tag{7.5}$$

with F defined in (6.8). Since $\mathcal{V}$ is a viscosity solution of (7.3), $\overline{V}$ is a viscosity solution of

$$F(D^2\overline{V}(x), D\overline{V}(x), \pi, x) = 0 \quad \text{in } \mathcal{S}\backslash\{0\}. \tag{7.6}$$

Hence $(t,x) \mapsto \overline{V}(x)$ is also a viscosity solution of (7.5). Therefore, once the uniqueness of a solution to (7.5) is established, the statement of the proposition follows. The uniqueness of a viscosity solution of (7.5) is proved by adapting the Ishii technique as in the proof of Theorem 6.7. □

Proof of Theorem 7.2. Let $\mathcal{V}$ be a viscosity solution of (7.3). Let $x \in \mathcal{S}\backslash\{0\}$, $(\mathcal{L}, \mathcal{M})$ be an admissible policy for x, and $S(t)$ be the solution of (2.1–2.2). Proposition 7.4 implies

$$E[\log W(T) + \mathcal{V}(s(T)) - \pi T] \leq \log \rho(x) + \mathcal{V}(Y(x)).$$

Since $\mathcal{V}$ is bounded (Δ is compact) and $Y(t) \in \Delta$, we get

$$\frac{E \log W(T)}{T} \leq \pi + \frac{\log \rho(x) + \mathcal{V}(Y(x)) + \|\mathcal{V}\|_\infty}{T},$$

where $\|\mathcal{V}\|_\infty$ denotes the $L^\infty =$ norm of $\mathcal{V}$. When T goes to infinity, we obtain

$$\limsup_{T\to+\infty} \frac{E \log W(T)}{T} \leq \pi$$

for all $x \in \mathcal{S}\backslash\{0\}$ and $(\mathcal{L}, \mathcal{M})$ admissible. Hence

$$\sup_{(\mathcal{L},\mathcal{M})} \liminf_{T\to+\infty} \frac{E \log W(T)}{T} \leq \pi.$$

In order to prove the opposite inequality, we fix $T_0 > 0, \varepsilon > 0$ and $x \in \mathcal{S}\backslash\{0\}$. From Proposition 7.4, there exists an admissible policy $(\mathcal{L}, \mathcal{M})$ for the initial position $(1, 0, \dots, 0)$ such that

$$E[\log W(T_0) + \mathcal{V}(s(T_0)) - \pi T_0] \geq \mathcal{V}(0) - \varepsilon.$$

Due to the linearity of (2.1–2.2), the following inequality holds for the process S with policy $(\rho\mathcal{L}, \rho\mathcal{M})$ and initial position $(\rho, 0, \dots, 0)$:

$$E[\log W(T_0) + \mathcal{V}(s(T_0)) - \pi T_0] \geq \log \rho + \mathcal{V}(0) - \varepsilon. \tag{7.7}$$

Let $(\mathcal{L}^*, \mathcal{M}^*)$ be defined as follows:

$$\begin{cases} \mathcal{L}_i^*(t) & = \rho(x)\mathcal{L}_i(t), \\ \mathcal{M}_i^*(t) & = x_i + \rho(x)\mathcal{M}_i(t), \end{cases}$$

for $0 \leq t < T_0$, and

$$\begin{cases} \mathcal{L}_i^*(t) = \mathcal{L}_i^*(kT_0^-) + W(kT_0^-)\mathcal{L}_i(t - kT_0^-)(\theta_{kT_0}\omega) \\ \mathcal{M}_i^*(t) = \mathcal{M}_i^*(kT_0^-) + S_i(kT_0^-) + W(kT_0^-)\mathcal{M}_i(t - kT_0^-)(\theta_{kT_0}\omega), \end{cases}$$

for $kT_0 \leq t < (k+1)T_0$, $k = 1, 2, \dots$. Here θ_t is an operator on Ω such that the Brownian motion satisfies $w(s)(\theta_t\omega) = w(s+t)(\omega) - w(t)(\omega)$. The control $(\mathcal{L}^*, \mathcal{M}^*)$ consists of an instantaneous sale of stocks at each time proportional to T_0, followed by an ε-optimal policy on $[kT_0, (k+1)T_0)$. By construction, the distributions of $s(t + kT_0)$ and

$W(t+kT_0)/W(kT_0^-)$ for $t \in [0, T_0)$ do not depend on k. Using (7.7) and the boundness of $\mathcal{V}$, we get

$$\begin{aligned} E[\log W(kT_0^-) - \log \rho(x)] &= \sum_{l=0}^{k-1} E[\log W((l+1)T_0^-) - \log W(lT_0^-)] \\ &\geq \sum_{l=0}^{k-1} E[\pi T_0 + \mathcal{V}(0) - \mathcal{V}(s((l+1)T_0)) - \varepsilon] \\ &\geq \pi k T_0 - k(\varepsilon + 2\|\mathcal{V}\|_\infty). \end{aligned}$$

Dividing by kT_0 and taking the limit when k goes to infinity, we obtain

$$\liminf_{T\to+\infty, T=kT_0} \frac{E \log W(T^-)}{T} \geq \pi - \frac{\varepsilon + 2\|\mathcal{V}\|_\infty}{T_0}.$$

If $kT_0 \leq T < (k+1)T_0$, using the previous computations and the fact that, when $T > T'$, $E(\log W(T) - \log W(T')) \leq \kappa(T - T')$ due to (3.2) and (4.3), we obtain

$$\begin{aligned} E(\log W(T) - \log \rho(x)) &\geq E(\log W((k+1)T_0^-) - \log \rho(x)) \\ &\quad - E(\log W((k+1)T_0^-) - \log W(T)) \\ &\geq \pi(k+1)T_0 - (k+1)(\varepsilon + 2\|\mathcal{V}\|_\infty) - \kappa T_0 \\ &\geq \pi T - (k+1)(\varepsilon + 2\|\mathcal{V}\|_\infty) - \kappa T_0. \end{aligned}$$

Dividing by T, we get

$$\frac{E \log W(T)}{T} \geq \pi + \frac{\log \rho(x) - \kappa T_0}{T} - \frac{(k+1)}{k} \frac{(\varepsilon + 2\|\mathcal{V}\|_\infty)}{T_0}.$$

When T goes to infinity with T_0 and ε fixed, we get $k \to +\infty$ and

$$\liminf_{T\to+\infty} \frac{E \log W(T)}{T} \geq \pi - \frac{(\varepsilon + 2\|\mathcal{V}\|_\infty)}{T_0}.$$

Hence

$$\sup_{(\mathcal{L},\mathcal{M})} \liminf_{T\to+\infty} \frac{E \log W(T)}{T} \geq \pi - \frac{(\varepsilon + 2\|\mathcal{V}\|_\infty)}{T_0}.$$

Since the above inequality holds for all $T_0 > 0$ and $\varepsilon > 0$, the theorem is proved. □

Theorem 7.2 implies the uniqueness of the constant π such that a viscosity solution $\mathcal{V}$ of (7.3) exists. In the following lemma, we prove directly the uniqueness of the constant π in (7.6) by using the Ishii technique. This implies the same uniqueness result for equation (7.3), from the fact that $\mathcal{V} \circ Y$ is a (sub-, super-) solution of (7.6) if and only if $\mathcal{V}$ is a (sub-, super-) solution of (7.3). Hence, Theorem 7.2 may have been proved with the result of Proposition 7.4 for one $\mathcal{V}$ only. But this result can be obtained directly for any limit $\mathcal{V}$ of $\mathcal{W}^\delta = \mathcal{V}^\delta - \mathcal{V}^\delta(0)$ by passing to the limit in the weak dynamic programming principle (6.17).

LEMMA 7.5. *Let $\pi, \pi' \in \mathbb{R}$. If V is a bounded viscosity subsolution of $F(D^2V(x), DV(x), \pi, x) = 0$ in $\mathcal{S}\backslash\{0\}$ and V' is a bounded viscosity supersolution of $F(D^2V'(x), DV'(x), \pi', x) = 0$ in $\mathcal{S}\backslash\{0\}$, then $\pi \leq \pi'$.*

Proof. We argue by contradiction. Suppose that $\pi > \pi'$. Then there exist $\delta > 0, c, c' \in \mathbb{R}$ such that $\pi \geq \delta(V + c) > \delta(V' + c') \geq \pi'$ in $\mathcal{S}\backslash\{0\}$. Indeed, one can take δ, c, c' such that $c = (\pi/\delta) - \sup V$, $c' = (\pi'/\delta) - \inf V'$ and $\delta(V' - V - \inf V' + \sup V) < \pi - \pi'$. Therefore, $V + c$ is a viscosity subsolution and $V' + c'$ is a viscosity supersolution of (6.7). Theorem 6.7 implies $V + c \leq V' + c'$ in $\operatorname{int}(\mathcal{S})$ and we get a contradiction. □

Let $\overline{V}$ be a solution of (7.6). Define the following subsets of $\mathcal{S}\backslash\{0\}$:

$$\mathsf{B}_i = \left\{x \in \mathcal{S}\backslash\{0\},\ \ L_i\overline{V}(y) = \frac{\lambda_i + \mu_i}{\rho(x)}\right\}, \tag{7.8}$$

$$\mathsf{S}_i = \{y \in \mathcal{S}\backslash\{0\},\ \ M_i\overline{V}(y) = 0\}, \tag{7.9}$$

$$\mathsf{NT}_i = \big(\mathcal{S}\backslash\{0\}\big) \setminus \big(\operatorname{int}(\mathsf{B}_i) \cup \operatorname{int}(\mathsf{S}_i)\big), \tag{7.10}$$

$$\mathsf{NT} = \bigcap_{i=1}^{n} \mathsf{NT}_i, \tag{7.11}$$

where interiors are relative to $\mathcal{S}\backslash\{0\}$. Suppose that the boundaries of these sets are such that there exists a reflected diffusion $S^*(t)$ in NT solution of (2.1–2.2) where $\mathcal{M}_i$ and $\mathcal{L}_i$ are the local times at the boundaries with B_i and S_i respectively. Then an optimal stationary investment policy for problem (2.6) is obtained as follows. The set NT is the no-transaction region. Outside of NT, an instantaneous transaction brings the position $S(0)$ to the boundary of NT: Buy stock i in B_i, sell stock i in S_i. After the initial transaction, the agent position $S(t)$ remains in NT and further transactions occur only at the boundary. The optimal position process $S(t)$ coincides with $S^*(t)$ in NT (see Davis and Norman 1990 for the one-dimensional problem). This kind of Skohorod problem, namely the existence of a reflected diffusion, is studied for example in Soner and Shreve (1989), Shreve and Soner (1991), and Dupuis and Ishii (1993).

8. UNIQUENESS OF THE POTENTIAL FUNCTION

Lemma 7.5 shows that the standard Ishii technique leads to the uniqueness of a rate π such that a solution V of (7.6) exists, but not to the uniqueness of the potential function V. Moreover, if V is a solution of (7.6) and C is a constant, then $V + C$ is also a solution of (7.6). So the uniqueness of V can only be expected within an additive constant. In Bensoussan (1988, Chap. II), the uniqueness within an additive constant of a solution V of an ergodic Hamilton–Jacobi–Bellman equation of the form $\max_{u \in \mathcal{U}_{ad}}(A(u)V + c(u)) + \pi = 0$ is proved when $A(u)$ is strongly uniformly elliptic in x and u. Indeed, if this condition is fulfilled, then for any feedback policy $u(x)$, the associated controlled process is ergodic, then admits a positively lower bounded invariant measure, which implies that a solution V of the linear equation $A(u)V + c(u) + \pi = 0$ is unique within an additive constant. The proof of these results is done by using variational techniques: an invariant measure is a solution of a dual equation and the uniqueness is proved by "multiplying" the equation satisfied by V by the invariant measure. Viscosity techniques do not allow any of these two steps which have then to be replaced by the use of the maximum principle concept. Moreover, equation (7.6) is not even uniformly elliptic. However, if everything behaves as in the discounted case (see Davis and Norman 1990 and Akian et al. 1996), one might expect that for the optimal policy there exists a closed connected subdomain NT of $\mathcal{S}\backslash\{0\}$ such that, in its complementary NT^c, transactions occur instantaneously at time 0 and bring the process $S(t)$ on the boundary of NT, whereas in NT the process $S(t)$ is a diffusion reflected at the boundary (by transaction operations). Hence, positions in NT^c are transient and the process restricted to NT is ergodic. When this is the case, as detailed in Akian and Gaubert (2000), one can also expect the uniqueness of a solution V of (7.6) within an additive constant. We prove below this result for $n = 1$ only (one stock) and for equation (7.3) instead of equation (7.6), which is easier since Δ is compact. When $n \geq 2$, NT may have a nonregular boundary, which yields an additional difficulty. We first state some properties related to NT (see Figure 8.1). The points $\underline{y}$ and $\bar{y}$ defined in the following lemma are such that $\mathsf{NT} = [\underline{y}, \bar{y}]$ if $\mathcal{V}$ is concave. Otherwise, $[\underline{y}, \bar{y}]$ is the convex hull of NT only.

LEMMA 8.1. *Let $n = 1$, π be given by Theorem 7.2 and $\mathcal{V}$ be a solution of (7.3). Let*

$$\underline{y} = \sup\{y \in [0, 1], \mathcal{V}(y) = \log(1 + \nu_1 y) + \mathcal{V}(0)\}, \qquad \underline{y}' = \frac{(1+\nu_1)\underline{y}}{1 + \nu_1 \underline{y}},$$

$$\bar{y} = \inf\{y \in [0, 1], \mathcal{V}(y) = \mathcal{V}(1)\}.$$

Then, $\underline{y}$ and $\bar{y}$ do not depend on the solution $\mathcal{V}$ of (7.3) and

$$H(y) \leq \pi = H(\underline{y}') \qquad \forall y \in [0, \underline{y}'], \tag{8.1}$$

$$H(y) \leq \pi = H(\bar{y}) \qquad \forall y \in [\bar{y}, 1]. \tag{8.2}$$

Proof. We first prove (8.2) at least for some special $\bar{y}$, then we deduce (8.1) by symmetry. For $n = 1$, equation (7.3) becomes

$$\max\left(BV - \pi + H(y), \mathcal{F}_0\left(\frac{dV}{dy}, y\right)\right) = 0 \quad \text{in } \Delta = [0, 1], \tag{8.3}$$

with

$$BV = \frac{a_{11}}{2} y^2 (1-y)^2 \frac{d^2 V}{dy^2} + y(1-y)(\alpha_1 - r - a_{11} y)\frac{dV}{dy},$$

$$H(y) = r + (\alpha_1 - r)y - \frac{1}{2} a_{11} y^2,$$

$$\mathcal{F}_0(p, y) = \begin{cases} \max((\nu_1 y + 1)p - \nu_1, -p) & \text{if } y \in (0, 1), \\ p - \nu_1 & \text{if } y = 0, \\ -p & \text{if } y = 1, \end{cases}$$

where the expression of $\mathcal{F}_0(p, y)$ for $y = 0$ and 1 has been simplified compared to that of Corollary 6.6 as suggested by Remark 6.5. From this equation, we see that $\mathcal{V}$ is a

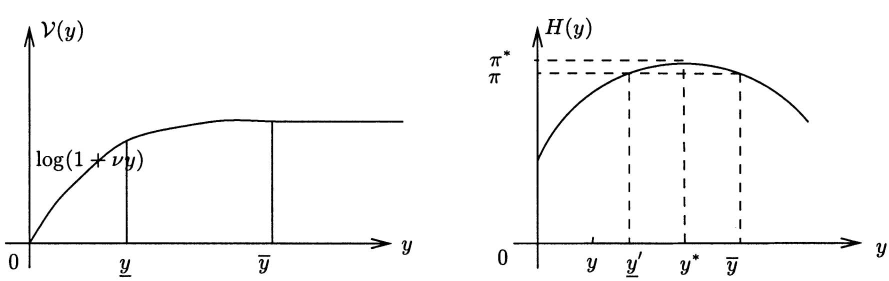

FIGURE 8.1 The rate π and the potential function $\mathcal{V}$ solution of equation (7.3), for $n = 1$.

viscosity supersolution of $-(d\mathcal{V}/dy) = 0$ in $(0, 1]$. This implies that $\mathcal{V}$ is nondecreasing on $[0, 1]$. Indeed, if $\mathcal{V}(y) < \mathcal{V}(z)$ for $y > z$, then for $[\mathcal{V}(y) - \mathcal{V}(z)]/(y - z) < \rho < 0$, the minimum of $\mathcal{V}(x) - \rho x$ in $[z, 1]$ is reached at $x \in (z, 1]$ and $(\rho, 0) \in \overline{J}_{\Delta}^{2,-}\mathcal{V}(x)$ which contradicts $-(d\mathcal{V}/dy) \leq 0$. From the definition of $\bar{y}$ and the nondecreasing property of $\mathcal{V}$, one deduces $\mathcal{V}(y) - \mathcal{V}(1) = 0, \forall y \in [\bar{y}, 1]$ and $\mathcal{V}(y) - \mathcal{V}(1) < 0, \forall y \in [0, \bar{y})$. Suppose first that $\bar{y} \neq 0$ and let us fix $\tilde{y} \in (0, \bar{y})$. Since $\mathcal{V}(\tilde{y}) < \mathcal{V}(\bar{y}) = \mathcal{V}(1)$, there exists $\varepsilon > 0$ such that $\mathcal{V}(\tilde{y}) - \varepsilon(\tilde{y} - \bar{y}) < \mathcal{V}(\bar{y})$. Hence, the maximum of the function $\mathcal{V}(y) - \varepsilon y$ in $[\tilde{y}, 1]$ is reached at $y \in (\tilde{y}, \bar{y}]$ ($\mathcal{V}$ is constant on $[\bar{y}, 1]$), and is thus a local maximum in $[0, 1]$. Using that $\mathcal{V}$ is a viscosity subsolution of (8.3), and applying Definition 6.1 to the test function $\phi(y) = \varepsilon y$, we get

$$\max(y(1 - y)(\alpha_1 - r - a_{11}y)\varepsilon - \pi + H(y), (\nu_1 y + 1)\varepsilon - \nu_1, -\varepsilon) \geq 0.$$

If ε is small enough ($\varepsilon < \nu_1/(\nu_1 + 1)$), this implies $\pi \leq H(y) + C\varepsilon$ for some constant C. Letting ε tend to zero and $\tilde{y}$ tend to $\bar{y}$, we get $\pi \leq H(\bar{y})$.

Suppose now that $\bar{y} \neq 1$. The constant function $\mathcal{V}(y) \equiv \mathcal{V}(1)$ is a viscosity supersolution of (8.3) in $(\bar{y}, 1]$, so

$$B\mathcal{V} - \pi + H(y) = -\pi + H(y) \leq 0 \qquad \forall y \in (\bar{y}, 1].$$

By continuity, we get $H(y) \leq \pi, \forall y \in [\bar{y}, 1]$. Therefore, (8.2) holds at least when $\bar{y} \neq 0$ and 1:

$$\text{(8.4)} \qquad H(y) \leq \pi \qquad \forall y \in [\bar{y}, 1] \text{ if } \bar{y} \neq 1, \qquad \text{and } H(\bar{y}) \geq \pi \quad \text{if } \bar{y} \neq 0.$$

In order to prove (8.1), we apply the increasing change of variable: $y' = \frac{(1+\nu_1)y}{1+\nu_1 y}$ in $[0, 1]$ and introduce the function

$$\mathcal{V}'(y') = \mathcal{V}(y) - \log(1 + \nu_1 y).$$

This function can indeed be obtained by the change of variables (5.1) and (5.11), when λ_i replaces $-\mu_i$; that is, $\mathcal{V}'(y') = V(1 - y', \ y'/(1 + \lambda_1))$, where $V(x) = \log \rho(x) + \mathcal{V}(Y(x))$ is a solution of (7.6) such that $V(\rho x) = \log \rho + V(x)$. Therefore, $\mathcal{V}$ is a viscosity solution of (8.3) if and only if $\mathcal{V}'$ is a viscosity solution of

$$\max\left(BV - \pi + H(y), \mathcal{F}_0'\left(\frac{dV}{dy}, y\right)\right) = 0 \quad \text{in } \Delta = [0, 1],$$

with

$$\mathcal{F}_0'(p, y) = \begin{cases} \max(p, -\nu_1 - p(1 + \nu_1 - \nu_1 y)) & \text{if } y \in (0, 1), \\ p & \text{if } y = 0, \\ -\nu_1 - p & \text{if } y = 1. \end{cases}$$

We may now either apply the same arguments as before, or notice that the function $\mathcal{V}'(1 - y)$ is a viscosity solution of (8.3) where r is replaced by $r' = \alpha_1 - \frac{1}{2}a_{11}$, the mean performance rate of stock 1, and α_1 is replaced by α_1' such that $\alpha_1' - \frac{1}{2}a_{11} = r$. We then obtain that $\mathcal{V}'$ is nonincreasing, or, equivalently, that $\mathcal{V}(y) - \log(1 + \nu_1 y)$ is nonincreasing. Hence,

$$\mathcal{V}(y) = \mathcal{V}(0) + \log(1 + \nu_1 y) \qquad \forall y \in [0, \underline{y}],$$

$$\mathcal{V}(y) < \mathcal{V}(0) + \log(1 + \nu_1 y) \qquad \forall y \in (\underline{y}, 1],$$

and $\underline{y} \leq \bar{y}$. We obtain also from (8.4) that

$$\text{(8.5)} \qquad H(y) \leq \pi \qquad \forall y \in [0, \underline{y}'] \text{ if } \underline{y} \neq 0, \qquad \text{and } H(\underline{y}') \geq \pi \quad \text{if } \underline{y} \neq 1.$$

It remains to prove (8.1) and (8.2) in some boundary cases. First, if $\bar{y} = 0$, then $\underline{y} = 0 \neq 1$, $\underline{y}' = 0$ and (8.5) implies $\pi \leq H(\bar{y})$. So the inequality $\pi \leq H(\bar{y})$ holds in all cases and similarly the inequality $\pi \leq H(\underline{y}')$ is always valid.

Second, suppose $\bar{y} = 1$. Using the previous results, we know that $\mathcal{V}(y) - \log(1 + \nu_1 y)$ is nonincreasing and so the minimum of this function in [0, 1] is attained in 1. Using now that $\mathcal{V}$ is a viscosity supersolution of (8.3) and applying Definition 6.1 to the test function $\phi(y) = \log(1 + \nu_1 y)$, we obtain

$$B\phi(1) - \pi + H(1) = H(1) - \pi \le 0.$$

So the inequality $H(y) \le \pi, \forall y \in [\bar{y}, 1]$, holds for all cases, and similarly the inequality $H(y) \le \pi, \forall y \in [0, \underline{y}']$, always holds.

Let us prove now the uniqueness of $\underline{y}$ and $\bar{y}$. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two solutions of (7.3) and denote by $\underline{y}_i$, $\bar{y}_i$ the values of $\underline{y}$, $\bar{y}$ for $\mathcal{V}_i$, $i = 1, 2$. If $\bar{y}_1 < \bar{y}_2$, for instance, then (8.2) implies that $H(y) \le \pi, \forall y \in [\bar{y}_1, \bar{y}_2]$, and $H(\bar{y}_1) = H(\bar{y}_2) = \pi$. But this contradicts the strict concavity of H ($a_{11} > 0$ by assumption). Thus $\bar{y}_1 = \bar{y}_2$ and similarly $\underline{y}_1 = \underline{y}_2$. □

THEOREM 8.2. *When $n = 1$, the solution $\mathcal{V}$ of* (7.3) *is unique within an additive constant.*

Proof. By Proposition 7.1, we know that there exists a solution $\mathcal{V}$ to (7.3), which is in addition concave. Moreover, since $\mathcal{V}(y)$ is the limit of $\mathcal{V}^{\delta_n}(y) - \mathcal{V}^{\delta_n}(0)$, and $V^{\delta}(x) = \log \rho(x) + \mathcal{V}^{\delta}(Y(x))$ is concave in $\mathcal{S}\setminus\{0\}$ by Proposition 4.2, then $V(x) = \log \rho(x) + \mathcal{V}(Y(x))$ is also concave. The concavity of $\mathcal{V}$ implies that NT is connected. We shall then prove Theorem 8.2 by comparing any solution of (7.3) to a particular solution $\mathcal{V}^*$ such that $V^*(x) = \log \rho(x) + \mathcal{V}^*(Y(x))$ is concave in $\mathcal{S}\setminus\{0\}$.

Let $\underline{y}$ and $\bar{y}$ be defined as in Lemma 8.1, and $\mathcal{V}$ and $\mathcal{V}'$ be two (bounded) viscosity solutions of (7.3). Denote $E = \arg\max_{y\in\Delta} w(y)$ with $w = \mathcal{V} - \mathcal{V}'$. Since $\mathcal{V} + C$ is also a solution of (7.3) for any constant C, we may suppose that $\sup_{y\in\Delta} w(y) = 0$. The set E is closed since w is continuous, and it is nonempty since Δ is compact. Then, $E \cap (0, 1)$ is closed in (0, 1) (but eventually empty).

• The main step of the proof is to show that

$$\text{if } \mathcal{V}' = \mathcal{V}^*, \text{ then } E \cap (0, 1) \text{ is open.} \tag{8.6}$$

By the same way, one will also show:

$$\begin{gathered}\text{if } \mathcal{V} = \mathcal{V}^*, \text{then } E \cap (\underline{y}, \bar{y}) \text{ is open, and for all } y \in E \cap (0, 1), \\ (\exists y_n \to y, \quad y_n \in E^c,\ y_n > y) \Rightarrow y \ge \bar{y} \\ (\exists y_n \to y, \quad y_n \in E^c,\ y_n < y) \Rightarrow y \le \underline{y}.\end{gathered} \tag{8.7}$$

Indeed, the two last implications imply that $E \cap (\underline{y}, \bar{y})$ is open.

Let $\tilde{y} \in \partial E \cap (0, 1)$, the boundary of E in (0, 1), that is $\tilde{y} \in E$ and there exists a sequence $y_n \in (0, 1) \cap E^c$ converging to $\tilde{y}$ when n goes to infinity. Taking a subsequence, we may suppose that for all n either $y_n > \tilde{y}$ or $y_n < \tilde{y}$.

Let us suppose first that $\tilde{y} < y_n < 1$ for all n. Since $y_n \notin E$, $w(y_n) < w(\tilde{y}) = 0$ and by the continuity of w there exists $\delta_n > 0$ such that $\sup_{B(y_n,\delta_n)} w < 0$, where $B(y, \delta)$ is the ball of center y and radius δ. The function $f_n(y) = \exp[-(\lambda_n/2)(y - y_n)^2]$ satisfies

$$\begin{aligned}Bf_n(y) = f_n(y)\bigg[&\frac{1}{2}a_{11}y^2(1-y)^2\lambda_n^2(y-y_n)^2 \\ &-\lambda_n\left(\frac{1}{2}a_{11}y^2(1-y)^2 + (y-y_n)y(1-y)(\alpha_1 - r - a_{11}y)\right)\bigg].\end{aligned}$$

For λ_n large enough, Bf_n is lower bounded on $[\tilde{y}/2, y_n-\delta_n]\subset(0,1)$ by some positive constant $\chi_n>0$. One can also choose χ_n small enough, so that $(1+\nu_1 y)(df_n/dy)\geq\chi_n>0$ on $[\tilde{y}/2, y_n-\delta_n]$. For such a fixed constant λ_n, and ε_n small enough, one has

$$\sup_{[y_n-\delta_n, y_n]} w+\varepsilon_n f_n \leq \sup_{B(y_n,\delta_n)} w+\varepsilon_n < w(\tilde{y})+\varepsilon_n f_n(\tilde{y}) = \varepsilon_n \exp\Big(-\frac{\lambda_n}{2}(\tilde{y}-y_n)^2\Big).$$

One chooses ε_n satisfying the previous equation together with $\varepsilon_n\lambda_n<1$. Since f_n increases on $[0, y_n]$, one has also $w(y)+\varepsilon_n f_n(y)\leq w(\tilde{y})+\varepsilon_n f_n(\tilde{y}), \forall y\in[0,\tilde{y}]$. Thus, the maximum of $w+\varepsilon_n f_n$ in $[0,y_n]$ can only be attained in $[\tilde{y}, y_n-\delta_n)$. But $w+\varepsilon_n f_n=\mathcal{V}-(\mathcal{V}'-\varepsilon_n f_n)$, and $\mathcal{V}'-\varepsilon_n f_n$ is a strict viscosity supersolution of

$$\max\Big(BV-\pi+H(y), (\nu_1 y+1)\frac{\partial V}{\partial y}-\nu_1\Big)=0 \quad \text{in } [\tilde{y}/2, y_n-\delta_n].$$

Hence, $\mathcal{V}$ cannot be a subsolution of this equation around $\tilde{y}$, so $\mathcal{V}$ is necessarily a subsolution (and thus a solution) of $-\partial V/\partial y=0$. Let us give a rigorous proof of this fact by using the Ishii technique. Considering (y^k, z^k) maximizing

$$w_k(y,z)=\mathcal{V}(y)-\mathcal{V}'(z)+\varepsilon_n f_n(z)-\frac{k}{2}|y-z|^2$$

in $[0,y_n]$, one obtains, when $k\to\infty$, that $k|y^k-z^k|^2\longrightarrow 0$, and $y^k, z^k\to\tilde{y}_n\in[\tilde{y}, y_n-\delta_n)$, where $\tilde{y}_n$ maximizes $w+\varepsilon_n f_n$. So the maximum of w_k is a local maximum, and there exist $Y^k, Z^k\in S^1$, such that $(k(y^k-z^k), Y^k)\in\overline{\mathcal{J}}^{2,+}\mathcal{V}(y^k)$, $(k(y^k-z^k), Z^k)\in\overline{\mathcal{J}}^{2,-}(\mathcal{V}'-\varepsilon_n f_n)(z^k)$, and

$$\begin{pmatrix} Y^k & 0\\ 0 & -Z^k\end{pmatrix}\leq k\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

For k large enough, $y^k, z^k\in(\frac{\tilde{y}}{2}, y_n-\delta_n)$ and

$$\max\Big(\frac{a_{11}}{2}(y^k(1-y^k))^2Y^k+y^k(1-y^k)(\alpha_1-r-a_{11}y^k)k(y^k-z^k)+H(y^k)-\pi,$$
$$(\nu_1 y^k+1)k(y^k-z^k)-\nu_1, -k(y^k-z^k)\Big)\geq 0$$

$$\max\Big(\frac{a_{11}}{2}(z^k(1-z^k))^2Z^k+z^k(1-z^k)(\alpha_1-r-a_{11}z^k)k(y^k-z^k)+H(z^k)-\pi+\varepsilon_n\chi_n,$$
$$(\nu_1 z^k+1)k(y^k-z^k)-\nu_1+\varepsilon_n\chi_n,\ -k(y^k-z^k)-\varepsilon_n\frac{\partial f_n}{\partial y}(z^k)\Big)\leq 0.$$

One deduces

$$\max(Ck|y^k-z^k|^2+C'|y^k-z^k|-\varepsilon_n\chi_n, Ck|y^k-z^k|^2-\varepsilon_n\chi_n, -k(y^k-z^k))\geq 0$$

and

$$k(y^k-z^k)+\varepsilon_n\frac{\partial f_n}{\partial y}(z^k)\geq 0.$$

So for k large enough, $p^k=k(y^k-z^k)\leq 0$, and there exist $(p^k, Y^k)\in\overline{\mathcal{J}}^{2,+}\mathcal{V}(y^k)$ and $(q^k, Z'^k)\in\overline{\mathcal{J}}^{2,-}\mathcal{V}'(z^k)$, such that $0\geq p^k\geq\varepsilon_n\lambda_n(z^k-y_n)f_n(z^k)\geq(\tilde{y}-y_n)$, $0\leq q^k=k(y^k-z^k)+\varepsilon_n\frac{\partial f_n}{\partial y}(z^k)\leq y_n-\tilde{y}$ ($\liminf z^k\geq\tilde{y}$ and $\varepsilon_n\lambda_n<1$). Taking the limit when n

goes to infinity, one obtains that $0 \in \overline{J}^{1,+}\mathcal{V}(\tilde{y})$ and $0 \in \overline{J}^{1,-}\mathcal{V}'(\tilde{y})$, that is there exist $z_n, z_n' \to \tilde{y}$, and $p_n, q_n \to 0$ such that

$$\limsup_{y \to z_n} \frac{\mathcal{V}(y) - \mathcal{V}(z_n) - p_n(y - z_n)}{|y - z_n|} \leq 0, \tag{8.8}$$

$$\liminf_{y \to z_n'} \frac{\mathcal{V}'(y) - \mathcal{V}'(z_n') - q_n(y - z_n')}{|y - z_n'|} \geq 0. \tag{8.9}$$

If $\mathcal{V}$ is concave, (8.8) implies that $\mathcal{V}(y) - p_n y$ is nonincreasing for $y \geq z_n$, then taking the limit when n goes to infinity, $\mathcal{V}$ is nonincreasing for $y \geq \tilde{y}$. But because $\mathcal{V}$ is a solution of (7.3), it is nondecreasing, so $\mathcal{V}$ is constant on $[\tilde{y}, 1]$. This implies $\tilde{y} \geq \bar{y}$. Similarly, if $\mathcal{V}'$ is concave, (8.9) implies that $\mathcal{V}'$ is nonincreasing, thus constant, and $\tilde{y} \geq \bar{y}$. But since $\mathcal{V} \leq \mathcal{V}'$, $\mathcal{V}(\tilde{y}) = \mathcal{V}'(\tilde{y})$, and $\mathcal{V}$ is nondecreasing, then $\mathcal{V} = \mathcal{V}'$ on $[\tilde{y}, 1]$, and we get a contradiction. So we have proved that if $\tilde{y} \in E \cap (0, 1)$ and $\mathcal{V}'$ is concave, then there does not exist $y_n \in (0, 1) \cap E^c$, $y_n > \tilde{y}$, $y_n \to \tilde{y}$, and that the result also holds if $\tilde{y} \in E \cap (0, \bar{y})$ and $\mathcal{V}$ is concave.

In order to get the same type of result on the left of $\tilde{y}$, we only have to use the same change of variable as in the proof of Lemma 8.1. One gets that if $\tilde{y} \in E \cap (0, 1)$ and $\mathcal{V}'(y) - \log(1 + \nu_1 y)$ is concave with respect to $y' = [(1 + \nu_1)y]/[1 + \nu_1 y]$, then there does not exist $y_n \in (0, 1) \cap E^c$, $y_n < \tilde{y}$, $y_n \to \tilde{y}$, and that the result also holds if $\tilde{y} \in E \cap (\underline{y}, 1)$ and $\mathcal{V}(y) - \log(1 + \nu_1 y)$ is concave with respect to y'. So (8.6) and (8.7) are proved.

• Let us suppose now that $\mathcal{V}' = \mathcal{V}^*$ and that $\mathcal{V}$ is another solution of (7.3) such that $\max(\mathcal{V} - \mathcal{V}^*) = 0$. By (8.6), $E \cap (0, 1)$ is open, but since it is also closed in (0, 1), and (0, 1) is connected, one deduces $E \cap (0, 1) = (0, 1)$ or $\emptyset$. If $E \cap (0, 1) = (0, 1)$, then $\mathcal{V} = \mathcal{V}^*$ on (0, 1) and by continuity $\mathcal{V} = \mathcal{V}^*$ on [0, 1].

Otherwise, one has $E \cap (0, 1) = \emptyset$, so the maximum of $\mathcal{V} - \mathcal{V}^*$ is attained in 0 or 1 only. Suppose for instance that $\mathcal{V}(1) = \mathcal{V}^*(1)$. Since $\mathcal{V}(y) < \mathcal{V}^*(y)$ for all y in (0, 1) and $\mathcal{V}^*$ is nondecreasing, one has $\mathcal{V}(y) < \mathcal{V}^*(y) \leq \mathcal{V}^*(1) = \mathcal{V}(1)$ for all y in (0, 1), then $\bar{y} = 1$ (by the uniqueness of $\bar{y}$). If in addition $\underline{y} = 1$, then by the uniqueness of $\underline{y}$, one gets again $\mathcal{V} = \mathcal{V}^*$ since they are both equal to the function $\log(1 + \nu_1 y) + \mathcal{V}(0)$.

Let us suppose now that $\underline{y} < 1$. Considering $E' = \arg\max(\mathcal{V}^* - \mathcal{V})$ and $\tilde{y} = \max E' \in E'$, the first implication of (8.7) says that if $\tilde{y} \in (0, 1)$, then $\tilde{y} \geq \bar{y} = 1$, which is impossible. Hence, $\tilde{y} = 0$ or 1. But since $\mathcal{V}^* - \mathcal{V}$ admits a strict local minimum in 1, it cannot have a maximum in 1, then $\tilde{y} \neq 1$. One gets $\tilde{y} = 0$, and $\mathcal{V} - \mathcal{V}^*$ attains its minimum in 0 only. Then, $\mathcal{V}(y) - \mathcal{V}(0) > \mathcal{V}^*(y) - \mathcal{V}^*(0), \forall y \in (0, 1]$, and $\mathcal{V}^*(y) - \mathcal{V}^*(0) - \log(1 + \nu_1 y) < \mathcal{V}(y) - \mathcal{V}(0) - \log(1 + \nu_1 y) \leq 0$, which implies $\underline{y} = 0$.

From $\underline{y} = 0$, $\bar{y} = 1$ and Lemma 8.1, we deduce $H(0) = H(1) = \pi$. Let us show that this is impossible. By the strict concavity of H, this implies $H(y) > \pi, \forall y \in (0, 1)$. Let $\varepsilon > 0$ and $f(y) = \log y(1 - y)$. The function $\mathcal{V} - \varepsilon f$ has a minimum in $y \in (0, 1)$. Since f is $\mathcal{C}^2$ in (0, 1) and $\mathcal{V}$ is a supersolution of (7.3), one gets $\varepsilon Bf - \pi + H(y) \leq 0$. But $B \log y = H(1) - H(y)$ and $B \log(1 - y) = H(0) - H(y)$, so that $\pi \geq \varepsilon H(1) + \varepsilon H(0) + (1 - 2\varepsilon)H(y)$, which leads to a contradiction when $\varepsilon < \frac{1}{2}$. □

9. COMPUTATION OF THE OPTIMAL POLICY

9.1. The Merton Problem

When the transaction costs are equal to zero, the optimal investment strategy is to keep a constant fraction y_i^* of total wealth in each risky asset (see Merton 1971;

Karatzas et al. 1986; Davis and Norman 1990; Fleming and Soner 1993; and Shreve and Soner 1994). Indeed, when $\lambda = \mu = 0$, equation (7.3) is equivalent to

$$\begin{cases} \mathcal{V}(y) = \text{constant}, \\ H(y) \leq \pi \qquad \forall y \in \Delta. \end{cases} \tag{9.1}$$

A solution π of (9.1) is nonunique, but this does not contradict the previous results because Theorems 6.4 and 7.2, or Lemma 7.5 holds only when $\lambda_i + \mu_i > 0$, $i = 1, \ldots, n$. The optimal growth rate is the minimal solution π of (9.1); that is, $\pi = \pi^*$ with

$$\pi^* = \max_{y \in \Delta} H(y). \tag{9.2}$$

Since H is strictly concave (the matrix a is positive), the optimal proportion is uniquely defined by

$$y^* = \arg\max_{y \in \Delta} H(y), \tag{9.3}$$

and the no-transaction region NT is reduced to $\{y^*\}$. For $n = 1$, this means that $\underline{y} = \bar{y} = y^*$ (see Lemma 8.1). Let $\tilde{y} = a^{-1}(\alpha - r)$ be the unique maximum point of H in $\mathbb{R}^n$. When $\tilde{y} \in \Delta$, $y^* = \tilde{y}$ and $\pi^* = r + \frac{1}{2}(\alpha - r) \cdot a^{-1}(\alpha - r)$. Otherwise y^* is on the boundary of Δ.

9.2. Explicit Solution of the One-Risky-Asset Problem

Here, we give a closed-form expression of the solution $(\mathcal{V}, \pi)$ of (7.3) in the case $n = 1$, using the results of Section 8. In Taksar et al. (1988), similar computations are done but for a slightly different model. To simplify, we drop the indices and write a for a_{11}, α for α_1, ν for ν_1, and so on. The ergodic variational inequality (7.3) reduces to (8.3). Let $\mathcal{V}$ be a solution of (8.3), $\underline{y}$ and $\bar{y}$ be defined as in Lemma 8.1, $\tilde{y} = (\alpha - r)/a$ be the optimum of H, and y^* be the Merton point defined in (9.3). From Lemma 8.1, we know that if $\tilde{y} \leq 0$, then $y^* = \underline{y} = \bar{y} = 0$ and $\mathcal{V}(y) = \mathcal{V}(1)$ for $y \in [0, 1]$. Similarly, if $\tilde{y} \geq 1$ then $y^* = \underline{y} = \bar{y} = 1$ and $\mathcal{V}(y) = \mathcal{V}(0) + \log(1 + \nu y)$ for $y \in [0, 1]$. In these two special cases, the optimal policy for the problem with transaction costs coincides with the Merton policy.

We restrict to the case $\tilde{y} = y^* \in (0, 1)$. We denote by π the constant in (8.3), and by π^* the Merton growth rate given by (9.2). From Lemma 8.1, we have

$$\{y \in (0, 1), \mathcal{V}(y) = \mathcal{V}(0) + \log(1 + \nu y)\} = [0, \underline{y}], \tag{9.4}$$

$$\{y \in (0, 1), \mathcal{V}(y) = \mathcal{V}(1)\} = [\bar{y}, 1], \tag{9.5}$$

and

$$H(\underline{y}') = H(\bar{y}) = \pi, \tag{9.6}$$

with $\underline{y}' = [(1 + \nu)\underline{y}]/(1 + \nu \underline{y})$. Therefore, $0 \leq \underline{y} \leq \underline{y}' \leq y^* \leq \bar{y} \leq 1$ with $y^* \in (0, 1)$, which implies $\underline{y} < \bar{y}$ and $\mathcal{V}$ is a viscosity solution of (8.3) in $(\underline{y}, \bar{y})$. Conversely, if $0 < \underline{y} < \bar{y} < 1$ satisfies (9.6), and $\mathcal{V}$ satisfies (9.4, 9.5), then $\mathcal{V}$ is a viscosity solution of (8.3) in $[0, 1]\backslash[\underline{y}, \bar{y}]$. If in addition, $\mathcal{V}$ is $\mathcal{C}^1$ in [0, 1] and satisfies

$$\text{(9.7)} \qquad \begin{cases} \dfrac{1}{2}ay^2(1-y)^2\dfrac{d^2V}{dy^2} + y(1-y)(\alpha - r - ay)\dfrac{dV}{dy} + H(y) - \pi = 0, \\ \dfrac{dV}{dy}(y) - \dfrac{\nu}{1+\nu y} \le 0 = \dfrac{dV}{dy}(\underline{y}) - \dfrac{\nu}{1+\nu \underline{y}}, \\ \dfrac{dV}{dy}(y) \ge 0 = \dfrac{dV}{dy}(\bar{y}), \end{cases}$$

in $(\underline{y}, \bar{y})$, then $\mathcal{V}$ is a viscosity solution of (8.3) in [0, 1]. Note that from (9.4–9.7), $\mathcal{V}$ is necessarily $\mathcal{C}^2$ in [0, 1] and is thus a classical solution of (8.3). We are thus looking for a $\mathcal{C}^2$ function $\mathcal{V}$ in $[\underline{y}, \bar{y}]$ solution of (9.7). Set $f(y) = d\mathcal{V}/dy$. Equation (9.7) can be rewritten as

$$\text{(9.8)} \qquad f'(y) + \frac{2(y^* - y)}{y(1-y)} f(y) = \left(\frac{2(\pi - r)}{a} - 2y^*y + y^2\right)\frac{1}{y^2(1-y)^2},$$

$$\text{(9.9)} \qquad f(y) - \frac{\nu}{1+\nu y} \le 0 = f(\underline{y}) - \frac{\nu}{1+\nu\underline{y}},$$

$$\text{(9.10)} \qquad f(y) \ge 0 = f(\bar{y}).$$

Let us integrate equation (9.8). The associated homogeneous equation writes $f'(y) = -[2(y^* - y)]/[y(1-y)]f(y)$, the solutions to which are $f(y) = Cy^{-2y^*}(1-y)^{-2(1-y^*)}$. We thus look for a solution of (9.8) of the form

$$\text{(9.11)} \qquad f(y) = \frac{g(y)}{y^{2y^*}(1-y)^{2(1-y^*)}},$$

where g is a function to determine. Plugging the expression of f in (9.8), we obtain

$$\text{(9.12)} \qquad g'(y) = y^{2y^*-2}(1-y)^{-2y^*}\left(\frac{2(\pi - r)}{a} + y^2 - 2y^*y\right),$$

integration of which leads to

$$g(y) = \begin{cases} C + \left(\dfrac{y}{1-y}\right)^{2y^*-1}\left(-y + \dfrac{2(\pi - r)}{a(2y^*-1)}\right) & \text{if } y^* \neq \frac{1}{2}, \\ C - y + \dfrac{2(\pi - r)}{a}\log\dfrac{y}{1-y} & \text{if } y^* = \frac{1}{2}, \end{cases}$$

for some constant C. Consequently, in $(\underline{y}, \bar{y})$,

$$\text{(9.13)} \quad f(y) = \frac{dV}{dy} = \begin{cases} Cy^{-2y^*}(1-y)^{2(y^*-1)} + \dfrac{1}{y(1-y)}\left(-y + \dfrac{2(\pi - r)}{a(2y^*-1)}\right) & \text{if } y^* \neq \frac{1}{2}, \\ \dfrac{1}{y(1-y)}\left(C - y + \dfrac{2(\pi - r)}{a}\log\dfrac{y}{1-y}\right) & \text{if } y^* = \frac{1}{2}. \end{cases}$$

The function f is required to satisfy the boundary conditions (9.9 and 9.10). The constants $\underline{y}$, $\bar{y}$, and π are related by (9.6); that is

$$r + (\alpha - r)\bar{y} - \frac{1}{2}a\bar{y}^2 = \pi = r + (\alpha - r)\underline{y}' - \frac{1}{2}a\underline{y}'^2.$$

Hence, using y^* and π^*,

$$\bar{y} = y^* + \sqrt{\frac{2(\pi^* - \pi)}{a}}$$

$$\underline{y}' = \frac{(1+\nu)\underline{y}}{1+\nu\underline{y}} = 2y^* - \bar{y} = y^* - \sqrt{\frac{2(\pi^* - \pi)}{a}}.$$

Conversely,

$$\frac{2(\pi - r)}{a} = y^{*2} - (\bar{y} - y^*)^2 = \bar{y}(2y^* - \bar{y}) = \bar{y}\underline{y}'. \tag{9.14}$$

Condition (9.10) leads to

$$C = \begin{cases} \dfrac{1}{1-2y^*}\bar{y}^{2y^*}(1-\bar{y})^{2(1-y^*)} & \text{if } y^* \neq \dfrac{1}{2}, \\ \bar{y}\left(1 - (1-\bar{y})\log\dfrac{\bar{y}}{1-\bar{y}}\right) & \text{if } y^* = \dfrac{1}{2}. \end{cases} \tag{9.15}$$

Using condition (9.9) with $\underline{y} = \underline{y}'/(1+\nu-\nu\underline{y}')$, we get

$$C = \begin{cases} \dfrac{\underline{y}'^{2y^*}(1-\underline{y}')^{2(1-y^*)}}{(1-2y^*)(1+\nu)^{2y^*-1}} & \text{if } y^* \neq \dfrac{1}{2}, \\ \underline{y}'\left(1 - (1-\underline{y}')\log\left(\dfrac{\underline{y}'}{(1-\underline{y}')(1+\nu)}\right)\right) & \text{if } y^* = \dfrac{1}{2}. \end{cases} \tag{9.16}$$

Conversely, if f is given by (9.13) in $(\underline{y}, \bar{y})$, with C defined in (9.15) and satisfying also condition (9.16), then f is a solution of (9.8–9.10). Indeed, it is sufficient to prove $f(y) \geq 0$ on $(\underline{y}, \bar{y})$. By symmetry, f will also satisfy $f(y) - (\nu/1+\nu y) \leq 0$ on $(\underline{y}, \bar{y})$ (see the argument in Lemma 8.1). But in view of (9.12), $g'(y) \leq 0$ on $[\underline{y}', \bar{y}]$ and $g'(y) \geq 0$ elsewhere, and in view of (9.11, 9.15, 9.16), $g(\underline{y}) > 0$ and $g(\bar{y}) = 0$. Since $\underline{y} < \underline{y}'$, we infer $g(y) \geq 0$ on $[\underline{y}, \bar{y}]$, hence $f(y) \geq 0$ on $[\underline{y}, \bar{y}]$. Using (9.14) and (9.15), we can express f (and thus $\mathcal{V}$) in terms of $\bar{y}$ and y^* only. Combining equations (9.15) and (9.16), we obtain the following equation in $\bar{y}$:

$$\begin{cases} 1+\nu = \left(\dfrac{2y^* - \bar{y}}{\bar{y}}\right)^{\frac{2y^*}{2y^*-1}}\left(\dfrac{1-2y^*+\bar{y}}{1-\bar{y}}\right)^{\frac{2(1-y^*)}{2y^*-1}} & \text{if } y^* \neq \dfrac{1}{2}, \\ \log(1+\nu) = 2\log\left(\dfrac{1-\bar{y}}{\bar{y}}\right) + \dfrac{2\bar{y}-1}{\bar{y}(1-\bar{y})} & \text{if } y^* = \dfrac{1}{2}. \end{cases} \tag{9.17}$$

Setting $\bar{y} = y^* + h$ and Taylor expanding equation (9.17) for small h and ν, we get

$$h \sim \left(\frac{3y^{*2}(1-y^*)^2}{4}\nu\right)^{1/3}$$

for all $y^* \in (0, 1)$. In other words, the size of the no-transaction region is on the order of the cubic root of the transaction cost. The same estimation was obtained in Atkinson and Wilmott (1995), but without a closed-form solution of the variational inequality. Indeed (9.17) defines ν as an analytical function of $\bar{y}$ for $|\bar{y} - y^*| < \min(y^*, 1 - y^*)$, which has a zero of multiplicity 3 in y^*. This function is positive and increasing for $\bar{y} \in (y^*, \min(2y^*, 1))$ and tends to infinity when $\bar{y}$ tends to $\min(2y^*, 1)$. This implies that $\bar{y}$ is defined by (9.17) as an increasing analytical function of $\nu^{1/3}$ for $\nu \in (0, +\infty)$.

Let us summarize the results of this section.

PROPOSITION 9.1. *Let $n = 1$, y^* be the Merton point given in* (9.3), *π the optimal growth rate for the transaction costs case, and $\mathcal{V}$ a solution of* (7.3). *We have*

$$\begin{cases} \pi = H(0) = r, \\ \mathcal{V}(y) = \mathcal{V}(1) \quad \forall y \in [0, 1] \end{cases} \qquad \textit{if } y^* = 0,$$

and

$$\begin{cases} \pi = H(1) = \alpha - \frac{a}{2}, \\ \mathcal{V}(y) = \mathcal{V}(0) + \log(1 + \nu y) \quad \forall y \in [0, 1] \end{cases} \qquad \textit{if } y^* = 1.$$

Finally, if $y^* \in (0, 1)$, π is given by (9.14) *with $\bar{y}$ defined by* (9.17) *and $\mathcal{V}$ is given by*

$$\begin{cases} \mathcal{V}(y) = \log\left(\dfrac{1 + \nu y}{1 + \nu \underline{y}}\right) + \mathcal{V}(\underline{y}) & \textit{for } y \in [0, \underline{y}], \\ \mathcal{V}(y) = \mathcal{V}(\bar{y}) & \textit{for } y \in [\bar{y}, 0], \end{cases}$$

and

$$\mathcal{V}(y) = \begin{cases} \dfrac{\bar{y}(1-\bar{y})}{(1-2y^*)^2}\left(\dfrac{y(1-\bar{y})}{(1-y)\bar{y}}\right)^{1-2y^*} + \log(1-y) & \textit{if } y^* \neq \frac{1}{2}, \\ \qquad + \dfrac{2(\pi - r)}{a(2y^* - 1)} \log \dfrac{y}{1-y} & \\ \bar{y}\left(1 - (1-\bar{y}) \log \dfrac{\bar{y}}{1-\bar{y}}\right) \log \dfrac{y}{1-y} + \log(1-y) & \textit{if } y^* = \frac{1}{2}, \\ \qquad + \dfrac{\pi - r}{a}\left(\log \dfrac{y}{1-y}\right)^2 & \end{cases}$$

for $y \in [\underline{y}, \bar{y}]$ with $\underline{y} = [2y^* - \bar{y}]/(1 + \nu(1 - 2y^* + \bar{y}))$.

9.3. Numerical Solution of the Variational Inequality

We turn to the numerical solution of the ergodic variational inequality (7.3), which is equivalent to (see Remark 6.5)

$$\max\left\{ B\mathcal{V} - \pi + H(y), \max_{\substack{1 \le i \le n \\ y_1 + \cdots + y_n \neq 1}} (P_i \mathcal{V} - \nu_i), \max_{\substack{1 \le i \le n \\ y_i \neq 0}} Q_i \mathcal{V} \right\} = 0 \quad \text{in } \Delta,$$

with B, P_i, Q_i defined in (6.13, 6.15, 6.16). We begin by performing a change of variables which transforms the simplex Δ into $[0, 1]^n$:

$$z_1 = y_1 + \cdots + y_n, \qquad z_i = \frac{y_i + \cdots + y_n}{y_{i-1} + \cdots + y_n} \quad i = 2, \ldots, n.$$

We thus obtain for $\mathcal{U}(z) = \mathcal{V}(y)$ an equation of the form:

$$\max_{a \in \mathcal{A}(z)} (C_a \mathcal{U} - \pi 1_{a=0} + G_a(z)) = 0 \quad \text{in } [0,1]^n, \tag{9.18}$$

where $0 \in \mathcal{A}(z) \subset \mathcal{A}$ is a finite set of admissible controls, $G_0(z) = H(y)$, C_0 is a second-order elliptic operator, and C_a, $a \neq 0$, are first-order operators.

Discretization. In order to solve equation (9.18), we discretize the first and second derivatives of $\mathcal{U}$ by using a consistent finite difference approximation (see Akian et al. 1996). We obtain an equation of the form

$$\max_{a \in \mathcal{A}(z)} \left(C_a^h U^h(z) - \pi^h 1_{a=0} + G_a(z) \right) = 0 \quad \text{in } \Omega_h, \tag{9.19}$$

where $h > 0$ is a discretization step and $\Omega_h \subset [0,1]^n$ is the discretization grid. The function U^h is defined on Ω_h and interpolated linearly with respect to each coordinate in $[0,1]^n$. If the discrete operators C_a^h satisfy the discrete maximum principle (DMP), then (9.19) is monotone in the sense of Barles and Souganidis (1991). Moreover (9.19) can then be interpreted as a Dynamic Programming Equation for an ergodic discrete control problem. When h is small, the associated optimal discrete process $s^h(t)$ has the same ergodicity properties as the optimal continuous process $s(t)$. Then the existence and the uniqueness of π^h and U^h (within an additive constant) can be proved by using similar arguments as in the continuous case (see Akian and Gaubert 2000). This means that equation (9.19) is stable. If this is the case and if the functions U^h are uniformly bounded, the comparison result for π (Lemma 7.5) and the arguments used in the proof of Barles and Souganidis (1991, Thm. 2.1) imply the convergence of π^h toward π as h tends to 0. Since a solution $\mathcal{U}$ of (9.18) is unique only within an additive constant, these arguments are not enough to prove the convergence of U^h. However, a solution of (9.19) has its first derivatives bounded by a constant independent of h, and thus the functions U^h are necessarily uniformly equicontinuous. If we impose in addition one of the two conditions $U^h(0) = 0$ or $\int_{[0,1]^n} U^h(z)dz = 0$, then the sequence U^h is also uniformly bounded. By the Ascoli theorem, there exists a subsequence of U^h converging uniformly toward a function U on $[0,1]^n$. The arguments of Barles and Souganidis (Thm. 2.1) imply that U is a viscosity solution of (9.18). The additional condition $U(0) = 0$ or $\int_{[0,1]^n} U(z)\,dz = 0$ is also fulfilled. If the uniqueness within an additive constant of a solution of (7.3) is proved, as we did for $n = 1$, then U^h converges uniformly toward the unique solution $\mathcal{U}$ of (9.18) such that $\mathcal{U}(0) = 0$ or $\int_{[0,1]^n} \mathcal{U}(z)dz = 0$.

In practice, we use a classical finite difference approximation scheme in a regular grid Ω_h following Kushner's approximation method (Kushner and Dupuis 1992; Akian et al. 1996). The operators C_a^h, $a \neq 0$, satisfy the DMP. But, because of the presence of mixed derivatives and the degeneracy of C_0 at the boundary, the DMP is not satisfied for C_0^h; in particular, equation (9.19) may not be stable, even for small step h. However C_0^h is the sum of a symmetric negative definite operator and an operator that satisfies the DMP; this seems to ensure the stability of (9.19) as is confirmed by numerical experiments.

Solution of the Discrete Equation. Equation (9.19) is solved by using the FMGH (Full-Multigrid-Howard) algorithm based on the "Howard algorithm" (policy iteration) and the multigrid method (see Akian 1990a, 1990b; Akian et al. 1996 for the elliptic case).

The Howard algorithm that we consider here (see Howard 1960) consists of an iteration algorithm on the control and value functions, starting from a_0 or (U_0, π_0):

$$
\begin{aligned}
&\text{for } n \geq 1 \qquad a_n(z) \in \arg\max_{a \in \mathcal{A}(z)} (C_a^h U_{n-1}(z) - \pi_{n-1} 1_{a=0} + G_a(z)) \quad \forall z \in \Omega_h, \\
&\text{for } n \geq 0, \qquad (U_n, \pi_n) \text{ is the solution of}
\end{aligned}
$$

$$
\begin{cases} C_{a_n}^h U_n - \pi_n 1_{a_n=0} + G_{a_n} = 0 \quad \text{in } \Omega_h, \\ \sum_{z \in \Omega_h} m(z) U_n(z) = 0, \end{cases} \tag{9.20}
$$

where $m(.)$ is a positive function on Ω_h with sum 1. When the operators C_a^h, $a \in \mathcal{A}$, are of the form $k(M_a^h - I)$, where $k > 0$, I is the identity matrix, and M_a^h are irreducible Markov matrices, then the sequence (π_n) increases and converges to the solution π^h of equation (9.19) and (U_n) converges toward U^h. However, in the general case, the Howard algorithm described above may not converge. It should be replaced by a more sophisticated algorithm called a "multichain policy iteration method" for which the increasing convergence of π_n toward the optimal ergodic performance π^h holds under more general conditions (see Denardo and Fox 1968).

In the Multigrid-Howard algorithm, the exact solution of the linear equation (9.20) is replaced by an iterative algorithm with initial value (U_{n-1}, π_{n-1}), consisting of multigrid iterations (see, e.g., McCormick 1987) on the functions U_n, followed by an updating of π_n such that

$$
\sum_{z \in \Omega_h,\ a_n(z)=0} C_0^h U_n(z) - \pi_n + G_0(z) = 0.
$$

The FMGH algorithm that is implemented here to solve equation (9.19) with discretization step h is recursively defined as follows: Apply a fixed number of Multigrid-Howard iterations starting from an initial value (U_0, π_0, a_0), obtained itself by interpolating the result of the FMGH algorithm with step $2h$. This algorithm solves equation (9.19) with a computing time in the order of the number of discretization points.

An Example of a Two-Risky-Assets Portfolio. Consider a portfolio with one bank account and two risky assets, and the following values of the parameters: $r = 7\%$, $\alpha = (9\%, 11\%)$, $a_{11} = (0.22)^2 = 0.048$, $a_{12} = a_{21} = 0.028$, $a_{22} = (0.32)^2 = 0.102$, $\nu_1 = \nu_2 = 1\%$ for both assets. Solving equation (7.3) for $n = 2$, we obtain $\pi = 0.0785$ which is, as expected, lower than the Merton optimal performance $\pi^* = 0.0788$. The discrete control selecting the equation which satisfies the maximum in (7.3) allows us to construct the following subsets of Δ:

$$
\begin{aligned}
\mathsf{B}_i &= \{y \in \Delta,\ P_i \mathcal{V}(y) = \nu_i\}, \\
\mathsf{S}_i &= \{y \in \Delta,\ Q_i \mathcal{V}(y) = 0\}, \\
\mathsf{NT}_i &= \Delta \setminus (\operatorname{int}(\mathsf{B}_i) \cup \operatorname{int}(\mathsf{S}_i)), \\
\mathsf{NT} &= \bigcap_{i=1}^{n} \mathsf{NT}_i,
\end{aligned}
$$

where interiors are relative to Δ. The boundaries of these sets are displayed in Figure 9.1. For simplicity we have kept here the same notation for the images of the sets defined in (7.8–7.11) by the change of variables (5.11).

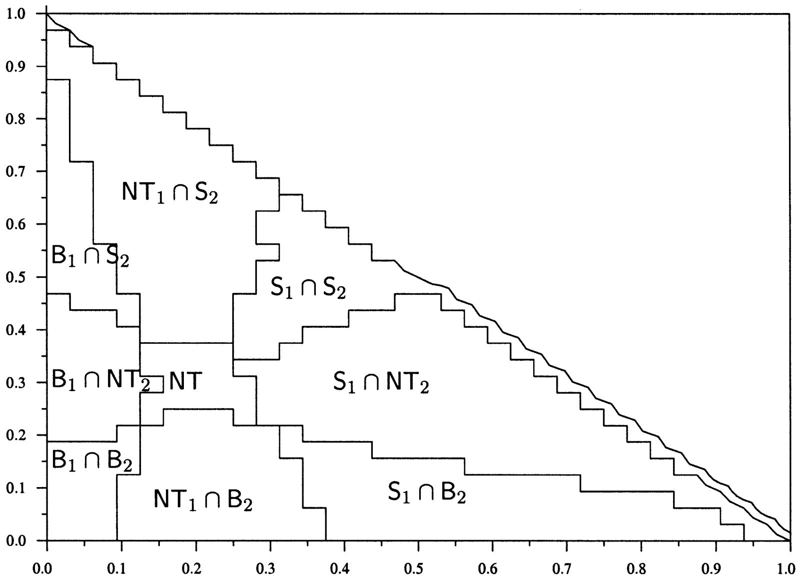

FIGURE 9.2 Optimal transaction policy for a two risky assets portfolio.

Here, the optimal Merton proportion is $y^* = (0.22, 0.33)$. As in the one risky asset case, one can prove (see Akian et al. 1996, Prop. 7.1) that y^* is located inside a polygon whose vertices are the intersections between the boundaries of the transaction regions $(\partial \mathsf{S}_1 \cap \partial \mathsf{S}_2, \partial \mathsf{S}_1 \cap \partial \mathsf{B}_2, \partial \mathsf{B}_1 \cap \partial \mathsf{S}_2, \partial \mathsf{B}_1 \cap \partial \mathsf{B}_2)$. In Figure 9.2, one can notice that y^* is also located in the no-transaction region NT.

REFERENCES

AKIAN, M. (1990a): Analyse de l'algorithme multigrille FMGH de résolution d'équations d'Hamilton-Jacobi-Bellman; in *Analysis and Optimization of Systems*, eds. A. Bensoussan and J. Lions; vol. 144 of *Lecture Notes in Control and Information Sciences*. Berlin: Springer Verlag, 113–122.

AKIAN, M. (1990b): Méthodes multigrilles en contrôle stochastique. Thèse, Université Paris-IX Dauphine, Paris.

AKIAN, M., and S. GAUBERT (2001): A Spectral Theorem for Convex Monotone Homogeneous Maps, Satellite Workshop on Max-Plus Algebras; in *Proceedings of IFAC SSSC '01*, to appear.

AKIAN, M., J. L. MENALDI, and A. SULEM (1996): On an Investment-Consumption Model with Transaction Costs, *SIAM J. Control Optim.* 34(1), 329–364.

AKIAN, M., A. SULEM, and P. SÉQUIER (1995a): A Finite Horizon Multidimensional Portfolio Selection Problem with Singular Transactions; in *Proceedings CDC*, New Orleans, IEEE.

AKIAN, M., A. SULEM, P. SÉQUIER, and A. ABOULALAA (1995b): A Finite Horizon Portfolio Selection Problem with Multi Risky Assets and Transaction Costs: The Domestic Asset Allocation Example; in *Proceedings Association Française de Finance*, Paris, IEEE.

ATKINSON, C., and P. WILMOTT (1995): Portfolio Management with Transaction Costs: An Asymptotic Analysis of the Morton and Pliska Model, *Math. Finance* 5(4), 357–367.

BARLES, G. (1994): *Solutions de viscosité des équations de Hamilton-Jacobi*; vol. 17 of *Mathématiques & Applications-SMAI*. Berlin: Springer Verlag.

BARLES, G., and B. PERTHAME (1987): Discontinuous Solutions of Deterministic Optimal Stopping Time Problems, *M2AN* 21(4), 557–579.

BARLES, G., and B. PERTHAME (1990): Comparison Principle for Dirichlet Type Hamilton-Jacobi Equations and Singular Perturbations of Degenerated Elliptic Equations, *Appl. Math. Optim.* 21, 21–44.

BARLES, G., and P. E. SOUGANIDIS (1991): Convergence of Approximation Schemes for Fully Nonlinear Second-Order Equations, *Asymptotic Analysis* 4, 271–283.

BENSOUSSAN, A. (1988): *Perturbation Methods in Optimal Control*. Chichester: Wiley/Gauthier-Villars.

BIELECKI, T., and S. PLISKA (2000): Risk Sensitive Asset Management with Transaction Costs, *Finance Stoch.* 4, 1–33.

BUCKLEY, I., and R. KORN (1997): Optimal Cash Management for Equity Index in the Presence of Fixed and Proportional Costs; preprint.

CHANCELIER, J.-P., B. ØKSENDAL, and A. SULEM (2001): Combined Stochastic Control and Optimal Stopping, with Application to Portfolio Optimization under Fixed Transaction Costs. *Math. Finance*. Moscow: Steklov Mathematical Institute.

CONSTANTINIDES, G. M. (1986): Capital Market Equilibrium with Transaction Costs, *J. Political Econ.* 94(4), 842–862.

CRANDALL, M. G., H. ISHII, and P. L. LIONS (1992): User's Guide to Viscosity Solutions of Second Order Partial Differential Equations, *Bull. Amer. Math. Soc.* 27, 1–67.

DAVIS, M., and A. NORMAN (1990): Portfolio Selection with Transaction Costs, *Math. Oper. Res.* 15, 676–713.

DENARDO, E., and B. FOX (1968): Multichain Markov Renewal Programs, *SIAM J. Appl. Math.* 16, 468–487.

DUPUIS, P., and H. ISHII (1993): SDEs with Oblique Reflection on Nonsmooth Domains, *Ann. Probab.* 21(1), 554–580.

FITZPATRICK, B., and W. H. FLEMING (1991): Numerical Methods for an Optimal Investment-Consumption Model, *Math. Operations Res.* 16, 823–841.

FLEMING, W. H., and H. M. SONER (1993): *Controlled Markov Processes and Viscosity Solutions*. New York: Springer Verlag.

FRAMSTAD, N., B. ØKSENDAL, and A. SULEM (2001): Optimal Consumption and Portfolio in a Jump Diffusion Market with Proportional Transaction Costs, *J. Math. Econ.*, forthcoming.

HOWARD, R. A. (1960): *Dynamic Programming and Markov Process*. Cambridge, Mass.: The MIT Press.

ISHII, H. (1985): Hamilton-Jacobi Equations with Discontinuous Hamiltonians on Arbitrary Open Sets, *Bull. Fac. Sci. Eng. Chuo Univ.* 28, 33–77.

ISHII, H., and P. L. LIONS (1990): Viscosity Solutions of Fully Nonlinear Second-Order Elliptic Partial Differential Equations, *J. Diff. Eq.* 83, 26–78.

KARATZAS, I., J. LEHOCZKY, S. SETHI, and S. SHREVE (1986): Explicit Solution of a General Consumption/Investment Problem, *Math. Operations Res.* 11(2), 261–294.

KORN, R. (1997): Portfolio Optimisation with Strictly Positive Transaction Costs and Impulse Control, *Finance Stoch.* 2, 85–114.

KUSHNER, H., and P. DUPUIS (1992): *Numerical Methods for Stochastic Control Problems in Continuous Time*. New York: Springer Verlag.

LIONS, P. L. (1983a): Optimal Control of Diffusion Processes and Hamilton-Jacobi-Bellman Equations. Part 1: The Dynamic Programming Principle and Applications, *Comm. Partial Diff. Eqs.* 8(10), 1101–1174.

LIONS, P. L. (1983b): Optimal Control of Diffusion Processes and Hamilton-Jacobi-Bellman Equations. Part 2: Viscosity Solutions and Uniqueness, *Comm. Partial Diff. Eqs.* 8(11), 1229–1276.

MAGILL, M. J. P., and G. M. CONSTANTINIDES (1976): Portfolio Selection with Transaction Costs, *J. Econ. Theory* 13, 245–263.

MCCORMICK, S. F., ed. (1987): *Multigrid Methods*; vol. 5 of *Frontiers in Applied Mathematics*. Philadelphia: SIAM.

MERTON, R. C. (1971): Optimum Consumption and Portfolio Rules in a Continuous Time Model, *J. Econ. Theory* 3, 373–413.

MEYER, P. A. (1976): Un cours sur les intégrales stochastiques; in *Séminaire de Probabilités*, vol. 511 of *Lectures Notes in Mathematics*. Berlin: Springer Verlag, 245–400.

MORTON, A. J., and S. PLISKA (1995): Optimal Portfolio Management with Fixed Transaction Costs. *Math. Finance* 5(4), 337–356.

ØKSENDAL, B., and A. SULEM (1999): Optimum Consumption and Portfolio with Both Fixed and Proportional Transaction Costs; preprint series 19, University of Oslo, Department of Mathematics.

SHREVE, S. E., and H. M. SONER (1991): A Free Boundary Problem Related to Singular Stochastic Control: Parabolic Case, *Comm. PDE* 16, 373–424.

SHREVE, S. E., and H. M. SONER (1994): Optimal Investment and Consumption with Transaction Costs, *Ann. Appl. Prob.* 4(3), 692–909.

SHREVE, S. E., H. M. SONER, and V. XU (1991): Optimal Investment and Consumption with Two Bonds and Transaction Costs, *Math. Finance* 1(3), 53–84.

SONER, H. (1986a): Optimal Control with State Space Constraints I, *SIAM J. Control Optim.* 24, 552–562.

SONER, H. (1986b): Optimal Control with State Space Constraints II, *SIAM J. Control Optim.* 24, 1110–1122.

SONER, H., and S. SHREVE (1989): Regularity of the Value Function of a Two-Dimensional Singular Stochastic Control Problem, *SIAM J. Control Optim.* 27(4), 876–907.

TAKSAR, M., M. J. KLASS, and D. ASSAF (1988): A Diffusion Model for Optimal Portfolio Selection in the Presence of Brokerage Fees, *Math. Operations Res.* 13, 277–294.

TOURAIN, A., and T. ZARIPHOPOULOU (1997): Viscosity Solutions and Numerical Schemes for Investment/Consumption Models with Transaction Costs; in *Numerical Methods in Finance*, eds. L. Rogers and D. Talay. Cambridge University Press, 245–269.

WHITTLE, P. (1990): *Risk-Sensitive Optimal Control*. Chichester: Wiley.

ZARIPHOPOULOU, T. (1992): Investment-Consumption Model with Transaction Fees and Markov Chain Parameters, *SIAM J. Control Optim.* 30, 613–636.

ZARIPHOPOULOU, T. (1994): Investment and Consumption Models with Constraints, *SIAM J. Control Optim.* 32, 59–84.

ZHU, H. (1991): Variational Inequalities and Dynamic Programming for Singular Stochastic Control, PhD thesis, Brown University.

ZHU, H. (1992): Generalized Solution in Singular Stochastic Control: The Nondegenerate Problem, *Appl. Math. Optim.* 25, 225–245.