Linear systems in (max,+) algebra

Max Plus*
INRIA
Le Chesnay, France

Proceedings of the 29th Conference on Decision and Control
Honolulu, Dec. 1990

Abstract

In this paper, we study the general system of linear equations in the $(\max,+)$ algebra. We introduce a *symmetrization* of this algebra and a new notion called *balance* which generalizes classical equations. This construction results in the *linear closure* of the $(\max,+)$ algebra in the sense that every non-degenerate system of linear balances has a unique solution given by Cramer's rule.

1 Introduction

The $(\max,+)$ algebra plays a crucial role in at least two fields:

- path algebra (research of the path of maximal weight in a graph).
- performance evaluation of Discrete Event Dynamic Systems (DEDS).

In this paper, we examine a fundamental problem in this algebra: solving systems of linear equations.

Let us start by introducing the notation used throughout this paper. We shall denote $\max$ by $\oplus$ (i.e. $\max(a,b)$ is noted $a \oplus b$), and use $\otimes$ instead of the usual addition $+$ (e.g. $2 \otimes 3 = 5$). $-\infty$ (also denoted by ε) is the *null* element for $\oplus$ ($x \oplus -\infty = x$) and is absorbing for the product ($-\infty \otimes x = -\infty$). 0 is the *unit element*: $0 \otimes x = x$. $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the "$(\max,+)$ algebra", or simply $\mathbb{R}_{\max}$. Usual computational rules hold in $\mathbb{R}_{\max}$ (for instance $a \otimes (b \oplus c) = (b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)$). This in particular allows us to define and manipulate vectors and matrices as usual. For simplicity, we sometimes omit $\otimes$ (we write ab instead of $a \otimes b$).

A general account of this kind of algebraic structures can be found in Gondran and Minoux [7] for the graph theoretic point of view and Cohen, Moller, Quadrat and Viot [4] for the Discrete Event Systems point of view.

For more than thirty years, it has been known that the implicit vector equation $x = A \otimes x \oplus b$ (A being a $n \times n$ matrix) can be solved by iteration, leading to the study of $A^* = \mathrm{Id} \oplus A \oplus A^2 \oplus \ldots$. Other vector equations of the type $H \otimes x = b$ (H not being necessarily a square matrix) also can be dealt with by using residuation theory (see Blyth [1]). But for the most general system of n linear equations with n unknowns

$$A \otimes x \oplus b = C \otimes x \oplus d \tag{1}$$

where A and C are $n \times n$ matrices with entries in $\mathbb{R}_{\max}$ and b,d are vectors of $(\mathbb{R}_{\max})^n$, no result existed until now to the best of the authors' knowledge. In section 2, we first explain why the general equation (1) is essential for the study of Discrete Event Systems. Then, we embed the $(\max,+)$ algebra into a *symmetrized algebra* (cf. Section 3.2), where the *balance* relation Δ plays the role of equality (Definition 3.1). The original elements are identified with *positive* elements in this new algebra. We associate the system of balances $(A \ominus C)x \;\Delta\; (d \ominus b)$ with system (1). Among the many solutions balances may have, a restricted class can be associated with solutions in the $(\max,+)$ algebra: these are *signed* solutions (i.e. positive, *negative*, or null). The main result of this paper states that *non-degenerate systems of linear balances always have a unique signed solution, given by Cramer's rule* (Theorem 6.1). When this solution is positive, it determines the unique solution of system (1).

Example 1.1 Find the solutions of:

$$\begin{cases} \max(x, y-4, 1) = \max(x-1, y+1, 2) \\ \max(x+3, y+2, -5) = \max(y+2, 7) \end{cases} \tag{2}$$

or in matrix form

$$\begin{bmatrix} 0 & -4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -\infty & 2 \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 7 \end{bmatrix} .$$

This problem is solved in Section 6.1. Before going into further details, let us make a simple remark: if $a > b$, the equation $a = x \oplus b$ is equivalent to $a = x$. This suggests the

$$\text{naive rule:} \qquad \text{“} a \ominus b = a \;\text{ if }\; a > b \text{”}. \tag{3}$$

We can now try to "solve" system (2) using this naive rule:

$$(2) \Leftrightarrow \begin{cases} (i) & x \oplus (-4)y \oplus 1 = (-1)x \oplus 1y \oplus 2 \\ (ii) & 3x \oplus 2y \oplus (-5) = 2y \oplus 7 \end{cases}$$

$$(i) \Rightarrow [0 \ominus (-1)]x = [1 \ominus (-4)]y \oplus (2 \ominus 1)$$
$$\Rightarrow (i') : \; x = 1y \oplus 2$$

$$(i') \text{ and } (ii) \;\Rightarrow\; 3(1y \oplus 2) \oplus 2y \oplus (-5) = 2y \oplus 7$$

$$\Rightarrow (4 \oplus 2 \ominus 2)y = 7 \ominus (-5) \ominus 5 \Rightarrow 4y = 7 \Rightarrow y = 3 .$$

Together with $(i)'$, we get $x = 1 \otimes 3 \oplus 2 = 4$, and it is immediate to check that $(x,y) = (4,3)$ is a solution of system (2).

Our goal is to make it clear when and why these calculations are valid.

---

*This is a collective name for a working group on Discrete Event Systems Theory, at INRIA (META2 Project) composed of Marianne Akian, Guy Cohen, Stéphane Gaubert, Ramine Nikoukhah, and Jean Pierre Quadrat. Guy Cohen is also with Ecole des Mines.
Address for correspondence:
S. Gaubert, INRIA, Domaine de Voluceau-Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France. e-mail: gaubert@seti.inria.fr *(moved to Stephane.Gaubert@inria.fr).*

2 Linear System Theory for Discrete Event Systems

To see why equations of type (1) are fundamental in the $(\max, +)$ algebra and its applications, we need to review some of the work done in developing a system theory in this algebra. In the context of DEDS, Cohen, Dubois, Moller, Quadrat and Viot (see [2, 4]) have developed a system theory, analogous to the conventional system theory for linear differential and recurrent equations. They have shown that a restricted class of deterministic Petri nets, timed event graphs, can be described by linear recurrent equations in the $\mathbb{R}_{\max}$ algebra. This class can be used to model flexible workshops, some distributed processing systems, and in particular systems involving synchronization constraints. A complete study, leading to the concepts of state-space representation and transfer function can be found in [4].

The study of general linear equations of type (1) appears to be the theoretical background needed to deal with the following interesting topics:

Notions of rank Many notions of rank and of linear dependence in vector structures over the $(\max, +)$ algebra can be found in the literature. The following is a conventional one.

Definition 2.1 *$\{u_i\}_{i\in I}$ is* free *iff the canonical map $\{\lambda_i\}_{i\in I} \mapsto \bigoplus_{i\in I} \lambda_i u_i$ is one to one.*

This condition is practically "never" fulfilled. The following notion of weak independence has been studied by Moller [9] and Wagneur [13].

Definition 2.2 *$\{u_i\}_{i\in I}$ is* weakly independent *iff no vector u_i is spanned by the others $\{u_j\}_{j\neq i}$.*

Weak independence has some "pathological" features: for instance, there exists an infinite family weakly independent in $\mathbb{R}^3_{\max}$ (see Cuninghame Green [5]).

From the early work of Gondran and Minoux [6], there is a more appealing definition of linear dependence, bearing a strong resemblance with classical linear algebra:

Definition 2.3 *$\{u_i\}_{i\in I}$ is* dependent *iff there exists a partition $I = J \cup K$ and a non-trivial family of scalars $\{\lambda_i\}$ such that $\bigoplus_{j\in J} \lambda_j u_j = \bigoplus_{k\in K} \lambda_k u_k$.*

We have: $\{u_i\}$ free $\Rightarrow$ $\{u_i\}$ non-dependent $\Rightarrow$ $\{u_i\}$ weakly independent. In the case of Definition 2.3, the column vector defined by $\Lambda = (\lambda_i)$ is solution of an equation of the type $U\Lambda = U'\Lambda$ which is a homogeneous form of equation (1).

Controllability, Observability It is well known that the only invertible matrices with entries in $\mathbb{R}_{\max}$ can be written as $M = DS$ where D is a diagonal matrix with invertible entries and S is a permutation matrix (see e.g. [5]). This means that endomorphisms of $(\mathbb{R}_{\max})^n$ are "never" invertible. As a consequence, defining an effective notion of controllability or observability is far from being obvious and the only notions studied so far are structural (see [3]). A sharper theory should be based on the notion 2.3 of dependence.

Minimality Some attempts (cf. Olsder [10]) have already been made suggesting that minimal realizations may be related to "two-sided" ARMA models:

$$\begin{aligned} &Y(n) \oplus A_1^+ Y(n-1) \oplus \ldots \oplus A_k^+ Y(n-k) \oplus B_0^- U(n) \oplus \ldots \\ &\ldots \oplus B_k^- U(n-k) = A_1^- Y(n-1) \oplus \ldots \oplus A_k^- Y(n-k) \oplus \\ &\qquad \oplus B_0^+ U(n) \oplus \ldots \oplus B_k^+ U(n-k) \end{aligned}$$

which clearly have the form of equation (1).

3 Symmetrizing the $(\max, +)$ algebra

A natural approach to our problem may have been to embed the $(\max, +)$ algebra into a structure in which every non-trivial scalar linear equation has at least one solution. Indeed, solving $a \oplus x = \varepsilon$ means *symmetrizing* a. Because $\max$ is idempotent (i.e. $a \oplus a = \max(a, a) = a$), the following remark shows how hopeless it is to adapt the classical symmetrization of monoids (e.g. the way we build $\mathbb{Z}$ from $\mathbb{N}$) to the $\mathbb{R}_{\max}$ context:

Proposition 3.1 *Every idempotent group is reduced to the null element.*

Proof Assume the group $(\mathcal{G}, \oplus)$ is idempotent with null element ε. Let b be the symmetric element of $a \in \mathcal{G}$. Then $a = a \oplus \varepsilon = a \oplus (a \oplus b) = (a \oplus a) \oplus b = a \oplus b = \varepsilon$. ■

3.1 The algebra of pairs

Let us now consider the set of pairs $\mathbb{R}^2_{\max}$ endowed with the natural *dioid*[1] structure:

$$\begin{aligned} (x', x'') \oplus (y', y'') &= (x' \oplus y, x'' \oplus y'') \\ (x', x'') \otimes (y', y'') &= (x'y' \oplus x''y'', x'y'' \oplus x''y') \end{aligned}$$

with $(\varepsilon, \varepsilon)$ as the null element and $(0, \varepsilon)$ as the unit element.

Let $x = (x', x'')$ and define the *minus* sign as $\ominus x = (x'', x')$. The *absolute value* of x is denoted by $|x| = x' \oplus x''$. The *balance operator* $(\cdot)^\bullet$ is defined by $x^\bullet = x \ominus x = (|x|, |x|)$. It is immediate to check that $|\ |$ and $^\bullet$ are dioid morphisms (respectively onto $\mathbb{R}_{\max}$ and the diagonal of $\mathbb{R}^2_{\max}$). In addition, we have the following properties:

$$\begin{array}{lll} (i) & & a^\bullet = (\ominus a)^\bullet \\ (ii) & \text{idempotence} & a^{\bullet\bullet} = a^\bullet \\ (iii) & \text{absorpsion} & ab^\bullet = (ab)^\bullet \\ (iv) & \text{involution} & \ominus(\ominus a) = a \\ (v) & \text{additive morphism} & \ominus(a \oplus b) = (\ominus a) \oplus (\ominus b) \\ (vi) & \text{sign rule} & \ominus(a \otimes b) = (\ominus a) \otimes b\ . \end{array} \quad (4)$$

In particular $(iv) - (vi)$ allow us to write as usual $a \oplus (\ominus b) = a \ominus b$.

[1] Recall that a *dioid* is a set $\mathcal{D}$ together with two laws $\oplus$ and $\otimes$ such that (i): $(\mathcal{D}, \oplus)$ is associative, commutative, idempotent (i.e. $\forall a \quad a \oplus a = a$), with *null* element ε, (ii): $(\mathcal{D}, \otimes)$ is associative with *unit* element e, (iii): product is distributive over addition and (iv): the null element is absorbing ($\forall a \quad a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$).

3.2 Quotient structure

Definition 3.1 [Balance relation] *Let $x = (x', x'')$ and $y = (y', y'')$. We say that x balances y (denoted by $x \ \Delta \ y$) if and only if $x' \oplus y'' = x'' \oplus y'$.*

It is fundamental to notice that Δ is *not* transitive. For instance, consider $(0,1) \ \Delta \ (1,1)$, $(1,1) \ \Delta \ (1,0)$ but $(0,1) \not\Delta (1,0)$! Since Δ cannot be an equivalence relation, there is no point to define the quotient structure of $\mathbb{R}^2_{\max}$ by Δ (opposed to the conventional algebra in which $\mathbb{N}^2/\Delta \simeq \mathbb{Z}$). However, we can introduce a new relation $\mathcal{R}$ on $\mathbb{R}^2_{\max}$

$$(x', x'') \ \mathcal{R} \ (y', y'') \Leftrightarrow \begin{cases} x' \neq x'', y' \neq y'', \text{and } x' \oplus y'' = x'' \oplus y' \\ (x', x'') = (y', y'') \quad \text{otherwise} \end{cases}$$

which is an equivalence relation, stronger than Δ. It is easy to check that $\mathcal{R}$ is compatible with the structure laws of $\mathbb{R}^2_{\max}$, with the balance relation Δ and also with the $\ominus$, $|\ |$ and $^\bullet$ operators.

Definition 3.2 *We set $\mathbb{S}_{\max} = \mathbb{R}^2_{\max}/_{\mathcal{R}}$ and we call it the* symmetrized algebra *of* $\mathbb{R}_{\max}$.

We distinguish three kinds of equivalence classes:

$$\begin{array}{ll} \overline{(t, -\infty)} = \{(t, x'');\ x'' < t\} & \text{called positive} \\ \overline{(-\infty, t)} = \{(x', t);\ x' < t\} & \text{called negative} \\ \overline{(t, t)} = \{(t, t)\} & \text{called balanced.} \end{array}$$

By associating $\overline{(t, -\infty)}$ with $t \in \mathbb{R}_{\max}$, we can identify $\mathbb{R}_{\max}$ with the subdioid of positive or null classes, $\mathbb{R}^\oplus_{\max}$. The set of negative or null classes (of the form $\ominus x$ for $x \in \mathbb{R}^\oplus_{\max}$) will be denoted by $\mathbb{R}^\ominus_{\max}$, the set of balanced classes (of the form $x^\bullet$) by $\mathbb{R}^\bullet_{\max}$. This yields the decomposition

$$\mathbb{S}_{\max} = \mathbb{R}^\oplus_{\max} \cup \mathbb{R}^\ominus_{\max} \cup \mathbb{R}^\bullet_{\max} \tag{5}$$

ε being the *only* element common to any two of these three sets. This should be compared with $\mathbb{Z} = \mathbb{N}^+ \cup \mathbb{N}^-$.

These conventions allow us to write $3 \ominus 2$ instead of $\overline{(3, -\infty)} \oplus \overline{(-\infty, 2)}$. We thus have $3 \ominus 2 = \overline{(3,2)} = \overline{(3, -\infty)} = 3$. More generally, calculations in $\mathbb{S}_{\max}$ can be summarized as follows

$$\begin{array}{ll} a \ominus b = a & \text{if } a > b \\ b \ominus a = \ominus a & \text{if } a > b \\ a \ominus a = a^\bullet \ . & \end{array} \tag{6}$$

This includes and generalizes the initial naive rule (3).

Because of its importance, we introduce the notation $\mathbb{R}^\vee_{\max}$ for the set $\mathbb{R}^\oplus_{\max} \cup \mathbb{R}^\ominus_{\max}$. The elements of $\mathbb{R}^\vee_{\max}$ are called *signed* elements. They are either positive, negative or null. We have:

Proposition 3.2 $\mathbb{R}^\vee_{\max} \setminus \{\varepsilon\} = \mathbb{S}_{\max} \setminus \mathbb{R}^\bullet_{\max}$ *is the set of all invertible elements of* $\mathbb{S}_{\max}$.

Proof $t \otimes (-t) = (\ominus t) \otimes (\ominus - t) = 0$ for $t \in \mathbb{R}_{\max} \setminus \{\varepsilon\}$ obviously shows that every non-null element of $\mathbb{R}^\vee_{\max}$ is invertible. Moreover, formula (4,(iii)) shows that $\mathbb{R}^\bullet_{\max}$ is absorbing for the product. Thus, $x^\bullet y \neq 0$ for all $y \in \mathbb{S}_{\max}$ since $0 \notin \mathbb{R}^\bullet_{\max}$. ∎

Remark 3.1 It can be proved that $\mathcal{R}$ is the weakest equivalence relation stronger than Δ. In this sense $\mathcal{R}$ is natural.

Remark 3.2 There is a nicer algebraic way to introduce the relation $\mathcal{R}$. Let $\text{Sol}(a) = \{x \in \mathbb{R}^2_{\max};\ x \ \Delta \ a\}$, then it can easily be verified that

$$x \ \mathcal{R} \ y \iff \text{Sol}(x) = \text{Sol}(y) \tag{7}$$

which makes it clear that $\mathcal{R}$ is an equivalence relation. This leads to a very simple proof of the compatibility of $\mathcal{R}$ with addition. Because the following propositions are equivalent:

$$\begin{array}{l} x \in \text{Sol}(a \oplus c) \\ x \ \Delta \ a \oplus c \\ x \ominus c \ \Delta \ a \\ x \ominus c \in \text{Sol}(a) \end{array}$$

$\text{Sol}(a) = \text{Sol}(b)$ implies that $\text{Sol}(a \oplus c) = \text{Sol}(b \oplus c)$.

Remark 3.3 The equivalent formulation (7) allows extending *symmetrization* to more general dioids than the $(\max, +)$ algebra. In fact, it is always possible to define the quotient of the additive monoid of pairs by the map $a \mapsto \text{Sol}(a)$, but this quotient may not be compatible with multiplication! What is specific to the total order structure of $\mathbb{R}_{\max}$ is the decomposition (5). Since our goal here is to give an existence and uniqueness theorem, we only consider the case of a totally ordered multiplicative group, the generic case of which being the $(\max, +)$ algebra. But a more general theory can be developed along the same lines.

4 Linear balances

4.1 General properties

Before solving general linear balances, we need to explain why balances in $\mathbb{S}_{\max}$ generalize equations in $\mathbb{R}_{\max}$. The main algebraic features of balances are:

Properties 4.1

(i)	$a \ \Delta \ a$	(reflexivity)
(ii)	$a \ \Delta \ b \Leftrightarrow b \ \Delta \ a$	(symmetry)
(iii)	$a \ \Delta \ b \ \Leftrightarrow \ a \ominus b \ \Delta \ \varepsilon$	
(iv)	$a \ \Delta \ b, \ c \ \Delta \ d \ \Rightarrow \ a \oplus c \ \Delta \ b \oplus d$	
(v)	$a \ \Delta \ b \ \Rightarrow \ ac \ \Delta \ bc$	

Let us prove (v): $a \ \Delta \ b \ \Leftrightarrow a \ominus b \in \mathbb{R}^\bullet_{\max}$ and as $\mathbb{R}^\bullet_{\max}$ is absorbing, $(a \ominus b)c = ac \ominus bc \in \mathbb{R}^\bullet_{\max}$, i.e. $ac \ \Delta \ bc$. ∎

Although Δ is not transitive, when some variables are signed, we can manipulate balances in the same way as we manipulate equations:

Property 4.2 [Weak substitution]

$$\begin{cases} x \ \Delta \ a \\ cx \ \Delta \ b \end{cases} \text{and } x \in \mathbb{R}^\vee_{\max} \Rightarrow ca \ \Delta \ b$$

Proof We have $x \in \mathbb{R}^\oplus_{\max}$ or $x \in \mathbb{R}^\ominus_{\max}$. Assume for instance that $x \in \mathbb{R}^\oplus_{\max}$, $x = \overline{(x', \varepsilon)}$. With obvious notations: $x' \oplus a'' = a'$ and $c'x' \oplus b'' = c''x' \oplus b'$. Adding $c'a'' \oplus c''a''$ to the last equality, we get $c'x' \oplus c'a'' \oplus c''a'' \oplus b'' = c''x' \oplus c'a'' \oplus c''a'' \oplus b'$, which yields $c'a' \oplus c''a'' \oplus b'' = c''a' \oplus c'a'' \oplus b'$, i.e. $ca \ \Delta \ b$. ∎

By taking $c = 0$, the weak substitution property 4.2 becomes:

Property 4.3 [Weak transitivity]
$a \,\nabla\, x,\; x \,\nabla\, b$ and $x \in \mathbb{R}^{\vee}_{\max} \Rightarrow a \,\nabla\, b$

We conclude by a simple remark which allows translating balances into equalities:

Property 4.4 [Reduction of balances]
$x \,\nabla\, y$ and $(x, y) \in (\mathbb{R}^{\vee}_{\max})^2 \quad \Rightarrow x = y$

It is immediate to extend balances to the vector case. Properties (4.1,$i - v$), 4.2, 4.3 and 4.4 still hold when a, b, x, y and c are matrices with appropriate dimensions, provided we replace "$\in \mathbb{R}^{\vee}_{\max}$" by "every entry $\in \mathbb{R}^{\vee}_{\max}$". Therefore, we say a vector is *signed* iff every entry is signed.

4.2 From equations to balances

We now consider a solution x of the equation (1) in $\mathbb{R}_{\max}$. We have $Ax \oplus b \,\nabla\, Cx \oplus d$ (reflexivity), and by (4.1,iii):

$$(A \ominus C)x \oplus (b \ominus d) \,\nabla\, \varepsilon \;. \tag{8}$$

Conversely, assuming that x is a positive solution of (8), we get $Ax \oplus b \,\nabla\, Cx \oplus d$ with $Ax \oplus b$ and $Cx \oplus d \in \mathbb{R}^{\oplus}_{\max} \subset \mathbb{R}^{\vee}_{\max}$. Using 4.4, we get $Ax \oplus b = Cx \oplus d$. So, we have:

Proposition 4.5 *The set of solutions of the general linear system of equations (1) in $\mathbb{R}_{\max}$ and the set of* positive *solutions of the associated linear balance (8) in $\mathbb{S}_{\max}$ coincide.*

Thus, the original problem reduces to studying linear balances in $\mathbb{S}_{\max}$.

Remark 4.1 The case where a solution x of (8) has some negative and some positive entries is also of interest. We write $x = x^+ \ominus x^-$ with $x^+, x^- \in (\mathbb{R}^{\oplus}_{\max})^n$. Partitioning the columns of A and C according to the sign of the entries of x: $A = A^+ \oplus A^-$,$C = C^+ \oplus C^-$ (in such a way that $Ax = A^+x^+ \ominus A^-x^-$ and $Cx = C^+x^+ \ominus C^-x^-$), we can affirm the existence of a solution to the new problem

$$A^+x^+ \oplus C^-x^- \oplus b = A^-x^- \oplus C^+x^+ \oplus d \;.$$

4.3 The scalar linear balance

Theorem 4.6 *Let $a \in \mathbb{R}^{\vee}_{\max} \setminus \{\varepsilon\}$ and $b \in \mathbb{R}^{\vee}_{\max}$, then the balance*

$$ax \oplus b \,\nabla\, \varepsilon \tag{9}$$

has the unique signed solution: $x^{\flat} = \ominus a^{-1}b$.

Proof From properties 4.1,v and 4.1,iii, $ax \oplus b \,\nabla\, \varepsilon$ is equivalent to $x \,\nabla\, \ominus a^{-1}b$. Using the reduction property 4.4 and $\ominus a^{-1}b \in \mathbb{R}^{\vee}_{\max}$, we get $x = \ominus a^{-1}b$. ∎

Remark 4.2 Non-trivial linear balances always have solutions in $\mathbb{S}_{\max}$, that is why $\mathbb{S}_{\max}$ may be considered as a *linear closure* of $\mathbb{R}_{\max}$.

Remark 4.3 We can describe all the solutions of (9). For all $t \in \mathbb{R}_{\max}$, we have obviously $at^{\bullet} \,\nabla\, \varepsilon$. Adding this balance to $ax^{\flat} \oplus b \,\nabla\, \varepsilon$, we get $a(x^{\flat} \oplus t^{\bullet}) \oplus b \,\nabla\, \varepsilon$. Thus,

$$x_t = x^{\flat} \oplus t^{\bullet} \tag{10}$$

is solution of (9). If $t \geq |x^{\flat}|$, then $x_t = t^{\bullet}$ is balanced. Conversely, it can be checked that every solution of (9) may be written as in (10). The *unique* signed solution $x^{\flat}$ is also the least solution.

Remark 4.4 If $b \notin \mathbb{R}^{\vee}_{\max}$, we lose uniqueness of signed solutions. Every x such that $|ax| \leq |b|$ (i.e. $|x| \leq |a^{-1}b|$) is solution of balance (9).

Remark 4.5 If $a \notin \mathbb{R}^{\vee}_{\max}$, we again lose uniqueness. Assume $b \in \mathbb{R}^{\vee}_{\max}$ (otherwise, the balance holds for all value of x), then every x such that $|ax| \geq |b|$ is a solution.

5 A fundamental identity

Before dealing with general systems, we need to extend the determinant machinery to the $\mathbb{S}_{\max}$ context. We define the sign of a permutation σ by $\mathrm{sgn}(\sigma) = 0$ if σ is even and $\mathrm{sgn}(\sigma) = \ominus 0$ if σ is odd. Then the determinant of an $n \times n$ matrix $A = (a_{i,j})$ is given (as usual) by

$$\det A = \bigoplus_{\sigma} \mathrm{sgn}(\sigma) \bigotimes_{i=1}^{n} a_{i,\sigma(i)} \;.$$

$\det$ remains an n-linear antisymmetric function of the rows (or columns). $\det A$ is balanced (but non-null in general) if two rows (or columns) of the matrix A are identical. We denote by A^{adj} the transpose of the matrix of cofactors ($[A^{\mathrm{adj}}]_{i,j} = \mathrm{cof}_{j,i}(A)$), and by Id the identity matrix (with 0 on the diagonal and ε elsewhere). The following is just a "combinatorial" identity, that can be shown by adapting a result by Reutenauer & Straubing [12] or the usual demonstration:

Theorem 5.1 $AA^{\mathrm{adj}} \;\nabla\; \det A\,.\mathrm{Id}\;.$

Remark 5.1 The formulation of Reutenauer and Straubing consists in defining a "positive determinant" $\det^+ A$ (where the sum is taken over all even permutations) and a "negative" determinant $\det^- A$ (odd permutations). The matrix of "positive" cofactors is defined by

$$[A^{\mathrm{adj}+}]_{i,j} = \begin{cases} \det^+ A(j|i) & \text{if } i+j \text{ even} \\ \det^- A(j|i) & \text{if } i+j \text{ odd} \end{cases}$$

where $A(i|j)$ denotes the matrix A from which row i and column j are removed, and the matrix of "negative" cofactors $A^{\mathrm{adj}-}$ is defined similarly. With these notations, Theorem 5.1 can be rewritten as follows:

$$AA^{\mathrm{adj}+} \oplus \det{}^- A.\mathrm{Id} = AA^{\mathrm{adj}-} \oplus \det{}^+ A.\mathrm{Id}\;.$$

This formula does not use the $\ominus$ sign and is valid in any semiring. The symmetrized algebra appears as a natural way of handling (and proving in an algebraic way) such identities.

6 Solving systems of linear balances

6.1 Cramer's rule

Because of the remarks of section 4, we only consider the solutions of balances in $(\mathbb{R}^{\vee}_{\max})^n$, that is *signed* solutions. We can now state the fundamental result for the existence and uniqueness of signed solutions of linear systems:

Theorem 6.1 (Cramer system) *Let A be an $n \times n$ matrix with entries in $\mathbb{S}_{\max}$ and $b \in (\mathbb{S}_{\max})^n$. Then every signed solution of*

$$Ax \;\nabla\; b \tag{11}$$

satisfies:

$$\det A \,.x \;\nabla\; A^{\mathrm{adj}} b \;. \tag{12}$$

Conversely, assume that $A^{\mathrm{adj}} b$ is signed and $\det A$ is invertible, then the "Cramer solution" $x^\flat = (\det A)^{-1} A^{\mathrm{adj}} b$ is the unique signed solution of (11).

Proof Assume $\det A$ is invertible and $A^{\mathrm{adj}} b$ is signed. By right-multiplying the identity $AA^{\mathrm{adj}} \;\nabla\; \det A \;\mathrm{Id}$ by $(\det A)^{-1} b$ we easily see that the Cramer signed solution $x^\flat$ satisfies (11). This proves the converse implication. We shall consider the direct implication only when $\det A$ is invertible. The proof is by induction on the size of the matrix. Let us prove (12) for the last row, i.e. $\det A \, x_n \;\nabla\; (A^{\mathrm{adj}} b)_n$. Developing with respect to the last column, $\det A = \bigoplus_{k=1}^n a_{k,n} \mathrm{cof}_{k,n}(A)$ we get that at least one term is invertible, say $a_{1,n}\mathrm{cof}_{1,n}(A)$. We now partition in an obvious way A, b and x:

$$A = \begin{bmatrix} A_{1,1} & a_{1,n} \\ A' & A_{n,n} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b' \end{bmatrix}, \quad x = \begin{bmatrix} x' \\ x_n \end{bmatrix}$$

where $A_{1,1}$ is an $1 \times (n-1)$ matrix, A' is an $(n-1)\times(n-1)$ matrix, etc...

$$Ax \;\nabla\; b \iff \begin{cases} A_{1,1}x' \oplus a_{1,n}x_n \;\nabla\; b_1 & (\alpha) \\ A'x' \oplus A_{n,n}x_n \;\nabla\; b' & (\beta) \end{cases}$$

Since $\det A' = (\ominus 0)^{n+1}\mathrm{cof}_{1,n}(A)$ is invertible, we apply the induction hypothesis to

$$(\beta) \Leftrightarrow (\beta') : \quad A'x' \;\nabla\; b' \ominus A_{n,n}x_n$$

which implies that $x' \;\nabla\; (\det A')^{-1} A'^{\mathrm{adj}}(b' \ominus A_{n,n}x_n)$. Using the weak substitution property 4.2, we replace $x' \in (\mathbb{R}^\vee_{\max})^{n-1}$ in (α):

$$A_{1,1}(\det A')^{-1} A'^{\mathrm{adj}}(b' \ominus A_{n,n}x_n) \oplus a_{1,n}x_n \;\nabla\; b_1$$

i.e.

$$\begin{aligned} &[\det A' \,.a_{1,n} \ominus A_{1,1}A'^{\mathrm{adj}}A_{n,n}]x_n \;\nabla \\ &\qquad \det A' \,.b_1 \ominus A_{1,1}A'^{\mathrm{adj}}b' \;. \end{aligned}$$

Here, we recognize the developments of $(\ominus 0)^{n+1}\det A$ and $(\ominus 0)^{n+1}(A^{\mathrm{adj}}b)_n$. Thus

$$\det A \,.x_n \;\nabla\; (A^{\mathrm{adj}}b)_n \;.$$

Since the same result holds when developing with respect to any column k other than n, this concludes the proof. ∎

Remark 6.1 Let D_{x_i} be the determinant of the matrix obtained by replacing the i-th column of A with b, then $(A^{\mathrm{adj}}b)_i = D_{x_i}$. Assume $\det A$ invertible, then the equation (12) is equivalent to:

$$(\forall i) \;\; x_i \;\nabla\; (\det A)^{-1} D_{x_i} \;.$$

If $A^{\mathrm{adj}}b \in (\mathbb{R}^\vee_{\max})^n$, then $x_i = (\det A)^{-1}D_{x_i}$, which is exactly the classical i-th Cramer formula.

Example 6.1 Let us go back to our original problem (Example 1). The balance corresponding to equation (2) is

$$\begin{bmatrix} 0 & \ominus 1 \\ 3 & 2^\bullet \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \nabla \begin{bmatrix} 2 \\ 7 \end{bmatrix} \tag{13}$$

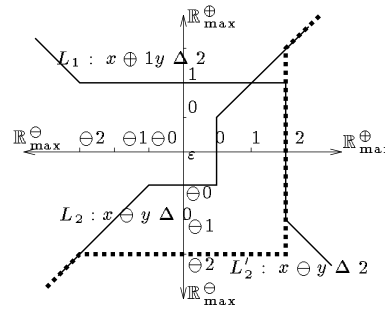
with determinant $D = 4$ (invertible).

$$D_x = \begin{vmatrix} 2 & \ominus 1 \\ 7 & 2^\bullet \end{vmatrix} = 8, \quad D_y = \begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} = 7$$

$$A^{\mathrm{adj}}b = \begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix} \in (\mathbb{R}^\vee_{\max})^2 \;.$$

So, $x = \frac{D_x}{D} = 8 - 4 = 4, y = \frac{D_y}{D} = 7 - 4 = 3$ gives the unique positive solution in $\mathbb{S}_{\max}$ of balance (13). Thus, it is the unique solution of equation (2) in $\mathbb{R}_{\max}$.

Example 6.2 In the two dimensional case, the condition $A^{\mathrm{adj}}b \in (\mathbb{R}^\vee_{\max})^n$ has a very clear geometric interpretation. The following picture represents the solutions to balances in the plane of signed coordinates $(\mathbb{R}^\vee_{\max})^2$.



From Theorem 6.1, we easily see that the two lines L_1 and L_2 meet at a single point: $(1, 1)$. However, L'_2, which is "parallel" to L_2 has a degenerate intersection with L_1 because the second Cramer determinant of system L_1, L'_2 is balanced.

Remark 6.2 $\det A$ being invertible is *not* a necessary condition for system $Ax \;\nabla\; b$ to have a signed solution for all values of b ! Consider

$$A = \begin{bmatrix} 0 & 0 & \varepsilon \\ 0 & 0 & \varepsilon \\ 0 & 0 & \varepsilon \end{bmatrix}$$

$\det A = \varepsilon$. Let $t \in \mathbb{R}^\vee_{\max}$ such that $|b_i| \le |t|$ for all coordinate i, and let $x = \begin{bmatrix} t & \ominus t & \varepsilon \end{bmatrix}^t$. Then $Ax \;\nabla\; b$.

Remark 6.3 As already noticed by Gondran and Minoux (see [6]), determinants have a natural interpretation in terms of assignment problems. So the Cramer computations have the same complexity as $n+1$ assignment problems, which can be solved using flow algorithms.

6.2 General case

We can even solve $Ax \;\nabla\; b$ in some degenerate cases:

Theorem 6.2 *Assume that $\det A \neq \varepsilon$ (but possibly $\det A \;\nabla\; \varepsilon$) then for all values of b there exists a signed solution x of $Ax \;\nabla\; b$ such that $|x| = |\det A|^{-1}|A^{\mathrm{adj}}b|$.*

It is remarkable that the classical Gauss-Seidel and Jacobi algorithms can be adapted to the $\mathbb{S}_{\max}$ case, for which we have convergence after n iterations (!). This in particular provides an algorithmic proof of Theorem 6.2. We write $A = D \oplus U \oplus L$, with U upper-triangular, L lower-triangular and D diagonal. Let us introduce the notation "$x \;|\nabla|\; y$" for $x \nabla y$ and $|x| = |y|$. We now state:

Theorem 6.3 [Jacobi algorithm] *Assume the domination property* $|\det A| = |\bigotimes_{i=1}^{n} a_{i,i}| \neq \varepsilon$ *then:*
1/ There exists a (perhaps non-unique) sequence of signed vectors $\{x^p\}$ *such that*
$(i) \quad \varepsilon = x^0 \leq x^1 \leq \ldots \leq x^p \leq \ldots$
$(ii) \quad Dx^{p+1} \;|\nabla|\; \ominus (U \oplus L)x^p \oplus b$.
2/ Such a sequence is stationary after n *iterations (*$x^n = x^{n+1} = \ldots$*) and* x^n *is a solution of* $Ax \;\nabla\; b$
3/ $|x^n| = |\det A|^{-1}|A^{\mathrm{adj}}b|$.

Sketch of proof 1/ can be shown by an induction argument which is omited due to the lack of space. To understand why x^p is stationary, we introduce $\hat{x}^p = |x^p|$. (ii) yields $\hat{x}^{p+1} = M\hat{x}^{p+1} \oplus |D|^{-1}|b|$ with $M = |D|^{-1}|U \oplus L|$. We have $\hat{x}^{p+1} = (\mathrm{Id} \oplus M \oplus \ldots \oplus M^p)|D|^{-1}|b|$. The domination hypothesis implies that M has *no circuits with weight* > 0, which implies that the series $M^* = \mathrm{Id} \oplus M \oplus M^2 \oplus \ldots \oplus M^p \oplus \ldots$ is stationary after step $n-1$, i.e. $\hat{x}^n = \hat{x}^{n+1} = \ldots$ (this is a classical result, cf. [7], p.72, Theorem 1). Because x^n and x^{n+1} are signed, $x^n \leq x^{n+1}$ and $|x^n| = |x^{n+1}|$ imply $x^n = x^{n+1}$. Replacing in (ii), we get $Dx^n \;\nabla\; \ominus (U \oplus L)x^n \oplus b$ which is equivalent to $Ax^n \;\nabla\; b$. This concludes the proof of $2/$. Statement $3/$ follows:

Lemma 6.4 $(|D|^{-1}|U \oplus L|)^*|D|^{-1} = |\det A|^{-1}|A^{\mathrm{adj}}|$.

This can be deduced from a theorem due to Yoeli ([14], Theorem 4). ∎

Example 6.3 We apply the Jacobi algorithm to

$$\begin{bmatrix} 5 & \ominus 0 & 3 \\ 1 & 3 & \ominus 1 \\ 3 & \ominus 2 & 1^\bullet \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\nabla\; \begin{bmatrix} \ominus 1 \\ 4 \\ 0 \end{bmatrix}$$

with $|\det A|^{-1}|A^{\mathrm{adj}}b| = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^t$.

$$\begin{cases} 5x_1^1 \;|\nabla|\; \ominus 1 \\ 3x_2^1 \;|\nabla|\; 4 \\ 1^\bullet x_3^1 \;|\nabla|\; 0 \end{cases} \Rightarrow \begin{cases} x_1^1 = \ominus -4 \\ x_2^1 = 1 \\ x_3^1 = -1 \text{ or } \ominus -1, \text{ say } x_3^1 = -1 \end{cases}$$

$$\begin{cases} 5x_1^2 \;|\nabla|\; 0x_2^1 \ominus 3x_3^1 \ominus 1 = \ominus 2 \\ 3x_2^2 \;|\nabla|\; \ominus 1x_1^1 \oplus 1x_3^1 \oplus 4 = 4 \\ 1^\bullet x_3^2 \;|\nabla|\; \ominus 3x_1^1 \oplus 2x_2^1 \oplus 0 = 3 \end{cases} \Rightarrow \begin{cases} x_1^2 = \ominus -3 \\ x_2^2 = 1 \\ x_3^2 = 2 \text{ or } \ominus 2, \text{ say } x_3^2 = 2 \end{cases}$$

$$\begin{cases} 5x_1^3 \;|\nabla|\; \ominus 5 \\ 3x_2^3 \;|\nabla|\; 4 \\ 1^\bullet x_3^3 \;|\nabla|\; 3 \end{cases} \text{and } x_3^3 \geq x_3^2 \Rightarrow \begin{cases} x_1^3 = \ominus 0 \\ x_2^3 = 1 \\ x_3^3 = 2 \end{cases}$$

Different choices for x_3^1 and x_3^2 yield another solution: $x_1^3 = 0, x_2^3 = 1, x_3^3 = \ominus 2$.

For the homogeneous system, the analogy with the classical situation is complete. The following result generalizes a theorem of Gondran and Minoux [6]:

Theorem 6.5 [Homogeneous case] *Let* A *be an* $n \times n$ *matrix with entries in* $\mathbb{S}_{\max}$*. Then the equation* $Ax \;\nabla\; \varepsilon$ *has a signed non-null solution if and only if* $\det A \;\nabla\; \varepsilon$.

References

[1] T.S. Blyth. “Matrices over ordered algebraic structures”, *J. of London Mathematical Society*, Vol. 39, pp. 427–432, 1964.

[2] G. Cohen, D. Dubois, J.P. Quadrat and M. Viot. “Analyse du comportement périodique des systèmes de production par la théorie des dioides”, *INRIA Report No. 191*, Le Chesnay, France, 1983.

[3] G. Cohen, P. Moller, J.P. Quadrat and M. Viot. “Linear system theory for discrete-event systems”, *23rd IEEE Conf. on Decision and Control*, Las Vegas, Nevada, 1984.

[4] G. Cohen, P. Moller, J.P. Quadrat and M. Viot. “Algebraic Tools for the Performance Evaluation of Discrete Event Systems”, *IEEE Proceedings: Special issue on Discrete Event Systems*, 77, No. 1, 1989.

[5] R.A. Cuninghame-Green, “*Minimax Algebra*”, Lectures notes in Economics and Mathematical Systems No. 166, Springer, Berlin, 1979.

[6] M. Gondran and M. Minoux, “L’indépendance linéaire dans les dioïdes”, *E.D.F., Bulletin de la Direction des Etudes et Recherches*, Série C, Mathématiques, Informatique, No. 1, pp. 67–90, 1978.

[7] M. Gondran and M. Minoux, *“Graphes et algorithmes”*, Eyrolles, Paris, France, 1979.

[8] M. Plus, “L’algèbre $(\max, +)$ et sa symétrisation ou l’algèbre des équilibres”, *Comptes Rendus à l’Académie des Sciences*, Section Automatique, 1990 (to appear).

[9] P. Moller, “Notions de rang dans les dioides vectoriels”, *in* “*Algèbres Exotiques et Systèmes à Evénements Discrets*”, Séminaire CNRS/CNET/INRIA, 3-4 Juin 1987, CNET Issy les Moulineaux.

[10] G.J. Olsder, R.E. de Vries. “On an analogy of minimal realizations in conventional and discrete-event dynamic systems”, *in “Algèbres Exotiques et Systèmes à Evénements Discrets”*, Séminaire CNRS/CNET/INRIA, 3-4 Juin 1987, CNET Issy les Moulineaux.

[11] G.J. Olsder, C. Roos, “Cramer and Cayley-Hamilton in the max-algebra”, *Linear Algebra and its Applications*, 101, pp. 87–108, 1988.

[12] C. Reutenauer, H. Straubing. “Inversion of matrices over a commutative semiring”, *J. Algebra*, 88, No. 2, 1984, pp. 350–360.

[13] E. Wagneur. “Finitely generated moduloids: the existence and unicity problem for bases”, *in Analysis and Optimization of Systems,* J.L. Lions, A. Bensoussan Eds, Lecture notes in Control and Information Sciences No. 111, pp. 967–976, Springer, 1988.

[14] M. Yoeli. “A note on a generalization of Boolean matrix theory”, *Amer. Math. Monthly*, No. 68, pp. 552–557, 1961.

Erratum and Further References
September 6, 1994

— Remark 3.1. Read *congruence* instead of *equivalence relation*.

— The theory initiated in this paper has been developed in:

1. *Synchronization and Linearity*, F. Baccelli, G. Cohen, G.J. Olsder and J.P. Quadrat, Wiley, 1992.

2. Théorie des systèmes linéaires dans les dioïdes, S. Gaubert, *Thèse de l’École des Mines de Paris*, July, 1992.