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# SPECTRAL ASYMPTOTICS OF THE HELMHOLTZ MODEL IN FLUID–SOLID STRUCTURES

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## SUMMARY

A model representing the vibrations of a coupled fluid–solid structure is considered. This structure consists of a tube bundle immersed in a slightly compressible fluid. Assuming periodic distribution of tubes, this article describes the asymptotic nature of the vibration frequencies when the number of tubes is large. Our investigation shows that classical homogenization of the problem is not sufficient for this purpose. Indeed, our end result proves that the limit spectrum consists of three parts: the macro-part which comes from homogenization, the micro-part and the boundary layer part. The last two components are new. We describe in detail both macro- and micro-parts using the so-called Bloch wave homogenization method. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: homogenization; bloch waves; spectral analysis; fluid–solid structures

## 1. INTRODUCTION

The subject matter of discussion in the present work is the behaviour of vibrations of a coupled system of fluid–solid structures governed by the so-called Helmholtz model. This model along with its physical motivation has been studied in some detail in [1]. For the purpose of this paper, we recall it with necessary background. Following is the physical picture to be kept in mind.

A fluid of constant density $\rho$ occupies a container. The fluid assumed to be non-viscous is slightly compressible so that disturbances in it propagate with a finite speed $c_0$. In particular, its velocity potential $u_0$ satisfies the usual wave equation:

$$\frac{\partial^2 u_0}{\partial t^2} - c_0^2 \Delta u_0 = 0 \quad \text{in the fluid part} \tag{1}$$

For simplicity, we impose Dirichlet boundary condition, i.e.

$$u_0 = 0 \quad \text{on the walls of the container} \tag{2}$$

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A bundle of vibrating tubes is immersed into the fluid. Before immersion, each tube independently executes a simple harmonic vibrating motion. After immersion, the movement of the tubes is obviously constrained by the force exerted by the surrounding fluid. Assuming that the tubes and the container are long enough, we can write down a two-dimensional model valid on the cross-section of the tubes and the container. Indeed, denoting by $\mathbf{r}_{0j}(t)$ the transverse displacement vector of the $j$th tube, we have

$$m\frac{\mathrm{d}^2\mathbf{r}_{0j}}{\mathrm{d}t^2} + k\mathbf{r}_{0j} = \int p_0\mathbf{n}\,\mathrm{d}\gamma \quad \forall j \tag{3}$$

where $m$, $k$ are positive constants representing the mass per unit length and the stiffness of the tubes and $p_0 = p_0(x,t)$ is the pressure in the fluid. The integral in the above equation is on the boundary of the $j$th tube and $\mathbf{n}$ is the unit normal on the boundary directed outward from the fluid region. Apart from (3), there is yet another coupling between the fluid and the tube movements and it is the following:

$$\frac{\partial u_0}{\partial n} = \mathbf{r}_{0j}\cdot\mathbf{n} \quad \text{on the boundary of the } j\text{th tube} \tag{4}$$

To complete the picture, we impose that $p_0$ and $u_0$ are related by a Bernoulli-type equation, namely

$$p_0 = -\rho\frac{\partial u_0}{\partial t} + c(t) \quad \text{in the fluid part} \tag{5}$$

where $c(t)$ is a function depending only on time $t$. As usual, to get an eigenvalue problem, one seeks solutions which are sinusoidal in time, that is

$$u_0 = u(x)\mathrm{e}^{\mathrm{i}\omega t}, \quad p_0 = p(x)\mathrm{e}^{\mathrm{i}\omega t}, \quad \mathbf{r}_{0j} = \mathbf{r}_j\mathrm{e}^{\mathrm{i}\omega t} \tag{6}$$

where $\omega$ denotes the frequency of the coupled structure.

To write down the resulting eigensystem, we describe first the geometry of the cross-section. Let $\Omega$ be a connected and bounded open set in $\mathbb{R}^N$ with a smooth boundary $\Gamma$. We wish to perform our analysis when the tubes are periodically arranged and their number is large. To this end, let us introduce the unit cube $Y = ]0,1[^N$ and let $H$ be a smooth, connected, simply connected and closed subset of $Y$. For each value of a small parameter $\varepsilon > 0$, let $Y_i^\varepsilon$ and $H_i^\varepsilon$ be the translates of the cell $\varepsilon Y$ and the hole $\varepsilon H$ by the vector $\varepsilon i$ ($i \in \mathbb{Z}^N$), respectively. We denote the boundary of $H_i^\varepsilon$ by $\Gamma_i^\varepsilon$. For the analysis below, we need to assume that $\Omega$ is such that no hole $H_i^\varepsilon$ meets the boundary $\Gamma$. We are interested in those holes which are contained in $\Omega$. We denote them by $\{H_i^\varepsilon \mid i \in Q^\varepsilon\}$ where $Q^\varepsilon \subset \mathbb{Z}^N$. It is easily seen that their total number $n(\varepsilon)$ ($=$cardinality of $Q^\varepsilon$) is of the order $\varepsilon^{-N}|\Omega|$, $|\Omega|$ being the measure of $\Omega$. Let us finish the description of the geometry by noting that the cross-section of the fluid part is given by

$$\Omega^\varepsilon = \Omega \Big\backslash \bigcup_{i\in Q^\varepsilon} H_i^\varepsilon \tag{7}$$

An example of a domain satisfying the above requirement is the rectangle

$$\Omega = \prod_{j=1}^{N} ]0, L_j[, \quad L_j \in \mathbb{N} \tag{8}$$

In such a case, we can take $\varepsilon = 1/n$, $n \in \mathbb{N}$.

 

With these notations, we are in a position to formulate the eigenvalue problem corresponding to (1)–(6) and it is as follows: Find $\omega_\varepsilon \in \mathbb{C}$ for which there exists a non-zero function $u^\varepsilon$ and $\mathbf{t}^\varepsilon = \{\mathbf{t}_i^\varepsilon\}_{i\in Q^\varepsilon} \in \mathbb{C}^{Nn(\varepsilon)}$ satisfying

$$\begin{aligned} &c_0^2 \Delta u^\varepsilon + \omega_\varepsilon^2 u^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon \\ &c_0^2 \frac{\partial u^\varepsilon}{\partial n} = \frac{1}{\varepsilon} \mathbf{t}_i^\varepsilon \cdot \mathbf{n} \quad \text{on } \Gamma_i^\varepsilon, \quad i \in Q^\varepsilon \\ &(k - m\omega_\varepsilon^2)\mathbf{t}_i^\varepsilon = (\rho c_0^2/\varepsilon^{(N-1)})\omega_\varepsilon^2 \int_{\Gamma_i^\varepsilon} u^\varepsilon \mathbf{n}\, \mathrm{d}\gamma \quad \forall i \in Q^\varepsilon \\ &u^\varepsilon = 0 \quad \text{on } \Gamma \end{aligned} \tag{9}$$

The appearance of various factors involving powers of $\varepsilon$ is due to the fact that the tubes are of size $O(\varepsilon)$. Consequently, we have $\partial/\partial x_j = O(\varepsilon^{-1})$ on $\Gamma_j^\varepsilon$ and the Lebesgue measure on $\Gamma_j^\varepsilon$ is $O(\varepsilon^{N-1})$. Let us also observe that $\omega_\varepsilon \neq 0$; for if not, we see easily that $u^\varepsilon = 0$ and $\mathbf{t}^\varepsilon = 0$. An existence result concerning problem (9) which has already been proved in [1] will be recalled below.

For the moment, let us set

$$\sigma^\varepsilon = \left\{ \frac{1}{\omega_\varepsilon^2} \,\middle|\, \exists (u^\varepsilon, \mathbf{t}^\varepsilon) \neq (0,0);\ (\omega_\varepsilon^2, u^\varepsilon, \mathbf{t}^\varepsilon) \text{ solves } (9) \right\} \tag{10}$$

Let $\sigma_\infty$ be the set of limit points of subsequences drawn from $\sigma^\varepsilon$ as $\varepsilon \to 0$ and we call $\sigma_\infty$ the *limit spectrum*. The main goal of this paper is to characterize $\sigma_\infty$ in simple terms. More precisely, we seek a characterization of $\sigma_\infty$ in the spirit of the results of Allaire and Conca [2, 3]. In these works just cited, a model similar to (9) (called Laplace model) has been considered wherein the fluid is incompressible (i.e. the propagation speed $c_0 = \infty$). This means that there is no vibration at all in the fluid part. The compressibility of the fluid allows vibrations in the fluid region also and there is an interaction between these fluid vibrations and the tube vibrations. This introduces richness in the structure of the spectrum $\sigma^\varepsilon$ and the purpose of this work is to analyse it as $\varepsilon \to 0$ and highlight the new features. Arguing heuristically along the same lines, we see that there will be further interaction between the $\varepsilon$-periodic structure defined by the tubes and the fluid if we vary $c_0$ with $\varepsilon$ and take $c_0 = O(\varepsilon)$. This aspect which is not considered here will be the subject of discussion of a future publication.

Now, we compare this type of results with other similar results found in the literature. Analysis of (9) as $\varepsilon \to 0$ can be termed as homogenization of eigenvalues, a topic which was treated earlier in [4, 5]. In these works, point-wise convergence of the eigenvalues has been established and the limits have been identified as the eigenvalues of the homogenized problem. This gives a characterization of only a part of the limit spectrum (in the above sense) corresponding to the class of problems considered in [4, 5]. A full characterization of the limit spectrum was missing. Here, in this paper, a complete characterization of $\sigma_\infty$ is given. Roughly speaking, it consists of two parts (not necessarily disjoint): *interior spectrum* $\sigma_{\text{int}}$ and *boundary layer spectrum* $\sigma_{\text{bdry}}$. The so-called boundary layer spectrum consists of limits of eigenvalues whose eigenvectors concentrate near the boundary $\Gamma$ as $\varepsilon \to 0$; all other limits are included in the interior spectrum.

There are at least two ways of homogenization of spectral problems of type (9): the classical method of Bensoussan *et al.* [6] which provides $\sigma_{\text{homo}}$ (also called *macro-part* of the spectrum) and the non-standard one [1] which gives $\sigma_{\text{Bloch}}$ (also referred to as *micro-part* of the spectrum). They differ in the manner in which the original problem is scaled and consequently they provide apparently contradictory conclusions regarding the limit spectrum [1]. This puzzle has been resolved in [2, 7, 8], where the authors introduce the limit set $\sigma_\infty$ (instead of point-wise limit of the eigenvalues) and combine the two previous methods and put forward what they call Bloch wave homogenization method. One of their conclusions is that $\sigma_{\text{int}}$ is made up of both $\sigma_{\text{homo}}$ and $\sigma_{\text{Bloch}}$ in the case of Laplace model. In this paper, we plan to apply the method of [2] to the spectral problem (9) with necessary modifications.

The plan of our paper is as follows: in Section 2, we carry out two-scale convergence analysis to find $\sigma_{\text{homo}}$. In Section 3, Bloch wave homogenization is used to describe $\sigma_{\text{Bloch}}$. The final part is reserved to establish that the limit spectrum is made up of $\sigma_{\text{homo}}$, $\sigma_{\text{Bloch}}$ and $\sigma_{\text{bdry}}$. Each section contains heuristic arguments before the presentation of rigorous results.

## 2. CLASSICAL HOMOGENIZATION OF THE HELMHOLTZ MODEL

As mentioned previously, this work is aimed at the asymptotic analysis of the Helmholtz model (particularly the eigenfrequencies described by it) introduced in the Introduction. This model has been proposed and studied in detail in the book [1]. For convenience, we rewrite (9) in the following form after trivial modifications in the notation (we set $\mathbf{s}_i^\varepsilon = \varepsilon^{N-1}\mathbf{t}_i^\varepsilon$): Find $[\omega_\varepsilon^2, u^\varepsilon, \{\mathbf{s}_i^\varepsilon\}] \in \mathbb{R} \times H^1(\Omega^\varepsilon) \times \mathbb{R}^{Nn(\varepsilon)}$, $u^\varepsilon \not\equiv 0$ satisfying

$$
\begin{aligned}
&c_0^2 \Delta u^\varepsilon + \omega_\varepsilon^2 u^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon \\
&c_0^2 \frac{\partial u^\varepsilon}{\partial n} = \frac{1}{\varepsilon^N} \mathbf{s}_i^\varepsilon \cdot \mathbf{n} \quad \text{on } \Gamma_i^\varepsilon, \quad i \in Q^\varepsilon \\
&(k - m\omega_\varepsilon^2)\mathbf{s}_i^\varepsilon = \rho c_0^2 \omega_\varepsilon^2 \int_{\Gamma_i^\varepsilon} u^\varepsilon \mathbf{n} \, \mathrm{d}\gamma \quad \forall i \in Q^\varepsilon \\
&u^\varepsilon = 0 \quad \text{on } \Gamma
\end{aligned}
\tag{11}
$$

Existence of a complete set of eigenfrequencies and eigenvectors to problem (11) is obtained in the book cited above and we will recall it below. The goal of this article is to describe the limiting behaviour of the eigenfrequencies $\{\omega_\varepsilon^2\}$ as $\varepsilon \to 0$ and to present a simplified picture of the entire set $\sigma_\infty$ of limit points. In this section, such a task is carried out to characterize a subset of the limit set.

Let us now discuss the various issues and difficulties involved in the above process. Usually, eigenvalue problems such as (11) are treated by considering the associated Green operator $\widetilde{T}^\varepsilon$ which will be introduced below. The spectrum $\sigma(\widetilde{T}^\varepsilon)$ consists of the reciprocal of the eigenfrequencies $\omega_\varepsilon^2$ and hence we direct all our future efforts to the study of the limiting behaviour of $\sigma(\widetilde{T}^\varepsilon)$ as $\varepsilon \to 0$. For this purpose, we introduce the limit spectrum by defining

$$\sigma_\infty = \{\lambda \in \mathbb{R} \mid \exists \lambda^n \in \sigma(\widetilde{T}^{\varepsilon_n}) \text{ such that } \lambda^n \to \lambda \text{ as } \varepsilon_n \to 0\}$$

This is the same as the limit set defined in the Introduction. In this work, we try to give a characterization of $\sigma_\infty$ in simple terms. The peculiarity of the problem on hand is that the Green operator $\widetilde{T}^\varepsilon$ is not self-adjoint with respect to the standard scalar product. However, it is self-adjoint with respect to a weighted scalar product which depends on the sound speed $c_0$. This has already been noted in [1]. Thus $\sigma(\widetilde{T}^\varepsilon)$ and hence $\{\omega_\varepsilon^2\}$ are subsets of $\mathbb{R}$. This is why, we have taken $\omega_\varepsilon^2$ to be real in (11) without loss of generality.

Our approach to analyse the asymptotic behaviour of $\sigma(\widetilde{T}^\varepsilon)$ consists, as a first step, of studying the convergence properties of the operators $\widetilde{T}^\varepsilon$ themselves. The first obstacle, in this regard, is the variable nature of the domain and the co-domain of $\widetilde{T}^{\varepsilon}$. Hence we seek to change $\widetilde{T}^{\varepsilon}$ to another operator $T^{\varepsilon}$ with fixed domain and co-domain. Of course, in doing so, we should satisfy the following conditions:

(a) Spectrum should not be disturbed.

(b) $T^{\varepsilon}$ must converge strongly to an operator $T$. In such a case, it follows that $\sigma(T) \subseteq \sigma_{\infty}$ since the spectrum depends in a lower semi-continuous manner with respect to strong convergence of operators. Weak convergence of operators is not sufficient as it does not have any implication on the point-wise convergence of the spectrum.

(c) The definition of $T^{\varepsilon}$ is desired to be such that homogenization techniques can be applied to study its behaviour as $\varepsilon \to 0$.

There is no unique way of carrying out the above task; indeed we will construct a sequence of such operators of which the first one is constructed in this section. It is done by enlarging the state spaces $L^2(\Omega^{\varepsilon})$ and $\mathbb{R}^{Nn(\varepsilon)}$ of fluid potentials and tube displacements in a suitable way. In [2], the space $\mathbb{R}^{Nn(\varepsilon)}$ is enlarged to $L^2(\Omega)^N$ by identifying $\mathbb{R}^{Nn(\varepsilon)}$ with the subspace of functions in $L^2(\Omega)^N$ which are piece-wise constants with respect to the mesh $\{Y_i^{\varepsilon}\}_{i\in Q^{\varepsilon}}$. Viewed in this manner, a natural choice of the space enlarging $L^2(\Omega^{\varepsilon})$ seems to be $L^2(\Omega)$ with the identification that $L^2(\Omega^{\varepsilon})$ coincides with the space of functions in $L^2(\Omega)$ vanishing on the holes $\{H_i^{\varepsilon} \mid i \in Q^{\varepsilon}\}$. However, as shown below, this does not satisfy our requirement (b). The reason is that the corresponding eigenvectors oscillate on two-scales and the space $L^2(\Omega)$ is too small to capture them. The space $L^2(\Omega)$ is therefore further enlarged to the following one

$$L^2(\Omega; L^2_{\#}(Y)) = \{v \in L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^N)), v(x,y) \text{ is } Y\text{-periodic in } y\}$$

In this write-up, the symbol # is used to mean periodicity of the functions under consideration. The important point is that we are able to satisfy conditions (a)–(c) with the above choice.

It is to be pointed out that the above enlarged state space has been used in [7, 8] in the context of the so-called Bloch wave homogenization method applied to the wave equation. It is somewhat surprising to know that such a relaxation is needed already in the case of the usual homogenization of the present problem.

*Remark 2.1.* The iso-spectral property stated in condition (a) above is a wonderful property. It provides an alternative way to view the spectrum of the Helmholtz model. As we see below, an operator having this iso-spectral property is not unique. Indeed, we will construct a sequence of such operators $T^{\varepsilon,K}$, $K \in \mathbb{N}$ in Section 3. In this section, we construct first of these operators which corresponds to $K = 1$. Using homogenization techniques, one can analyse each one of these operators and this will yield complementary information on the asymptotic behaviour of the original spectrum. This step is a new input into the asymptotic analysis of eigenvalue problems when compared with the classical works in this area [4, 5], which considered only the associated Green operator. We recall that these Green operators converge uniformly and the eigenvectors converge strongly in the classical work cited above and this is used to prove that the limit of the $n$th eigenvalue is equal to the $n$th eigenvalue of the homogenized operator. In the present case, however, there is no uniform convergence; indeed, as we shall see, the limit operator is not compact even though each $T^{\varepsilon,K}$ is. Further, there is no strong convergence of the associated eigenvectors. The main advantage of the new idea mentioned above is to take into account the oscillations of the eigenvectors at various length scales.

 

After this long description of the ideas and methods, let us carry out the programme step by step.

### 2.1. *Green operator in the Helmholtz model*

In this paragraph, we introduce the Green operator $\widetilde{T}^\varepsilon$ associated with the problem (11). We state its properties and recall a result giving existence and characterization of all solutions of (11). For the proof of all assertions made here, the reader is referred to [1]. Define

$$\widetilde{T}^\varepsilon : L^2(\Omega^\varepsilon) \times \mathbb{R}^{Nn(\varepsilon)} \to L^2(\Omega^\varepsilon) \times \mathbb{R}^{Nn(\varepsilon)}$$

$$\widetilde{T}^\varepsilon[f, \{\mathbf{s}_i\}] = \left[\frac{1}{c_0^2} u^\varepsilon, \{\boldsymbol{\sigma}_i\}\right] \tag{12}$$

where

$$\boldsymbol{\sigma}_i = \frac{\rho}{k\varepsilon^N} \int_{\Gamma_i^\varepsilon} u^\varepsilon \mathbf{n} \, d\gamma + \frac{m}{k} \mathbf{s}_i \quad \forall i \in Q^\varepsilon$$

and $u^\varepsilon \in H^1(\Omega^\varepsilon)$ is the unique solution of

$$\begin{aligned} -\Delta u^\varepsilon &= f \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u^\varepsilon}{\partial n} &= \mathbf{s}_i \cdot \mathbf{n} \quad \text{on } \Gamma_i^\varepsilon, \quad i \in Q^\varepsilon \\ u^\varepsilon &= 0 \quad \text{on } \Gamma \end{aligned} \tag{13}$$

The motivation to introduce the operators $\widetilde{T}^\varepsilon$ comes from the fact that its spectrum describes the vibration frequencies of the Helmholtz model. In fact, the spectrum of $\widetilde{T}^\varepsilon$ consists of reciprocals of $\omega_\varepsilon^2$. More precisely, we have:

*Theorem 2.2.* (i) *Let* $[\omega_\varepsilon^2, u^\varepsilon, \{\mathbf{s}_i^\varepsilon\}] \in \mathbb{R} \times H^1(\Omega^\varepsilon) \times \mathbb{R}^{Nn(\varepsilon)}$ *solve* (11) *with* $u^\varepsilon \not\equiv 0$. *Then* $\omega_\varepsilon \neq 0$ *and* $(1/\omega_\varepsilon^2)$ *is an eigenvalue of* $\widetilde{T}^\varepsilon$ *and the pair* $[u^\varepsilon, (1/\omega_\varepsilon^2)\{\mathbf{s}_i^\varepsilon\}]$ *is a corresponding eigenvector.*

(ii) *Conversely, if* $[u^\varepsilon, \{\mathbf{s}_i^\varepsilon\}] \in H^1(\Omega^\varepsilon) \times \mathbb{R}^{Nn(\varepsilon)}$ *is an eigenvector of* $\widetilde{T}^\varepsilon$ *with eigenvalue* $\lambda^\varepsilon$, *then* $\lambda^\varepsilon > 0$ *and* $[1/\lambda^\varepsilon, u^\varepsilon, (1/\lambda^\varepsilon)\{\mathbf{s}_i^\varepsilon\}]$ *is a solution of* (11).

Next, we mention some of the main properties of $\widetilde{T}^\varepsilon$. Since the trace map $H^1(\Omega) \to L^2(\partial\Omega)$ is compact, it follows that $\widetilde{T}^\varepsilon$ is compact. It has been already noted in [1] that $\widetilde{T}^\varepsilon$ is self-adjoint with respect to the following weighted inner product on $L^2(\Omega^\varepsilon) \times \mathbb{R}^{Nn(\varepsilon)}$:

$$\langle [f, \{\mathbf{s}_i\}], [g, \{\mathbf{t}_i\}] \rangle_w = \int_{\Omega^\varepsilon} f g \, dx + \frac{k\varepsilon^N}{\rho c_0^2} \mathbf{s}_i \cdot \mathbf{t}_i \tag{14}$$

In the above relation and in the sequel, the usual summation convention with respect to repeated indices is adopted. Integrating by parts, it is easily verified that $\widetilde{T}^\varepsilon$ is positive definite in the following sense

$$\langle [\widetilde{T}^\varepsilon[f, \{\mathbf{s}_i\}], \ [f, \{\mathbf{s}_i\}] \rangle_w = \frac{1}{c_0^2} \left[ \int_{\Omega^\varepsilon} |\nabla u^\varepsilon|^2 \, dx + \frac{m\varepsilon^N}{\rho} \mathbf{s}_i \cdot \mathbf{s}_i \right]$$

Here, $u^\varepsilon$ is the solution of (13) associated to $f$ and $\{\mathbf{s}_i\}$.

 

These informations on $\widetilde{T}^\varepsilon$ are sufficient to apply the spectral theory of compact self-adjoint operators and deduce the following existence result.

*Theorem 2.3. There exists a sequence of triplets* $\{[\omega^2_{\ell,\varepsilon}, u^\varepsilon_\ell, \{\mathbf{s}^{\,\varepsilon}_{\ell,i}\}]\}_{\ell=1}^\infty$ *such that*

(i) *For each* $\ell \geqslant 1$, $[\omega^2_{\ell,\varepsilon}, u^\varepsilon_\ell, \{\omega^2_{\ell,\varepsilon}\mathbf{s}^{\,\varepsilon}_{\ell,i}\}]$ *is a solution of* (11).

(ii) *The set* $\{[u^\varepsilon_\ell, \{\mathbf{s}^\varepsilon_{\ell,i}\}]\}_{\ell=1}^\infty$ *forms a basis of* $L^2(\Omega^\varepsilon)\times\mathbb{R}^{Nn(\varepsilon)}$ *which is orthonormal in the sense that*

$$\int_{\Omega^\varepsilon} u^\varepsilon_\ell u^\varepsilon_m \,\mathrm{d}x + \frac{k\varepsilon^N}{\rho c_0^2}\mathbf{s}^\varepsilon_{\ell,i}\cdot\mathbf{s}^\varepsilon_{m,i} = \delta_{\ell m} \quad \forall \ell, m \geqslant 1$$

(iii) $0 < \omega^2_{1,\varepsilon} \leqslant \omega^2_{2,\varepsilon} \leqslant \cdots \to \infty$.

(iv) *If* $[\omega^2_\varepsilon, u^\varepsilon, \{\mathbf{s}^\varepsilon_i\}]$ *is any solution of* (11), *then there exists* $\ell \geqslant 1$ *such that* $\omega^2_\varepsilon = \omega^2_{\ell,\varepsilon}$. *There are only finitely many* $\ell$ *with this property. Further, the pair* $[u^\varepsilon, \{\mathbf{s}^\varepsilon_i\}]$ *can be written as a linear combination of all pairs* $[u^\varepsilon_\ell, \{\mathbf{s}^\varepsilon_{\ell,i}\}]$ *for which* $\omega^2_{\ell,\varepsilon} = w^2_\varepsilon$. *Thus the above triplets provide the entire set of solutions to* (11).

As seen from definition (14), the weighted inner product depends on $c_0$ and this is very inconvenient if we wish to do asymptotics with respect to $c_0$. This difficulty can be overcome (see Remark 2.7 later on).

### 2.2. *Modified Green operator-I*

We present here the first canonical attempt to modify the Green operator of the Helmholtz model defined in Section 2.1 so that it has fixed domain and co-domain which are both equal to $L^2(\Omega)\times L^2(\Omega)^N$. It is shown that it does not satisfy our condition (b) introduced at the beginning of Section 2. Nevertheless, its analysis motivates yet another modification which meets our requirements and it will be presented in the next paragraph. We seek the modified operator in the form

$$T^\varepsilon_1 = \begin{pmatrix} E^\varepsilon_0 & 0 \\ 0 & E^\varepsilon \end{pmatrix} \widetilde{T}^\varepsilon \begin{pmatrix} P^\varepsilon_0 & 0 \\ 0 & P^\varepsilon \end{pmatrix} \tag{15}$$

where $E^\varepsilon_0$ and $E^\varepsilon$ are operators of identification discussed earlier

$$E^\varepsilon_0 : L^2(\Omega^\varepsilon) \to L^2(\Omega), \quad E^\varepsilon_0 f = \tilde{f} \ \text{(extension by zero outside } \Omega^\varepsilon)$$

$$E^\varepsilon : \mathbb{R}^{Nn(\varepsilon)} \to L^2(\Omega)^N, \quad E^\varepsilon\{\mathbf{s}_i\} = \sum_{i\in Q^\varepsilon} \mathbf{s}_i \chi_{Y^\varepsilon_i}(x)$$

In order to make $T^\varepsilon_1$ self-adjoint, we need to take $P^\varepsilon_0$ and $P^\varepsilon$ to be operators dual to $E^\varepsilon_0$ and $E^\varepsilon$, respectively. This leads us to define

$$P^\varepsilon_0 : L^2(\Omega) \to L^2(\Omega^\varepsilon), \quad P^\varepsilon_0 f = f|_{\Omega^\varepsilon}$$

$$P^\varepsilon : L^2(\Omega)^N \to \mathbb{R}^{Nn(\varepsilon)}, \quad P^\varepsilon(\mathbf{s}(x)) = \left\{ \frac{1}{|Y^\varepsilon_i|} \int_{Y^\varepsilon_i} \mathbf{s}(x)\,\mathrm{d}x \right\}_{i\in Q^\varepsilon}$$

 

It is then easily seen that

$$(E_0^\varepsilon)^* = P_0^\varepsilon \quad \text{and} \quad (E^\varepsilon)^* = \varepsilon^N P^\varepsilon \tag{16}$$

provided that the underlying spaces concerned are equipped with the usual inner products.

By using the definition of $\widetilde{T}^\varepsilon$, we can make the definition of $T_1^\varepsilon$ more explicit as follows:

$$T_1^\varepsilon : L^2(\Omega) \times L^2(\Omega)^N \to L^2(\Omega) \times L^2(\Omega)^N$$

$$T_1^\varepsilon[f, \mathbf{s}(x)] = \left[\frac{1}{c_0^2} E_0^\varepsilon u^\varepsilon, E^\varepsilon \boldsymbol{\sigma}^\varepsilon\right]$$

where $\boldsymbol{\sigma}^\varepsilon \in \mathbb{R}^{Nn(\varepsilon)}$ is defined by

$$\boldsymbol{\sigma}_i^\varepsilon = \frac{\rho}{k\varepsilon^N} \int_{\Gamma_i^\varepsilon} u^\varepsilon \mathbf{n} \, \mathrm{d}\gamma + \frac{m}{k}(P^\varepsilon \mathbf{s})_i \quad \forall i \in Q^\varepsilon$$

and $u^\varepsilon \in H^1(\Omega^\varepsilon)$ is the unique solution of

$$\begin{aligned} -\Delta u^\varepsilon &= P_0^\varepsilon f \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u^\varepsilon}{\partial n} &= (P^\varepsilon \mathbf{s})_i \cdot \mathbf{n} \quad \text{on } \Gamma_i^\varepsilon, \quad i \in Q^\varepsilon \\ u^\varepsilon &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{17}$$

Some of the properties of the operators $T_1^\varepsilon$ are given in our next result.

*Theorem 2.4.* (i) $T_1^\varepsilon$ *is a compact operator.*

(ii) $T_1^\varepsilon$ *is self-adjoint with respect to the following weighted inner product on* $L^2(\Omega) \times L^2(\Omega)^N$:

$$\langle [f, \mathbf{s}], [g, \mathbf{t}] \rangle_w = \int_\Omega f g \, \mathrm{d}x + \frac{k}{\rho c_0^2} \int_\Omega \mathbf{s}(x) \cdot \mathbf{t}(x) \, \mathrm{d}x \tag{18}$$

(iii) $T_1^\varepsilon$ *is positive definite; indeed we have*

$$\langle T_1^\varepsilon[f, \mathbf{s}], [f', \mathbf{s}'] \rangle_w = \frac{1}{c_0^2} \left[ \int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla u'^\varepsilon + \frac{m\varepsilon^N}{\rho} P^\varepsilon \mathbf{s} \cdot P^\varepsilon \mathbf{s}' \right] \tag{19}$$

*where* $u^\varepsilon$ *and* $u'^\varepsilon$ *are solution of* (17) *with data* $[f, \mathbf{s}']$ *and* $[f', \mathbf{s}']$, *respectively.*

(iv) $\widetilde{T}^\varepsilon$ *and* $T_1^\varepsilon$ *have same spectra.*

*Proof.* (i) The compactness of $T_1^\varepsilon$ follows from that of $\widetilde{T}^\varepsilon$ and (15) as $E^\varepsilon, E_0^\varepsilon, P^\varepsilon, P_0^\varepsilon$ are all continuous.

(ii) The self-adjointness of $T_1^\varepsilon$ with respect to the scalar product (18) results from that of $\widetilde{T}_1^\varepsilon$ with respect to the scalar product (14).

(iii) To prove (19), we multiply system (17) by $u'^\varepsilon$ and the system satisfied by $u'^\varepsilon$ by $u^\varepsilon$ and we integrate by parts. We also use (16).

(iv) Since $T_1^\varepsilon$ and $\widetilde{T}^\varepsilon$ are compact, 0 is always an element of their spectra. Thus, it is enough to consider non-zero elements in their spectra. By compactness, they are eigenvalues of the respective operators. Using the relations

$$P_0^\varepsilon E_0^\varepsilon = I \quad \text{on } L^2(\Omega^\varepsilon), \quad P^\varepsilon E^\varepsilon = I \quad \text{on } \mathbb{R}^{Nn(\varepsilon)} \tag{20}$$

 

we see that if $[f, \{\mathbf{s}_i\}] \in L^2(\Omega^\varepsilon) \times \mathbb{R}^{Nn(\varepsilon)}$ is an eigenvector of $\widetilde{T}^\varepsilon$ with eigenvalue $\lambda$, then it is easily verified that $[E_0^\varepsilon f, E^\varepsilon\{\mathbf{s}_i\}] \in L^2(\Omega) \times L^2(\Omega)^N$ is an eigenvector of $T_1^\varepsilon$ with the same eigenvalue $\lambda$.

In the reverse direction, let $[f, \mathbf{s}] \in L^2(\Omega) \times L^2(\Omega)^N$ be an eigenvector of $T_1^\varepsilon$ with eigenvalue $\lambda$. If we apply the operator

$$\begin{pmatrix} P_0^\varepsilon & 0 \\ 0 & P^\varepsilon \end{pmatrix}$$

to the eigenvalue relation for $T_1^\varepsilon$, we get an eigenvalue relation for $\widetilde{T}^\varepsilon$ with the same eigenvalue $\lambda$ and with eigenvector $[P_0^\varepsilon f, P^\varepsilon \mathbf{s}]$. Now, it is required to know that this vector is non-zero. Indeed, since $\lambda \neq 0$, it follows, from the definition of $T_1^\varepsilon$, that $[f, \mathbf{s}]$ is in the image of

$$\begin{pmatrix} E_0^\varepsilon & 0 \\ 0 & E^\varepsilon \end{pmatrix}$$

and so can be written as $[f, \mathbf{s}] = [E_0^\varepsilon g^\varepsilon, E^\varepsilon\{\mathbf{t}_i^\varepsilon\}]$ for some $[g^\varepsilon, \{\mathbf{t}_i^\varepsilon\}] \in L^2(\Omega^\varepsilon) \times \mathbb{R}^{Nn(\varepsilon)}$. Hence $[P_0^\varepsilon f, P^\varepsilon \mathbf{s}] = [g^\varepsilon, \{\mathbf{t}_i^\varepsilon\}]$ which is non-zero. □

Even though the modified Green operator $T_1^\varepsilon$ satisfies our primary requirement (a) (see the introduction of Section 2), it does not enjoy property (b). Indeed, taking $\mathbf{s} = 0$, we show that

$$T_1^\varepsilon[f, 0] \text{ does not converge strongly in } L^2(\Omega) \times L^2(\Omega)^N \tag{21}$$

To this end, let us denote the corresponding solution of (17) by $u_1^\varepsilon$, i.e.

$$\begin{aligned} -\Delta u_1^\varepsilon &= P_0^\varepsilon f \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u_1^\varepsilon}{\partial n} &= 0 \quad \text{on } \Gamma_i^\varepsilon, \quad i \in Q^\varepsilon \\ u_1^\varepsilon &= 0 \quad \text{on } \Gamma \end{aligned} \tag{22}$$

The convergence property of $T_1^\varepsilon[f, 0]$ is then governed by the behaviour of $u_1^\varepsilon$. This is a classical problem of homogenization with periodically perforated holes studied by Cioranescu and Saint Jean Paulin [9]. In order to present their results, let us first introduce the so-called homogenized matrix $A = [A_{mk}]$

$$A_{mk} = \int_{Y^*} (\nabla w_m + \mathbf{e}_m) \cdot (\nabla w_k + \mathbf{e}_k)\, \mathrm{d}y \quad 1 \leqslant m, \ k \leqslant N$$

where $w_m$, $1 \leqslant m \leqslant N$ is the unique solution (defined up to an additive constant) of the following cell-problem

$$\begin{aligned} -\Delta_y w_m &= 0 \quad \text{in } Y^* \\ \frac{\partial w_m}{\partial n_y} &= -n_m \quad \text{on } \partial H \\ w_m &\text{ is } Y^*\text{-periodic} \end{aligned} \tag{23}$$

We are now in a position to announce the homogenization result for problem (22).

*Theorem 2.5. There exists an extension operator $X^\varepsilon$ such that $X^\varepsilon u_1^\varepsilon \in H_0^1(\Omega)$ and $X^\varepsilon u_1^\varepsilon \rightharpoonup u_1$ in $H_0^1(\Omega)$ weak where $u_1$ satisfies the homogenized problem*

$$\begin{aligned} -\operatorname{div}(A\nabla u_1) &= \theta f \quad \text{in } \Omega \\ u_1 &= 0 \quad \text{on } \Gamma \end{aligned} \tag{24}$$

*Here, $\theta = |Y^*|$ is the volume of the fluid region in the unit cell Y.*

In the above notation, we have $E_0^\varepsilon u_1^\varepsilon = (X^\varepsilon u_1^\varepsilon)\chi_{Y^*}(\frac{x}{\varepsilon})$ and so $E_0^\varepsilon u_1^\varepsilon \rightharpoonup \theta u_1$ in $L^2(\Omega)$ weak. However,

$$\int_\Omega |E_0^\varepsilon u_1^\varepsilon|^2 \,\mathrm{d}x = \int_\Omega |X^\varepsilon u_1^\varepsilon|^2 \chi_{Y^*}\left(\frac{x}{\varepsilon}\right) \mathrm{d}x \to \theta \int_\Omega u_1^2 \,\mathrm{d}x$$

Thus the convergence of $E_0^\varepsilon u_1^\varepsilon$ in $L^2(\Omega)$ is only weak and not strong. Thus we reach conclusion (21).

### 2.3. *Modified Green operator-II*

Since the modification of the Green operator suggested in the previous paragraph is not satisfactory, we now seek yet another modification. In order to get motivated, we consider the two-scale behaviour of $E_0^\varepsilon u_1^\varepsilon$ where $u_1^\varepsilon$ is defined by (22). For the concept of two-scale convergence and its properties, we refer to Allaire [10]. We have

$$E_0^\varepsilon u_1^\varepsilon \rightharpoonup u_1(x)\chi_{Y^*}(y) \quad \text{weakly in } L^2(\Omega) \text{ in the sense of two scales}$$

More precisely, we have

$$\int_\Omega E_0^\varepsilon u_1^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x \to \int_{\Omega\times Y} u_1(x)\chi_{Y^*}(y)\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for all test functions $\phi \in \mathscr{D}(\Omega; C_\#^\infty(Y))$. Furthermore, we have

$$\int_\Omega (E_0^\varepsilon u_1^\varepsilon)^2 \,\mathrm{d}x \to \int_\Omega \theta u_1^2 \,\mathrm{d}x = \int_{\Omega\times Y} |u_1(x)\chi_{Y^*}(y)|^2 \,\mathrm{d}x\,\mathrm{d}y$$

Thus we conclude that there is strong convergence of $E_0^\varepsilon u_1^\varepsilon$ if we consider two scales. In this paragraph, we exploit the above property to our advantage. Since $u_1(x)\chi_{Y^*}(y) \in L^2(\Omega; L_\#^2(Y))$, the right choice for the state space of the fluid potential seems to be $L^2(\Omega; L_\#^2(Y))$ (and not $L^2(\Omega)$). The correct choice for the tube displacements was already made in Section 2.1 and it is $L^2(\Omega)^N$. From the definition of $E_0^\varepsilon$, it might look natural to consider $L^2(\Omega)$ as a subspace of $L^2(\Omega; L_\#^2(Y))$ by identifying $f(x)$ as a function independent of $y$. Since eigenvectors oscillate on scales finer than $\Omega$, the above identification will not be useful. That is why the following identification map $E_{1,\Omega}^\varepsilon$ (first introduced in [8]) is considered. It operates entirely in a different way.

$$\begin{aligned} &E_{1,\Omega}^\varepsilon : L^2(\Omega) \to L^2(\Omega; L_\#^2(Y)) \\ &(E_{1,\Omega}^\varepsilon f)(x,y) = \sum_i \chi_i^\varepsilon(x) f(x_i^\varepsilon + \varepsilon y) \end{aligned} \tag{25}$$

where $\chi_i^\varepsilon(x)$ is the characteristic function of the cell $Y_i^\varepsilon$ whose origin is $x_i^\varepsilon$. Thus, when $x$ varies in $Y_i^\varepsilon$, $E_{1,\Omega}^\varepsilon f(x,y)$ depends only on $y$ and it is a rescaled version of $f$ on $Y_i^\varepsilon$. The associated dual operator is also useful and it is defined as

$$P_{1,\Omega}^\varepsilon : L^2(\Omega; L^2_\#(Y)) \to L^2(\Omega)$$

$$(P_{1,\Omega}^\varepsilon \phi)(x) = \sum_i \chi_i^\varepsilon(x) \frac{1}{|Y_i^\varepsilon|} \int_{Y_i^\varepsilon} \phi\left(x', \frac{x}{\varepsilon}\right) \mathrm{d}x' \tag{26}$$

Summation in (25) and (26) is taken over all those cells $Y_i^\varepsilon$ which provide a covering of $\Omega$. If there is any need, we will take $f$ and $\phi$ to be zero outside $\Omega$. The expression given in (25) for $E_{1,\Omega}^\varepsilon$ is valid for $x \in \Omega$ and $y \in Y$. Its definition is then extended $Y$-periodically throughout $\mathbb{R}^N$.

Using these operators, we are now ready to introduce the following modification of the Green operator and this will be the object of main concern in this section. Define

$$T^\varepsilon : L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N \to L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N$$

$$T^\varepsilon = \begin{pmatrix} E_{1,\Omega}^\varepsilon E_0^\varepsilon & 0 \\ 0 & E^\varepsilon \end{pmatrix} \widetilde{T}^\varepsilon \begin{pmatrix} P_0^\varepsilon P_{1,\Omega}^\varepsilon & 0 \\ 0 & P^\varepsilon \end{pmatrix}$$

More explicitly, we have, for $[f, \mathbf{s}] \in L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N$,

$$T^\varepsilon[f, \mathbf{s}] = \left[\frac{1}{c_0^2} E_{1,\Omega}^\varepsilon E_0^\varepsilon u^\varepsilon, E^\varepsilon \boldsymbol{\sigma}^\varepsilon\right]$$

where $u^\varepsilon$ is the unique solution of

$$\begin{aligned} -\Delta u^\varepsilon &= P_0^\varepsilon P_{1,\Omega}^\varepsilon f \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u^\varepsilon}{\partial n} &= (P^\varepsilon \mathbf{s})_i \cdot \mathbf{n} \quad \text{on } \Gamma_i^\varepsilon, \quad i \in Q^\varepsilon \\ u^\varepsilon &= 0 \quad \text{on } \Gamma \end{aligned} \tag{27}$$

and

$$\boldsymbol{\sigma}_i^\varepsilon = \frac{\rho}{k\varepsilon^N} \int_{\Gamma_i^\varepsilon} u^\varepsilon \mathbf{n} \, \mathrm{d}\gamma + \frac{m}{k} (P^\varepsilon \mathbf{s})_i \quad \forall i \in Q^\varepsilon$$

Having introduced $T^\varepsilon$, it is our task to verify that it enjoys all the properties i.e. (a)–(c) introduced in the beginning of this section. In this paragraph, we verify (a) leaving the rest to the next one.

*Theorem 2.6.* (i) *$T^\varepsilon$ is a compact operator.*

(ii) *$T^\varepsilon$ is self-adjoint with respect to the following weighted inner product on $L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N$:*

$$\langle [f, \mathbf{s}], [g, \mathbf{t}] \rangle = \int_{\Omega \times Y} f g \, \mathrm{d}x \, \mathrm{d}y + \frac{k}{\rho c_0^2} \int_\Omega \mathbf{s}(x) \cdot \mathbf{t}(x) \, \mathrm{d}x \tag{28}$$

(iii) *$T^\varepsilon$ is positive definite; indeed we have*

$$\langle T^\varepsilon [f, \mathbf{s}], [f', \mathbf{s}'] \rangle = \frac{1}{c_0^2} \left[ \int_{\Omega\varepsilon} \nabla u^\varepsilon \cdot \nabla u'^\varepsilon + \frac{m\varepsilon^N}{\rho} (P^\varepsilon \mathbf{s})_i \cdot (P^\varepsilon \mathbf{s}')_i \right] \tag{29}$$

 

*where* $u^\varepsilon$ *and* $u'^\varepsilon$ *are solutions of* (27) *with data* $[f, \mathbf{s}]$ *and* $[f', \mathbf{s}']$, *respectively.*
(iv) $\widetilde{T}^\varepsilon$ *and* $T^\varepsilon$ *have same spectra.*

*Proof.* The proof proceeds along the lines of the proof of Theorem 2.4. In addition to (16) and (20), we have to use the following properties:

$$\begin{aligned} (E^\varepsilon_{1,\Omega})^* &= P^\varepsilon_{1,\Omega}, \quad (P^\varepsilon_{1,\Omega})^* = E^\varepsilon_{1,\Omega} \\ P^\varepsilon_{1,\Omega} E^\varepsilon_{1,\Omega} &= I \quad \text{on } L^2(\Omega) \end{aligned} \tag{30}$$

which are easily checked. □

Since $\sigma(T^\varepsilon)$ coincides with Helmholtz spectrum, we are motivated to study the asymptotic behaviour of $T^\varepsilon$ themselves and thereby establish conditions (b) and (c). Since the definition of $T^\varepsilon$ involves problem (27), our main concern will be the analysis of (27) as $\varepsilon \to 0$. By means of this, we will be able to establish the strong convergence of $T^\varepsilon$ towards a certain operator $T$ by passing to the limit in the bilinear form (29). In order to facilitate matters, in the remainder of this paragraph, we will present a description of $T^\varepsilon$ decoupling the contribution coming from the fluid part, the solid part and the interaction between them. By linearity, problem (27) can be decomposed into two sub-problems: $u^\varepsilon = u^\varepsilon_1 + u^\varepsilon_2$ where $u^\varepsilon_1$ and $u^\varepsilon_2$ are the solutions of

$$\begin{aligned} -\Delta u^\varepsilon_1 &= P^\varepsilon_0 P^\varepsilon_{1,\Omega} f \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u^\varepsilon_1}{\partial n} &= 0 \quad \text{on } \Gamma^\varepsilon_i, \quad i \in Q^\varepsilon \\ u^\varepsilon_1 &= 0 \quad \text{on } \Gamma \end{aligned} \tag{31}$$

$$\begin{aligned} -\Delta u^\varepsilon_2 &= 0 \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u^\varepsilon_2}{\partial n} &= (P^\varepsilon \mathbf{s})_i \cdot \mathbf{n} \quad \text{on } \Gamma^\varepsilon_i, \quad i \in Q^\varepsilon \\ u^\varepsilon_2 &= 0 \text{ on } \quad \Gamma \end{aligned} \tag{32}$$

Exploiting the above decomposition, it will be convenient to represent $T^\varepsilon$ in the form of a matrix of operators. More precisely, we write

$$T^\varepsilon = \begin{bmatrix} T^\varepsilon_{11} & T^\varepsilon_{12} \\ T^\varepsilon_{21} & T^\varepsilon_{22} \end{bmatrix} \tag{33}$$

with

$$\begin{aligned} T^\varepsilon_{11} : L^2(\Omega; L^2_\#(Y)) \to L^2(\Omega; L^2_\#(Y)), &\quad T^\varepsilon_{11} f = \frac{1}{c_0^2} E^\varepsilon_{1,\Omega} E^\varepsilon_0 u^\varepsilon_1 \\ T^\varepsilon_{12} : L^2(\Omega)^N \to L^2(\Omega; L^2_\#(Y)), &\quad T^\varepsilon_{12} \mathbf{s} = \frac{1}{c_0^2} E^\varepsilon_{1,\Omega} E^\varepsilon_0 u^\varepsilon_2 \\ T^\varepsilon_{21} : L^2(\Omega; L^2_\#(Y)) \to L^2(\Omega)^N, &\quad T^\varepsilon_{21} f = E^\varepsilon \left\{ \frac{\rho}{k\varepsilon^N} \int_{\Gamma^\varepsilon_i} u^\varepsilon_1 \mathbf{n} \, d\gamma \right\} \\ T^\varepsilon_{22} : L^2(\Omega)^N \to L^2(\Omega)^N, &\quad T^\varepsilon_{22} \mathbf{s} = E^\varepsilon \left\{ \frac{\rho}{k\varepsilon^N} \int_{\Gamma^\varepsilon_i} u^\varepsilon_2 \mathbf{n} \, d\gamma + \frac{m}{k} (P^\varepsilon \mathbf{s})_i \right\} \end{aligned} \tag{34}$$

It can be remarked that the operators $T^\varepsilon_{11}$ and $T^\varepsilon_{22}$ are Green operators representing pure fluid and solid vibrations, respectively, whereas the operators $T^\varepsilon_{12}$ and $T^\varepsilon_{21}$ represent interaction between them.

The bilinear form of $T^\varepsilon$ can also be split as follows:

$$\langle T^\varepsilon[f,\mathbf{s}],[f',\mathbf{s}']\rangle = \int_{\Omega\times Y}(T^\varepsilon_{11}f)f'\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y} f'(T^\varepsilon_{12}\mathbf{s})\,\mathrm{d}x\,\mathrm{d}y + \frac{k}{\rho c_0^2}\left[\int_\Omega (T^\varepsilon_{21}f)\cdot\mathbf{s}'\,\mathrm{d}x + \int_\Omega (T^\varepsilon_{22}\mathbf{s})\cdot\mathbf{s}'\,\mathrm{d}x\right] \tag{35}$$

Each of the integral in (35) can be expressed in terms of $(u^\varepsilon_1,u^\varepsilon_2)$, $(u'^\varepsilon_1,u'^\varepsilon_2)$ and $(\mathbf{s},\mathbf{s}')$. Indeed,

$$\int_{\Omega\times Y}(T^\varepsilon_{11}f)f'\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{c_0^2}\int_{\Omega^\varepsilon}\nabla u^\varepsilon_1\cdot\nabla u'^\varepsilon_1\,\mathrm{d}x \tag{36}$$

$$\int_{\Omega\times Y}(T^\varepsilon_{12}\mathbf{s})f'\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{c_0^2}\int_{\Omega^\varepsilon}\nabla u^\varepsilon_2\cdot\nabla u'^\varepsilon_1\,\mathrm{d}x$$

$$\int_\Omega (T^\varepsilon_{21}f)\mathbf{s}'\,\mathrm{d}x = \frac{\rho}{k}(P^\varepsilon\mathbf{s}')_i\cdot\int_{\Gamma^\varepsilon_i}u^\varepsilon_1\mathbf{n}\,\mathrm{d}\gamma = \frac{\rho}{k}\int_{\Omega^\varepsilon}\nabla u^\varepsilon_1\cdot\nabla u'^\varepsilon_2\,\mathrm{d}x \tag{37}$$

$$\int_\Omega (T^\varepsilon_{22}\mathbf{s})\cdot\mathbf{s}'\,\mathrm{d}x = \frac{\rho}{k}\int_{\Omega^\varepsilon}\nabla u^\varepsilon_2\cdot\nabla u'^\varepsilon_2\,\mathrm{d}x + \frac{m\varepsilon^N}{k}(P^\varepsilon\mathbf{s})_i\cdot(P^\varepsilon\mathbf{s}') \tag{38}$$

*Remark 2.7.* It was already observed that $T^\varepsilon$ is self-adjoint with respect to the weighted inner product (28) on $L^2(\Omega;L^2_\#(Y))\times L^2(\Omega)^N$. Since this inner product depends on $c_0$, it will be very inconvenient to work with it in case we wish to carry out asymptotics when $c_0$ varies with $\varepsilon$. Thus the question arises as to whether it is possible to work with standard inner product on $L^2(\Omega;L^2_\#(Y))\times L^2(\Omega)^N$. In general, if we change the inner product, then the self-adjointness is lost. In the present context, it turns out that we can change $T^\varepsilon$ to $T^\varepsilon_s$ such that $T^\varepsilon_s$ is self-adjoint with respect to the standard inner product and $T^\varepsilon_s$ also satisfies our requirements (a)–(c) introduced at the start of Section 2. In fact, such an operator $T^\varepsilon_s$ can be explicitly given, in matrix form, by

$$T^\varepsilon_s = \begin{bmatrix} T^\varepsilon_{11} & \left(\frac{\rho}{k}\right)^{1/2} c_0 T^\varepsilon_{12} \\ \left(\frac{k}{\rho}\right)^{1/2}\frac{1}{c_0}T^\varepsilon_{21} & T^\varepsilon_{22}\end{bmatrix}$$

Indeed, it follows from relations (36)–(38) that

$$(T^\varepsilon_{11})^* = T^\varepsilon_{11},\quad (T^\varepsilon_{22})^* = T^\varepsilon_{22}\quad\text{and}\quad (T^\varepsilon_{12})^* = \frac{k}{\rho c_0^2}T^\varepsilon_{21}$$

when $L^2(\Omega;L^2_\#(Y))$ and $L^2(\Omega)^N$ are equipped with the standard scalar products. Furthermore, it is easily verified that $T^\varepsilon$ and $T^\varepsilon_s$ are similar and so they have same spectra. More precisely, we have $P^{-1}T^\varepsilon_s P = T^\varepsilon$ where $P$ is the operator defined by

$$P = \begin{bmatrix} I & 0 \\ 0 & \left(\frac{k}{\rho}\right)^{1/2}\frac{1}{c_0}I\end{bmatrix}$$

In spite of the above considerations, throughout this article, we fix $c_0$ and continue to work with the weighted scalar product (28). □

*2.4. Homogenization of problem (31)*

Our main goal in Section 2 is to establish the strong convergence of $T^\varepsilon$ and identify the limit operator $T$. Since $T^\varepsilon$ is defined in terms of solutions of problems (31) and (32), our main concern will be to pass to the limit in these problems.

Because of self-adjointness of $T^\varepsilon$, it is sufficient to pass to the limit in the quadratic form of $T^\varepsilon$, namely $\langle T^\varepsilon[f,\mathbf{s}],[f,\mathbf{s}]\rangle$, to identify the limit operator $T$. This will also establish the weak convergence of $T^\varepsilon$ towards $T$. To prove the strong convergence, we have to pass to the limit in the bilinear form associated with $T^\varepsilon$, namely $\langle T^\varepsilon[f,\mathbf{s}],[f'^\varepsilon,\mathbf{s}'^\varepsilon]\rangle$ against weakly convergent sequences $[f'^\varepsilon,\mathbf{s}'^\varepsilon]$. It is then clear, from formula (29), that we will need the two-scale behaviour of the gradients of solutions of problems (31) and (32). Problem (32) was studied in detail in [2] and their results will be recalled in Section 2.5. For the moment, let us concentrate on the analysis of problem (31). As mentioned above, we will need a description of the behaviour of problem (31) as $\varepsilon\to 0$ when $f$ varies in $L^2(\Omega;L^2_\#(Y))$. For convenience, we rewrite the problem:

$$\begin{aligned} -\Delta u_1^\varepsilon &= P_0^\varepsilon P_{1,\Omega}^\varepsilon f^\varepsilon \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u_1^\varepsilon}{\partial n} &= 0 \quad \text{on } \Gamma_i^\varepsilon, \quad i\in Q^\varepsilon \\ u_1^\varepsilon &= 0 \quad \text{on } \Gamma \end{aligned} \tag{39}$$

Comparing this problem with the one treated in [9] we see that the right-hand side in (39) is more complicated and indeed simple weak convergence method developed in [9] does not seem to be adequate to handle (39). As seen below, it is convenient to apply the concept of two-scale convergence to the present situation. The crucial property needed in this process is given in our next result which is stronger than saying that $P_{1,\Omega}^\varepsilon \rightharpoonup I$ weakly as operators.

*Proposition 2.8. Let $f^\varepsilon\rightharpoonup f$ weakly in $L^2(\Omega;L^2_\#(Y))$. Then $P_{1,\Omega}^\varepsilon f^\varepsilon(x)\rightharpoonup f(x,y)$ weakly in the sense of two scales, i.e., we have*

$$\int_\Omega (P_{1,\Omega}^\varepsilon f^\varepsilon)(x)\chi\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x \to \int_{\Omega\times Y} f(x,y)\chi(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{40}$$

*for all $\chi\in\mathcal{D}(\Omega;C_\#^\infty(Y))$.*

*Proof.* First of all, it is straightforward to show

$$\|P_{1,\Omega}^\varepsilon f\|_{L^2(\Omega)} \leqslant \|f\|_{L^2(\Omega\times Y)} \quad \forall f\in L^2(\Omega;L^2_\#(Y))$$

Because of this, applying the usual density arguments, it is enough to take $f^\varepsilon(x,y)$ in the form of a tensor product: $f^\varepsilon(x,y)=\phi^\varepsilon(x)\psi^\varepsilon(y)$ with $\phi^\varepsilon\rightharpoonup\phi$ in $L^2(\Omega)$ weak and $\psi^\varepsilon\rightharpoonup\psi$ in $L^2_\#(Y)$ weak. By the same token, we can take $\chi(x,y)=\chi_1(x)\chi_2(y)$ in (40).

Next, we observe that

$$P_{1,\Omega}^\varepsilon(\phi\otimes\psi)(x) = E^\varepsilon(P^\varepsilon\phi)(x)\psi\left(\frac{x}{\varepsilon}\right)$$

 

where $E^\varepsilon$ and $P^\varepsilon$ are defined in Section 2.2. (Strictly speaking, they are defined on vector functions. However, the same definition holds for scalar functions as well.) Thus, we are reduced to showing that

$$\int_\Omega (E^\varepsilon P^\varepsilon \phi^\varepsilon)(x)\psi^\varepsilon\left(\frac{x}{\varepsilon}\right)\chi(x)\,dx \to \int_{\Omega\times Y} \phi(x)\psi(y)\chi(x)\,dx\,dy$$

for all $\chi \in \mathscr{D}(\Omega)$.

One more reduction is possible and it consists of using the following estimate of the commutator:

$$E^\varepsilon P^\varepsilon(\chi\phi^\varepsilon) - \chi E^\varepsilon P^\varepsilon \phi^\varepsilon = O(\varepsilon) \quad \text{in } L^2(\Omega)$$

Owing to the above estimate, it is sufficient to show

$$\int_\Omega (E^\varepsilon P^\varepsilon \phi^\varepsilon)(x)\psi^\varepsilon\left(\frac{x}{\varepsilon}\right)dx \to \int_{\Omega\times Y} \phi(x)\psi(y)\,dx\,dy \tag{41}$$

whenever $\phi^\varepsilon \rightharpoonup \phi$ in $L^2(\Omega)$ weak and $\psi^\varepsilon \rightharpoonup \psi$ in $L^2_\#(Y)$ weak. Since $E^\varepsilon P^\varepsilon \phi^\varepsilon$ is piece-wise constant, we can express

$$\int_\Omega (E^\varepsilon P^\varepsilon \phi^\varepsilon)(x)\psi^\varepsilon\left(\frac{x}{\varepsilon}\right)dx = \int_\Omega E^\varepsilon P^\varepsilon \phi^\varepsilon(x)\,dx \int_Y \psi^\varepsilon(y)\,dy$$

To complete the proo, we notice that

$$\int_\Omega E^\varepsilon P^\varepsilon \phi^\varepsilon(x)\,dx = \int_\Omega \phi^\varepsilon(x)\,dx$$

Thus we can pass to the limit easily and this proves (41) and hence the proposition. □

Before we move on to the application of the above result, let us record the following consequence which follows immediately from the duality (30). Indeed, if $f^\varepsilon \rightharpoonup f$ in $L^2(\Omega; L^2_\#(Y))$ weak, then for all $\chi \in L^2(\Omega)$ we have

$$\int_{\Omega\times Y} f^\varepsilon(x,y)(E^\varepsilon_{1,\Omega}\chi)(x,y)\,dx\,dy \to \int_{\Omega\times Y} f(x,y)\chi(x)\,dx\,dy$$

Applying the above proposition to (31), we will be in a position to describe its two-scale limit. Before announcing the result, let us write down the two-scale homogenized problem corresponding to (31).

$$\begin{aligned}
&\text{Find } [u_1,\hat{u}_1] \in H^1_0(\Omega) \times L^2(\Omega; H^1(Y^*)/\mathbb{R}) \text{ such that}\\
&-\Delta_y \hat{u}_1(x,y) = 0 \quad \text{in } \Omega \times Y^*\\
&-\mathrm{div}_x \int_{Y^*} (\nabla u_1(x) + \nabla_y \hat{u}_1(x,y))\,dy = \int_{Y^*} f(x,y)\,dy \quad \text{in } \Omega\\
&u_1(x) = 0 \quad \text{on } \Gamma\\
&[\nabla u_1(x) + \nabla_y \hat{u}_1(x,y)] \cdot \mathbf{n}_y = 0 \quad \text{on } \Omega \times \partial H\\
&y \in Y^* \to \hat{u}_1(x,y) \text{ is } Y\text{-periodic}
\end{aligned} \tag{42}$$

 

That the above system admits a unique solution can be seen via Lax–Milgram lemma. Further, the two components of the solution decouple. More precisely, $u_1$ is characterized as the solution of the following homogenized problem:

$$-\operatorname{div}(A\nabla u_1) = \int_{Y^*} f(x,y)\,\mathrm{d}y \quad \text{in } \Omega \qquad (43)$$
$$u_1 = 0 \quad \text{on } \Gamma$$

The second component $\hat{u}_1$ is then expressed in terms of $u_1$, separating the variables $x$ and $y$, as follows:

$$\hat{u}_1(x,y) = \sum_{m=1}^{N} w_m(y)\frac{\partial u_1}{\partial x_m}(x), \quad x\in\Omega,\ y\in Y^*, \qquad (44)$$

where $w_m$ is the test function defined by the cell problem (23).

With these notations, we can now state the main result on the two-scale convergence of the gradient of $u_1^\varepsilon$. We will be using, of course, the $H^1$-prolongation operator $X^\varepsilon$ of [9] which was already used in Theorem 2.5.

*Theorem 2.9. Let $f^\varepsilon \rightharpoonup f$ weakly in $L^2(\Omega; L^2_\#(Y))$. Then the solution $u_1^\varepsilon$ of* (31) *has the following behaviour*:

(i) *$X^\varepsilon u_1^\varepsilon \rightharpoonup u_1$ weakly in $H^1_0(\Omega)$ where $u_1$ is the solution of* (43).
(ii) *The gradient of $u_1^\varepsilon$ extended by zero outside $\Omega^\varepsilon$, denoted $\widetilde{\nabla u_1^\varepsilon}$, converges to $\chi_{Y^*}(y)(\nabla u_1(x) + \nabla_y \hat{u}_1(x,y))$ strongly in the sense of two-scales where $[u_1,\hat{u}_1]$ solves* (42).

*Proof.* First of all, it follows that $\|\nabla u_1^\varepsilon\|_{L^2(\Omega^\varepsilon)}$ is bounded independent of $\varepsilon$. Thus, for a subsequence, $X^\varepsilon u_1^\varepsilon \rightharpoonup u_1$ in $H^1_0(\Omega)$ weak. Since $\widetilde{\nabla u_1^\varepsilon}(x) = \chi_{Y^*}(\frac{x}{\varepsilon})\nabla(X^\varepsilon u_1^\varepsilon)$, the two-scale convergence theory shows that $\widetilde{\nabla u_1^\varepsilon}(x)$ converges weakly to $\chi_{Y^*}(y)(\nabla u_1(x) + \nabla_y\hat{u}_1(x,y))$ in the sense of two scales. The first task is to identify $[u_1,\hat{u}_1]$ with the solution of (42).

To this end, following the method of Allaire [10], we multiply (31) by $v(x)+\varepsilon\hat{v}(x,x/\varepsilon)$ with $v\in\mathscr{D}(\Omega)$ and $\hat{v}\in\mathscr{D}(\Omega; C^\infty_\#(Y))$ and integrate by parts. Passage to the limit in the left-hand side of the resulting equation is classical. Owing to Proposition 2.8, we can pass to the limit in the right side also and obtain

$$\int_{\Omega\times Y^*} (\nabla u_1(x) + \nabla_y\hat{u}_1(x,y))\cdot(\nabla v(x) + \nabla_y\hat{v}(x,y))\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega\times Y^*} f(x,y)v(x)\,\mathrm{d}x\,\mathrm{d}y \qquad (45)$$

It suffices to note that this is nothing but the variational formulation of (42).

To show that $\widetilde{\nabla u_1^\varepsilon}$ converges strongly in two-scales, we have to verify

$$\int_\Omega |\widetilde{\nabla u_1^\varepsilon}|^2\,\mathrm{d}x \to \int_{\Omega\times Y^*} |\nabla u_1(x) + \nabla_y\hat{u}_1(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \qquad (46)$$

In fact, it follows from (31) that $\int_\Omega |\widetilde{\nabla u_1^\varepsilon}|^2\,\mathrm{d}x = \int_\Omega \chi_{Y^*}(\frac{x}{\varepsilon})(P^\varepsilon_{1,\Omega}f^\varepsilon)(X^\varepsilon u_1^\varepsilon)\,\mathrm{d}x$ and so once again using Proposition 2.8, we get

$$\lim_{\varepsilon\to 0}\int_\Omega |\widetilde{\nabla u_1^\varepsilon}|^2\,\mathrm{d}x = \int_{\Omega\times Y^*} f(x,y)u_1(x)\,\mathrm{d}x\,\mathrm{d}y$$

That the above limit coincides with the limit in (46) can be seen by taking $[v, \hat{v}] = [u_1, \hat{u}_1]$ in (45). Thus the verification of (46) is over. □

According to the theory developed in [10], the strong convergence of $\widetilde{\nabla} u_1^\varepsilon$ implies

*Corollary 2.10. For any sequence* $\{\phi^\varepsilon\}$ *bounded in* $L^2(\Omega)^N$ *and converging weakly to* $\phi_0(x,y) \in L^2(\Omega \times Y)^N$ *in the sense of two scales, we have*

$$\widetilde{\nabla} u_1^\varepsilon \cdot \phi^\varepsilon \rightharpoonup \int_{Y^*} (\nabla u_1(x) + \nabla_y \hat{u}_1(x,y)) \cdot \phi_0(x,y)\,\mathrm{d}y$$

*in the space* $L^1(\Omega)$ *with respect to the weak topology* $\sigma(L^1, L^\infty)$.

### 2.5. *Homogenization of problem (32)*

For reasons explained in the beginning of Section 2.4, we seek to describe the two-scale behaviour of problem (32). This has already been done in [2]. Our aim in this paragraph is to merely recall their results and cast them in our notation.

For the sake of proving strong convergence, we need to consider (32) with varying $\mathbf{s}$ in $L^2(\Omega)^N$. For convenience of the reader, we rewrite the system below.

$$\begin{aligned} \Delta u_2^\varepsilon &= 0 \quad \text{in } \Omega \\ \frac{\partial u_2^\varepsilon}{\partial n} &= (P^\varepsilon \mathbf{s}^\varepsilon)_i \cdot \mathbf{n} \quad \text{on } \Gamma_i^\varepsilon, \quad i \in Q^\varepsilon \\ u_2^\varepsilon &= 0 \quad \text{on } \Gamma \end{aligned} \tag{47}$$

Let us start by noting down the associated two-scale homogenized system

$$\begin{aligned} &\text{Find } [u_2, \hat{u}_2] \in H_0^1(\Omega) \times L^2(\Omega; H^1(Y^*)/\mathbb{R}) \text{ satisfying} \\ &-\Delta_y \hat{u}_2(x,y) = 0 \quad \text{in } \Omega \times Y^* \\ &-\mathrm{div}_x \int_{Y^*} (\nabla u_2(x) + \nabla_y \hat{u}_2(x,y))\,\mathrm{d}y = |H| \mathrm{div}_x \mathbf{s} \quad \text{in } \Omega \\ &u_2(x) = 0 \quad \text{on } \Gamma \\ &[\nabla u_2(x) + \nabla_y \hat{u}_2(x,y) - \mathbf{s}(x)] \cdot \mathbf{n}_y = 0 \quad \text{on } \Omega \times \partial H \\ &y \in Y^* \to \hat{u}_2(x,y) \text{ is } Y\text{-periodic} \end{aligned} \tag{48}$$

A weak formulation of (48) can also be given and it is as follows

$$\begin{aligned} &\int_{\Omega \times Y^*} (\nabla u_2(x) + \nabla_y \hat{u}_2(x,y)) \cdot (\nabla v(x) + \nabla_y \hat{v}(x,y))\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = -\int_{\Omega \times H} \mathbf{s}(x) \cdot (\nabla v(x) + \nabla_y \hat{v}(x,y))\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{49}$$

for all $v \in H_0^1(\Omega)$ and $\hat{v} \in \mathscr{D}(\Omega; C_\#^\infty(Y))$.

 

In a classical way, the existence and the uniqueness of solution for (48) can be established via Lax–Milgram lemma. The structure of (48) is such that the two components $[u_2, \hat{u}_2]$ can be decoupled. Indeed, $u_2$ is characterized as the solution of the homogenized system

$$
\begin{aligned}
-\operatorname{div}(A\nabla u_2) &= \operatorname{div}((I-A)\mathbf{s}) \quad \text{in } \Omega \\
u_2 &= 0 \quad \text{on } \Gamma
\end{aligned}
\tag{50}
$$

The second component is then expressed as follows in which the variables $x$ and $y$ are separated:

$$
\hat{u}_2(x,y) = \sum_{m=1}^{N} w_m(y)\left(\frac{\partial u_2}{\partial x_m}(x) - s_m(x)\right) \quad x\in\Omega,\ y\in Y^* \tag{51}
$$

We are now ready to state the main results concerning problem (47).

*Theorem 2.11. Let* $\mathbf{s}^\varepsilon \rightharpoonup \mathbf{s}$ *in* $L^2(\Omega)^N$ *weak. Then the solution* $u_2^\varepsilon$ *of (47) has the following behaviour:*

(i) $X^\varepsilon u_2^\varepsilon \rightharpoonup u_2$ *weakly in* $H_0^1(\Omega)$ *where* $u_2$ *is the solution of (50).*

(ii) *The gradient of* $u_2^\varepsilon$ *extended by zero outside* $\Omega^\varepsilon$, *namely* $\widetilde{\nabla u_2^\varepsilon}$, *converges to* $\chi_{Y^*}(y)(\nabla u_2(x) + \nabla_y \hat{u}_2(x,y))$ *weakly in the sense of two-scales where* $\hat{u}_2$ *is given by (51).*

(iii) *If* $\mathbf{s}^\varepsilon \to \mathbf{s}$ *in* $L^2(\Omega)^N$ *strongly, then* $\widetilde{\nabla u_2^\varepsilon}$ *converges to* $\chi_{Y^*}(Y)(\nabla u_2(x) + \nabla_y \hat{u}_2(x,y))$ *strongly in the sense of two scales. In particular, we have the convergence of energies:*

$$
\int_{\Omega^\varepsilon} |\nabla u_2^\varepsilon|^2\, dx \to \int_{\Omega\times Y^*} |\nabla u_2(x) + \nabla_y \hat{u}_2(x,y)|^2\, dx\, dy
$$

*Corollary 2.12. If* $\mathbf{s}^\varepsilon \to \mathbf{s}$ *in* $L^2(\Omega)^N$ *strongly, then for any sequence* $\{\phi^\varepsilon\}$ *bounded in* $L^2(\Omega)^N$ *and converging weakly to* $\phi_0(x,y) \in L^2(\Omega\times Y)^N$ *in the sense of two-scales, we have*

$$
\widetilde{\nabla u_2^\varepsilon} \cdot \phi^\varepsilon \rightharpoonup \int_{Y^*} (\nabla u_2(x) + \nabla_y \hat{u}_2(x,y)) \cdot \phi_0(x,y)\, dy
$$

*in the space* $L^1(\Omega)$ *with respect to the weak topology* $\sigma(L^1, L^\infty)$. □

Let us end this paragraph with the following observations. Even though $\mathbf{s}^\varepsilon$ occurred only on the boundary of tubes in (47), the oscillation of the domain is such that at the limit we have contribution from $\mathbf{s}$ in the interior of the domain as a source term (cf. (48), (50) and (51)). Secondly, we underline the difference in the behaviour of (39) and (47): weak convergence of data implies strong convergence of the solution in (39) whereas it implies merely weak convergence of the solution in (47).

### 2.6. *Description of the macroscopic limit operator*

In this concluding paragraph of Section 2, we identify the limit operator $T$ and show that $T^\varepsilon \to T$ strongly as operators on $L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N$. Having analysed the problems (31) and (32) as $\varepsilon \to 0$ in the previous paragraphs, we are well equipped to carry out this task. We remark that it is enough to pass to the limit in the bilinear form associated with $T^\varepsilon$.

Thus, we are led to consider $\langle T^{\varepsilon}[f,\mathbf{s}],[f'^{\varepsilon},\mathbf{s}'^{\varepsilon}]\rangle$ where $[f,\mathbf{s}]$ belongs to $L^2(\Omega;L^2_{\#}(Y))\times L^2(\Omega)^N$ and

$$[f'^{\varepsilon},\mathbf{s}'^{\varepsilon}]\rightharpoonup[f',\mathbf{s}'] \quad \text{in } L^2(\Omega;L^2_{\#}(Y))\times L^2(\Omega)^N \text{ weak}$$

Our task will be accomplished if we pass to the limit and express

$$\lim_{\varepsilon\to 0}\langle T^{\varepsilon}[f,\mathbf{s}],[f'^{\varepsilon},\mathbf{s}'^{\varepsilon}]\rangle=\langle T[f,\mathbf{s}],[f'.\mathbf{s}']\rangle \tag{52}$$

for a suitably defined operator $T$.

To this end, let us denote by $u_1^{\varepsilon}$, $u_2^{\varepsilon}$, the solutions of problems (31) and (32) associated with $[f,\mathbf{s}]$. Let $u_1'^{\varepsilon}$, $u_2'^{\varepsilon}$ be the corresponding solutions associated with $[f'^{\varepsilon},\mathbf{s}'^{\varepsilon}]$. Our task is to pass to the limit in formulae (36)–(38). For this purpose, it is important to realize that the gradients of $u_1^{\varepsilon}$, $u_2^{\varepsilon}$ converge strongly in the sense of two scales. Let $[u_j,\hat{u}_j]$, $[u_j',\hat{u}_j']$, $j=1,2$ be the limits of $u_j^{\varepsilon}$, $u_j'^{\varepsilon}$, $j=1,2$ as per Theorems 2.9 and 2.11. In other words, they are solutions of (42) and (48) with data $[f,\mathbf{s}]$ and $[f',\mathbf{s}']$, respectively.

Using these facts, it is easy to establish the strong convergence of $T^{\varepsilon}$. Indeed, we use the expression (35) and pass to the limit in individual terms on the right side. Thus, we will be computing the strong limits of $T_{11}^{\varepsilon}$, $T_{12}^{\varepsilon}$, $T_{21}^{\varepsilon}$, $T_{22}^{\varepsilon}$ and hence $T^{\varepsilon}$. For instance, taking $\phi^{\varepsilon}(x)=\widetilde{\nabla u_1^{\varepsilon}}(x)$ and applying Corollary 2.10, we can pass to the limit in (36) and this gives

$$\int_{\Omega\times Y}(T_{11}^{\varepsilon}f)f'^{\varepsilon}\to\frac{1}{c_0^2}\int_{\Omega\times Y^*}(\nabla u_1(x)+\nabla_y\hat{u}_1(x,y))\cdot(\nabla u_1'(x)+\nabla_y\hat{u}_1'(x,y))\,\mathrm{d}x\,\mathrm{d}y$$

which, owing to (45), is equal to

$$\frac{1}{c_0^2}\int_{\Omega\times Y^*}f'(x,y)u_1(x)\,\mathrm{d}x\,\mathrm{d}y$$

This motivates us to define the operator $T_{11}$ as follows:

$$T_{11}:L^2(\Omega;L^2_{\#}(Y))\to L^2(\Omega;L^2_{\#}(Y)),\quad T_{11}f=\frac{1}{c_0^2}u_1(x)\chi_{Y^*}(y) \tag{53}$$

where $u_1$ is the solution of (43). The above arguments establish that

$$T_{11}^{\varepsilon}\to T_{11} \text{ strongly as operators on } L^2(\Omega;L^2_{\#}(Y))$$

In the same manner, we can demonstrate the strong convergence of $T_{12}^{\varepsilon}$, $T_{21}^{\varepsilon}$ and $T_{22}^{\varepsilon}$ and their limits, denoted as $T_{12}$, $T_{21}$ and $T_{22}$, respectively, are defined as follows:

$$T_{12}:L^2(\Omega)^N\to L^2(\Omega;L^2_{\#}(Y)),\quad T_{12}\mathbf{s}=\frac{1}{c_0^2}u_2(x)\chi_{Y^*}(y) \tag{54}$$

The strong limit of $T_{21}^{\varepsilon}$ is computed as follows: Passing to the limit in (37), we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega}(T_{21}^{\varepsilon}f)\mathbf{s}'^{\varepsilon}\,\mathrm{d}x=\frac{\rho}{k}\int_{\Omega\times Y^*}(\nabla u_1(x)+\nabla y\hat{u}_1(x,y))\cdot(\nabla u_2'(x)+\nabla y\hat{u}_2'(x,y))\,\mathrm{d}x\,\mathrm{d}y$$

 

which, because of (49), is equal to

$$= -\frac{\rho}{k}\int_{\Omega\times H} \mathbf{s}'(x)\cdot(\nabla u_1(x) + \nabla y\hat{u}_1(x,y))\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{\rho}{k}\left[-(1-\theta)\int_\Omega \mathbf{s}'(x)\cdot\nabla u_1(x)\,\mathrm{d}x + \int_\Omega \mathbf{s}'(x)\cdot\int_{\partial H}\hat{u}_1(x,y)\mathbf{n}(y)\,\mathrm{d}\gamma(y)\,\mathrm{d}x\right]$$

where we recall that the normal $\mathbf{n}$ is directed inwards $H$. This expression can further be simplified using (44) and the definition of the homogenized coefficients $A_{jk}$. After some algebra, we find

$$\lim_{\varepsilon\to 0}\int_\Omega (T_{21}^\varepsilon f)\mathbf{s}'^\varepsilon\,\mathrm{d}x = \frac{\rho}{k}\int_\Omega \mathbf{s}'(x)\cdot(A-I)\nabla u_1(x)\,\mathrm{d}x$$

Thus the strong limit of $T_{21}^\varepsilon$ is described as

$$T_{21} : L^2(\Omega; L^2_\#(Y)) \to L^2(\Omega)^N, \quad T_{21}f = \frac{\rho}{k}(A-I)\nabla u_1(x) \tag{55}$$

The calculation of the strong limit of $T_{22}^\varepsilon$ is analogous. Instead of (44), we use (51). This yields

$$\lim_{\varepsilon\to 0}\int_\Omega (T_{22}^\varepsilon \mathbf{s})\mathbf{s}'^\varepsilon\,\mathrm{d}x = \frac{\rho}{k}\int_\Omega \mathbf{s}'(x)\cdot\{(A-I)\nabla u_2 - (A-\theta I)\mathbf{s}\}\,\mathrm{d}x + \lim_{\varepsilon\to 0}\frac{m\varepsilon^N}{k}(P^\varepsilon \mathbf{s})_i\cdot(P^\varepsilon \mathbf{s}'^\varepsilon)_i$$

The above limit can be calculated because we have $\varepsilon^N P^\varepsilon = (E^\varepsilon)^*$ (cf. (16)) and

$$E^\varepsilon P^\varepsilon \to I \quad \text{strongly in } \mathscr{L}(L^2(\Omega)^N)$$

Using these, we obtain

$$\lim_{\varepsilon\to 0}\varepsilon^N(P^\varepsilon \mathbf{s})_i\cdot(P^\varepsilon \mathbf{s}'^\varepsilon)_i = \int_\Omega \mathbf{s}(x)\cdot\mathbf{s}'(x)\,\mathrm{d}x$$

All these computations show that the strong limit of $T_{22}^\varepsilon$ is given by

$$\begin{aligned} &T_{22} : L^2(\Omega)^N \to L^2(\Omega)^N \\ &T_{22}\mathbf{s} = \frac{\rho}{k}\{(A-I)\nabla u_2(x) - (A-\theta I)\mathbf{s}\} + \frac{m}{k}\mathbf{s} \end{aligned} \tag{56}$$

Here $u_2$ is the solution of problem (50).

Gathering all the information obtained above, we are now in a position to obtain (52). Indeed, passing to the limit in (35), we get

$$\lim_{\varepsilon\to 0}\langle T^\varepsilon[f,\mathbf{s}],[f'^\varepsilon,\mathbf{s}'^\varepsilon]\rangle = \frac{1}{c_0^2}\left[\int_{\Omega\times Y^*} f'(x,y)u(x,y)\chi_{Y^*}(y)\,\mathrm{d}x\,\mathrm{d}y\right.$$
$$\left. + \int_\Omega \mathbf{s}'(x)\cdot\{(A-I)\nabla u(x) - (A-\theta I)\mathbf{s}(x)\}\,\mathrm{d}x + \frac{m}{\rho}\int_\Omega \mathbf{s}'(x)\cdot\mathbf{s}(x)\,\mathrm{d}x\right]$$

where we have $u = u_1 + u_2$. The right hand side can be expressed as $\langle T[f,\mathbf{s}],[f',\mathbf{s}']\rangle$ in terms of the weighted inner product (28) on the product space $L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N$ where $T$ is given by

$$\begin{aligned} &T : L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N \to L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N \\ &T[f,\mathbf{s}] = \left[\frac{1}{c_0^2}u\chi_{Y^*}, \frac{\rho}{k}\{(A-I)\nabla u - (A-\theta I)\mathbf{s}\} + \frac{m}{k}\mathbf{s}\right] \end{aligned} \tag{57}$$

 

In the above definition, we recall that $u = u_1 + u_2$ is the unique solution of

$$\begin{aligned} -\mathrm{div}\,(A\nabla u) &= \int_{Y^*} f(x,y)\,\mathrm{d}y + \mathrm{div}\,((I-A)\mathbf{s}) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned} \tag{58}$$

We have thus proved:

*Theorem 2.13. The operators $T^{\varepsilon}_{11}$, $T^{\varepsilon}_{12}$, $T^{\varepsilon}_{21}$, $T^{\varepsilon}_{22}$ and $T^{\varepsilon}$ defined by* (33) *and* (34) *converge strongly and their limits are given, respectively, by* (53), (54), (55), (56) *and* (57). □

As an easy consequence of the above result, we reach the following main conclusion of this section:

*Theorem 2.14. Given $\lambda \in \sigma(T)$, there exists a sequence $\lambda^n \in \sigma(T^{\varepsilon_n})$ such that $\lambda^n \to \lambda$. In other words, $\sigma(T) \subseteq \sigma_\infty$.*

For the proof of the above theorem, we refer the reader to Proposition 2.1.11 of [2]. The above result says that the spectrum is lower semi-continuous in the operator when operator space is equipped with strong convergence. $T$, being the (strong) limit of self-adjoint operators $T^{\varepsilon}$, is itself self-adjoint with respect to the inner product (28). However, $T$ is non-compact, as is easily seen from (57). Unlike $T^{\varepsilon}$ (or $\widetilde{T}^{\varepsilon}$), the definition of $T$ does not involve any inhomogeneities and $\sigma(T)$ is easy to calculate. In this sense, the above result presents a simplified picture of a subset of $\sigma_\infty$. Since $T$ is obtained via classical homogenization techniques, its spectrum $\sigma(T)$ is sometimes denoted as $\sigma_{\text{homo}}$.

## 3. BLOCH WAVE HOMOGENIZATION OF THE HELMHOLTZ MODEL

The conclusion $\sigma(T) \subseteq \sigma_\infty$ that is reached in the last section has its advantages. It shows that a part (defined by $\sigma(T)$) of the complicated set $\sigma_\infty$ admits rather a simple description. Indeed, the problem (57), defining $T$, is straight forward to solve (unlike the problem (12) defining $\widetilde{T}^{\varepsilon}$) as the domain on which it is posed does not involve any complicated geometry. On the other hand, the result $\sigma(T) \subseteq \sigma_\infty$ is not entirely satisfactory because it is, in general, a strict inclusion (as the heuristics given below show) and so it does not provide a simplified picture of the entire set $\sigma_\infty$. The task ahead is to get simple characterizations of other elements of $\sigma_\infty$, if possible. In other words, we are especially interested in $\lambda \in \sigma_\infty$ but not in $\sigma(T)$. Of course, by definition, there exists a sequence $\lambda^n \in \sigma(T^{\varepsilon_n})$ which converges to $\lambda$. When $\lambda \notin \sigma(T)$, the normalized eigenvectors $[f^n, \mathbf{s}^n]$ of $T^{\varepsilon_n}$ associated with $\lambda^n$ should converge to zero weakly in $L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N$. (For otherwise, as $T^{\varepsilon_n}$ converges strongly, we can pass to the limit in the eigenvalue equation of $T^{\varepsilon_n}$ and we see that $\lambda$ is an eigenvalue for $T$.) This means that these eigenvectors, representing the vibrations of fluid–solid structure, exhibit oscillations which are not captured by problem (57).

On closer examination of problem (57), we infer that the oscillations captured by it are in the space $L^2(\Omega; L^2_\#(Y))$ and their effects can be seen in the definition of the homogenized matrix $A$ via the test functions $w_m$ defined by the cell problem (23). This, in particular, shows that the only oscillating motion that is accounted for in (57) is the one in which the fluid-solid portions in various cells $Y^{\varepsilon}_i$ in the domain move 'in phase'. Heuristically, one can imagine also configurations in which these portions exhibit motions 'out of phase'. One can distinguish three types of such motions.

*Case* (A). There is a possibility of grouping the (interior) cells in such a way that portions inside each group may perform motions 'out of phase' but different groups exhibit motions 'in phase'. It is possible that all portions oscillate 'out of phase' in which case the number of groups is just one. There is also another extreme case in which all portions move 'in phase' and in such a case, the number of groups is large and equal to the number of cells $Y_i^\varepsilon$. We realize, in no time, that this extreme case was treated in Section 2.

*Case* (B). Oscillations entirely concentrate near the boundary $\Gamma$.

*Case* (C). A mixture of configurations allowed by Cases (A) and (B) above.

It turns out that the frequencies of vibrations described in Case (C) are not needed to characterize $\sigma_\infty$; the oscillations described in Cases (A) and (B) are sufficient and they already exhaust $\sigma_\infty$. The above statements will be confirmed rigorously in the form of a Completeness Theorem which is proved in Section 4. Deferring a thorough study of Case (B) to a subsequent paper, it is the goal of the present section to implement mathematically the heuristic idea involved in Case (A).

Assuming, without loss of generality, that each group is a big cube of size $K\varepsilon$ consisting of $K^N$ cells $Y_i^\varepsilon$ ($K$ is a given integer $\geqslant 1$), the above idea amounts to allowing independent movements for the fluid–solid portions inside each $Y_i^\varepsilon$ and work with the reference cell $KY = \prod_{j=1}^N ]0,K[$. (The study completed is Section 2 corresponds to $K=1$). Once this is accepted in principle, the analysis parallels, to a certain extent, that of Section 2. However, there are important changes which we wish to highlight in this section.

Conclusion that is reached here can be roughly stated as follows: there exists a 'simple' operator $\widetilde{T}^K$ (whose structure can be made more explicit) such that $\sigma(\widetilde{T}^K) \subseteq \sigma_\infty$. We then vary $K \in \mathbb{N}$ to get

$$\bigcup_{K=1}^{\infty} \sigma(\widetilde{T}^K) \subseteq \sigma_\infty$$

This is precisely the set of frequencies described in Case (A) above.

This task has been accomplished by [2] in the case of the Laplace model. We have to generalize their method by making provisions for the vibrations of the fluid which are incorporated in the Helmholtz model apart from the tube vibrations.

### 3.1. *Notations*

Given an integer $K \in \mathbb{N}$, we introduce a reference cell $KY = \prod_{j=1}^N ]0,K[$ in which there are $K^N$ tubes $\{H_j\}$ indexed by multi-integers $j = (j_1, j_2, \ldots, j_N)$ such that each $j_\ell$ satisfies $0 \leqslant j_\ell \leqslant K-1$. All these tubes are identical to $H$ introduced in the introduction and they are periodically distributed inside $KY$. To each tube $H_j$, we associate the subcell $Y_j$ and fluid subcell $Y_j^*$ analogous to $Y$ and $Y^*$, respectively. We now regard our container $\Omega$ as a periodic domain with period $\varepsilon(KY)$. With regard to cells $\varepsilon(KY), \Omega$ is assumed to satisfy the conditions analogous to the ones introduced earlier when $K=1$ in Section 1. Let $\{Y_\ell^{\varepsilon,K}, \ell \in Q_K^\varepsilon\}$ be a listing of those cells $\varepsilon(KY)$ which are completely contained in $\Omega$. Let $n_K(\varepsilon) = |Q_K^\varepsilon|$ be the number of such cells.

For other notations and assumptions, we follow [2] as closely as possible and use them freely as and when necessary.

 

### 3.2. The operator $T^{\varepsilon,K}$

The purpose of the present paragraph is to introduce the spaces containing fluid–solid movements described in Case (A) above. Subsequent step would be to define the corresponding generalized Green operator. While doing this exercise, let us keep in mind conditions (a)–(c) introduced at the start of the previous section.

It is intuitively clear that independent movements of fluid–solid portions can be accommodated only if we enlarge suitably the corresponding state spaces. Let us first take up the fluid part. Recall that the original state space $L^2(\Omega^\varepsilon)$ for the fluid potential was enlarged to $L^2(\Omega; L^2_\#(Y))$ for the analysis of Section 2. We could prove strong convergence of the fluid potential in this space (see Theorem 2.9). Heuristically, the right choice for the state space in Case (A) seems to be

$$L^2(\Omega; L^2_\#(KY)) = \{v(x,y) \in L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^N)); v(x,\cdot) \text{ is } KY\text{-periodic}\}$$

The inner product in this space is the usual one, namely

$$\langle u, v\rangle = \frac{1}{|KY|}\int_{\Omega\times KY} u(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

The corresponding extension and projection operators can be defined analogous to (25) and (26):

$$\begin{aligned} E^\varepsilon_{K,\Omega} &: L^2(\Omega) \to L^2(\Omega; L^2_\#(KY)) \\ (E^\varepsilon_{K,\Omega} f)(x,y) &= \sum_m \chi^\varepsilon_m(x) f(x^\varepsilon_m + \varepsilon y) \end{aligned} \tag{59}$$

$$\begin{aligned} P^\varepsilon_{K,\Omega} &: L^2(\Omega; L^2_\#(KY)) \to L^2(\Omega) \\ (P^\varepsilon_{K,\Omega} v)(x) &= \sum_m \chi^\varepsilon_m(x) \frac{1}{|Y^{\varepsilon,K}_m|}\int_{Y^{\varepsilon,K}_m} v\left(x', \frac{x}{\varepsilon}\right)\mathrm{d}x' \end{aligned} \tag{60}$$

Summation in the above definitions is taken over all cells $Y^{\varepsilon,K}_m$ providing a cover for $\Omega$. ($f, v$ are assumed to be extended by zero outside $\Omega$). $\chi^\varepsilon_m(x)$ is the characteristic function of the cell $Y^{\varepsilon,K}_m$ whose origin is $x^\varepsilon_m$. The following important relations are easily verified:

$$(E^\varepsilon_{K,\Omega})^* = P^\varepsilon_{K,\Omega}, \quad P^\varepsilon_{K,\Omega} E^\varepsilon_{K,\Omega} = I \quad \text{on } L^2(\Omega)$$

Let us now turn our attention to the choice of the state space for tube displacements. Originally, the state space was $\mathbb{R}^{Nn(\varepsilon)}$ which was enlarged to $L^2(\Omega)^N$ in Section 2 to treat a single tube in the reference cell $Y$. Since now there are $K^N$ tubes, the following space suggests itself as a natural choice: $S = (L^2(\Omega)^N)^{K^N}$. Its elements are denoted as $\{\mathbf{s}_j(x)\}$. The corresponding extension and projection operators are defined by

$$E^\varepsilon_K : \mathbb{R}^{Nn(\varepsilon)} \to S, \quad E^\varepsilon_K\{\mathbf{s}_i\} = \{\mathbf{s}_j(x)\}$$

with

$$\mathbf{s}_j(x) = \sum_\ell \chi^\varepsilon_\ell(x)\mathbf{s}_i$$

$$P^\varepsilon_K : S \to \mathbb{R}^{Nn(\varepsilon)}, \quad P^\varepsilon_K\{\mathbf{s}_j(x)\} = \{\mathbf{s}_i\}$$

where

$$\mathbf{s}_i = \frac{1}{|Y_\ell^{\varepsilon,K}|} \int_{Y_\ell^{\varepsilon,K}} \mathbf{s}_j(x)\,\mathrm{d}x$$

Here, $i$ is related to $(\ell, j)$ according to formula (49) in [2], wherein the following relations are also established:

$$(E_K^\varepsilon)^* = \varepsilon^N P_K^\varepsilon, \quad P_K^\varepsilon E_K^\varepsilon = I \quad \text{on } \mathbb{R}^{Nn(\varepsilon)} \tag{61}$$

The usual scalar products on $S$ and $\mathbb{R}^{Nn(\varepsilon)}$ are understood here.

After the enlargement of the state spaces, it is now time to extend the operator $\widetilde{T}^\varepsilon$ such that conditions (a)–(c) of the Section 2 are verified. Thus we introduce

$$T^{\varepsilon,K} : L^2(\Omega; L^2_\#(KY)) \times S \to L^2(\Omega; L^2_\#(KY)) \times S \tag{62}$$

$$T^{\varepsilon,K} = \begin{pmatrix} E^\varepsilon_{K,\Omega} E^\varepsilon_0 & 0 \\ 0 & E^\varepsilon_K \end{pmatrix} \widetilde{T}^\varepsilon \begin{pmatrix} P^\varepsilon_0 P^\varepsilon_{K,\Omega} & 0 \\ 0 & P^\varepsilon_K \end{pmatrix}$$

or more explicitly, let $f \in L^2(\Omega; L^2_\#(KY))$ and $\{\mathbf{s}_j(x)\} \in S$ be given. We can solve the following boundary value problem uniquely

$$\begin{aligned} -\Delta u^{\varepsilon,K} &= P^\varepsilon_0 P^\varepsilon_{K,\Omega} f \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u^{\varepsilon,K}}{\partial n} &= P^\varepsilon_K \{\mathbf{s}_j(x)\} \cdot \mathbf{n} \quad \text{on } \Gamma^\varepsilon_i, \quad i \in Q^\varepsilon \\ u^{\varepsilon,K} &= 0 \quad \text{on } \Gamma \end{aligned} \tag{63}$$

Then by definition, we have

$$T^{\varepsilon,K}[f, \{\mathbf{s}_j(x)\}] = \left[ \frac{1}{c_0^2} E^\varepsilon_{K,\Omega} E^\varepsilon_0 u^{\varepsilon,K}, E^\varepsilon_K \{\boldsymbol{\sigma}^\varepsilon_i\} \right]$$

where $\{\boldsymbol{\sigma}^\varepsilon_i\} \in \mathbb{R}^{Nn(\varepsilon)}$ being associated to $\{\mathbf{s}_i\}$ by

$$\boldsymbol{\sigma}^\varepsilon_i = \frac{\rho}{k\varepsilon^N} \int_{\Gamma^\varepsilon_i} u^{\varepsilon,K} \mathbf{n}\,\mathrm{d}\gamma + \frac{m}{k} \mathbf{s}_i \quad \forall i \in Q^\varepsilon$$

where we have posed $\{\mathbf{s}_i\} = P^\varepsilon_K \{\mathbf{s}_j(x)\}$.

The following result giving some of the desired properties of $T^{\varepsilon,K}$ is analogous to Theorem 2.6 and is derived along similar lines.

*Theorem 3.1.* (i) $T^{\varepsilon,K}$ *is a compact operator.*
(ii) $T^{\varepsilon,K}$ *is self-adjoint with respect to the weighted inner product*

$$\langle [f, \{\mathbf{s}_j(x)\}], [g, \{\mathbf{t}_j(x)\}] \rangle = \frac{1}{|KY|} \int_{\Omega \times KY} f g \,\mathrm{d}x\,\mathrm{d}y + \frac{k}{\rho c_0^2} \int_\Omega \mathbf{s}_j(x) \cdot \mathbf{t}_j(x)\,\mathrm{d}x \tag{64}$$

 

(iii) $T^{\varepsilon,K}$ *is positive definite: indeed we have*

$$\langle T^{\varepsilon,K}[f,\{\mathbf{s}_j(x)\}],[f',\{\mathbf{s}'_j(x)\}]\rangle = \frac{1}{c_0^2}\int_{\Omega^\varepsilon} \nabla u^{\varepsilon,K}\cdot\nabla u'^{\varepsilon,K}\,\mathrm{d}x + \frac{m\varepsilon^N}{\rho c_0^2} P_K^\varepsilon\{\mathbf{s}'_j(x)\}\cdot P_k^\varepsilon\{\mathbf{s}_j(x)\} \tag{65}$$

*where* $u^{\varepsilon,K}$, $u'^{\varepsilon,K}$ *correspond to* $[f,\{\mathbf{s}_j(x)\}],[f',\{\mathbf{s}'_j(x)\}]$ *respectively via* (63).

(iv) $T^{\varepsilon,K}$ *and* $\widetilde{T}^\varepsilon$ *have same spectra.*

The above result shows, in particular, that condition (a) holds, i.e. that $\sigma(T^{\varepsilon,K})$ is identical with Helmholtz spectrum and so we are motivated to study the asymptotic behaviour of $T^{\varepsilon,K}$, fixing $K$ and letting $\varepsilon\to 0$ and thereby establish conditions (b) and (c). Since the definition of $T^{\varepsilon,K}$ involves problem (63), the main step is to analyse (63) as $\varepsilon\to 0$. As in Section 2, we will establish that $T^{\varepsilon,K}$ converges strongly to a certain limit operator $T^K$ by passing to the limit in the bilinear form (65). In order to simplify matters, we decompose the operators $T^{\varepsilon,K}$ by writing $u^{\varepsilon,K} = u_1^{\varepsilon,K} + u_2^{\varepsilon,K}$ where $u_1^{\varepsilon,K}, u_2^{\varepsilon,K}$ are solutions of

$$\begin{aligned} -\Delta u_1^{\varepsilon,K} &= P_0^\varepsilon P_{K,\Omega}^\varepsilon f \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u_1^{\varepsilon,K}}{\partial n} &= 0 \quad \text{on } \Gamma_i^\varepsilon \quad i\in Q^\varepsilon \\ u_1^{\varepsilon,K} &= 0 \quad \text{on } \Gamma \end{aligned} \tag{66}$$

$$\begin{aligned} -\Delta u_2^{\varepsilon,K} &= 0 \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u_2^{\varepsilon,K}}{\partial n} &= P_K^\varepsilon\{\mathbf{s}_j(x)\}\cdot\mathbf{n} \quad \text{on } \Gamma_i^\varepsilon, \quad i\in Q^\varepsilon \\ u_2^{\varepsilon,K} &= 0 \quad \text{on } \Gamma \end{aligned} \tag{67}$$

Based on the variables $(u_1^{\varepsilon,K}, u_2^{\varepsilon,K})$, we can express the operator $T^{\varepsilon,K}$ in the matrix form:

$$T^{\varepsilon,K} = \begin{bmatrix} T_{11}^{\varepsilon,K}, & T_{12}^{\varepsilon,K} \\ T_{21}^{\varepsilon,K}, & T_{22}^{\varepsilon,K} \end{bmatrix}$$

where the various entries are given by the rules below:

$$T_{11}^{\varepsilon,K} : L^2(\Omega; L^2_\#(KY)) \to L^2(\Omega; L^2_\#(KY))$$

$$T_{11}^{\varepsilon,K} f = \frac{1}{c_0^2} E_{K,\Omega}^\varepsilon E_0^\varepsilon u_1^{\varepsilon,K}$$

$$T_{12}^{\varepsilon,K} : S \to L^2(\Omega; L^2_\#(KY))$$

$$T_{12}^{\varepsilon,K}\{\mathbf{s}_j(x)\} = \frac{1}{c_0^2} E_{K,\Omega}^\varepsilon E_0^\varepsilon u_2^{\varepsilon,K}$$

$$T_{21}^{\varepsilon,K} : L^2(\Omega; L^2_\#(KY)) \to S$$

$$T_{21}^{\varepsilon,K} f = E_K^\varepsilon \left\{ \frac{\rho}{k\varepsilon^N} \int_{\Gamma_i^\varepsilon} u_1^{\varepsilon,K}\mathbf{n}\,\mathrm{d}\gamma \right\}$$

 

$$T_{22}^{\varepsilon,K} : S \to S$$

$$T_{22}^{\varepsilon,K}\{\mathbf{s}_j(x)\} = E_K^{\varepsilon}\left\{\frac{\rho}{k\varepsilon^N}\int_{\Gamma_i^{\varepsilon}} u_2^{\varepsilon,K}\mathbf{n}\,\mathrm{d}\gamma + \frac{m}{k}\mathbf{s}_i\right\}$$

with $\{\mathbf{s}_i\} = P_K^{\varepsilon}\{\mathbf{s}_j(x)\}$.

The bilinear form of $T^{\varepsilon,K}$ can also be split as follows:

$$\begin{aligned}\langle T^{\varepsilon,K}[f,\{\mathbf{s}_j(x)\}],[f',\{\mathbf{s}_j'(x)\}]\rangle &= \frac{1}{|KY|}\int_{\Omega\times KY}(T_{11}^{\varepsilon,K}f)f'\,\mathrm{d}x\,\mathrm{d}y\\ &+\frac{1}{|KY|}\int_{\Omega\times KY}(T_{12}^{\varepsilon,K}\{\mathbf{s}_j(x)\})f'\,\mathrm{d}x\,\mathrm{d}y\\ &+\frac{k}{\rho c_0^2}\int_{\Omega}(T_{21}^{\varepsilon,K}f)\cdot\{\mathbf{s}_j'(x)\}\,\mathrm{d}x\\ &+\frac{k}{\rho c_0^2}\int_{\Omega}(T_{22}^{\varepsilon,K}\{\mathbf{s}_j(x)\})\cdot\{\mathbf{s}_j'(x)\}\,\mathrm{d}x \end{aligned} \tag{68}$$

It will be useful if we express each of the above integrals in terms of $u_i^{\varepsilon,K}$, $u_i'^{\varepsilon,K}$ and $\{\mathbf{s}_j(x)\}$, $\{\mathbf{s}_j'(x)\}$:

$$\frac{1}{|KY|}\int_{\Omega\times KY}(T_{11}^{\varepsilon,K}f)f'\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{c_0^2}\int_{\Omega^{\varepsilon}}\nabla u_1^{\varepsilon,K}\cdot\nabla u_1'^{\varepsilon,K}\,\mathrm{d}x \tag{69}$$

$$\frac{1}{|KY|}\int_{\Omega\times KY}(T_{12}^{\varepsilon,K}\{\mathbf{s}_j(x)\})f'\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{c_0^2}\int_{\Omega^{\varepsilon}}\nabla u_1'^{\varepsilon,K}\cdot\nabla u_2^{\varepsilon,K}\,\mathrm{d}x$$

$$\int_{\Omega}(T_{21}^{\varepsilon,K}f)\cdot\{\mathbf{s}_j'(x)\}\,\mathrm{d}x = \frac{\rho}{k}\int_{\Omega^{\varepsilon}}\nabla u_1^{\varepsilon,K}\cdot\nabla u_2'^{\varepsilon,K}\,\mathrm{d}x,$$

$$\int_{\Omega}(T_{22}^{\varepsilon,K}\{\mathbf{s}_j(x)\})\cdot\{\mathbf{s}_j'(x)\}\,\mathrm{d}x = \frac{\rho}{k}\int_{\Omega^{\varepsilon}}\nabla u_2^{\varepsilon,K}\cdot\nabla u_2'^{\varepsilon,K}\,\mathrm{d}x + \frac{m\varepsilon^N}{k}P_K^{\varepsilon}\{\mathbf{s}_j(x)\}\cdot P_K^{\varepsilon}\{\mathbf{s}_j'(x)\} \tag{70}$$

*Remark 3.2.* Analogous to Remark 2.7, we can introduce the operator

$$T_s^{\varepsilon,K} = \begin{bmatrix} T_{11}^{\varepsilon,K} & (\frac{\rho}{k})^{1/2}c_0 T_{12}^{\varepsilon,K}\\ (\frac{k}{\rho})^{1/2}\frac{1}{c_0}T_{21}^{\varepsilon,K} & T_{22}^{\varepsilon,K}\end{bmatrix}$$

on $L^2(\Omega; L^2_{\#}(KY))\times S$. One can easily verify that $T^{\varepsilon,K}$ and $T_s^{\varepsilon,K}$ are similar (and hence their spectra coincide). Further $T_s^{\varepsilon,K}$ is self-adjoint with respect to the standard inner product

$$\langle[f,\{\mathbf{s}_j(x)\}],[g,\{\mathbf{t}_j(x)\}]\rangle = \frac{1}{|KY|}\int_{\Omega\times KY}fg\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega}\mathbf{s}_j(x)\cdot\mathbf{t}_j(x)\,\mathrm{d}x$$

In other words, we can remove $c_0$ from the definition (64) of the inner product and transfer it to the operator $T_s^{\varepsilon,K}$. This has advantages in case we wish to do asymptotics when $c_0$ also varies with $\varepsilon$.

 

### 3.3. *Homogenization of problem (66)*

The purpose of this paragraph is similar to that of Section 2.4; indeed we seek to describe the two-scale behaviour of the gradient of solution of problem (66). A look at expressions (68) and (69) shows that this will be necessary to pass to the limit in the bilinear form

$$\langle T^{\varepsilon,K}[f,\{\mathbf{s}_j(x)\}],[f'^{\varepsilon}(x),\{\mathbf{s}_j'^{\varepsilon}(x)\}]\rangle$$

with weakly convergent sequences $[f'^{\varepsilon},\{\mathbf{s}_j'^{\varepsilon}(x)\}]$. (Of course, a similar analysis of problem (67) is also necessary, but this will be done later). Once done, this will establish the strong convergence of $T_{11}^{\varepsilon,K}$ towards a limit $T_{11}^{K}$ as $\varepsilon \to 0$. In a subsequent paragraph, we will present a description of $T_{11}^{K}$ which will bring out its structure clearly in 'simple' terms.

Let us therefore rewrite equation (66) with varying right side $f^{\varepsilon} \in L^2(\Omega; L^2_{\#}(KY))$:

$$\begin{aligned} -\Delta u_1^{\varepsilon,K} &= P_0^{\varepsilon} P_{K,\Omega}^{\varepsilon} f^{\varepsilon} \quad \text{in } \Omega^{\varepsilon} \\ \frac{\partial u_1^{\varepsilon,K}}{\partial n} &= 0 \quad \text{on } \Gamma_i^{\varepsilon}, \quad i \in Q^{\varepsilon} \\ u_1^{\varepsilon,K} &= 0 \quad \text{on } \Gamma \end{aligned} \tag{71}$$

In order to pass to the limit in (71), we need the following proposition which is analogous to Proposition 2.8.

*Proposition 3.3. Let $f^{\varepsilon} \rightharpoonup f$ weakly in $L^2(\Omega; L^2_{\#}(KY))$. Then $P_{K,\Omega}^{\varepsilon} f^{\varepsilon} \rightharpoonup f(x,y)$ weakly in the sense of two-scales on $\Omega$ defined by the cell $KY$. More precisely,*

$$\int_{\Omega} (P_{K,\Omega}^{\varepsilon} f^{\varepsilon})(x)\chi\left(x,\frac{x}{\varepsilon}\right) \mathrm{d}x \to \frac{1}{|KY|}\int_{\Omega\times KY} f(x,y)\chi(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \forall \chi \in \mathscr{D}(\Omega; C^{\infty}_{\#}(KY)).$$

The subtle difference between the above result and Proposition 2.8 lies in the choice of the test functions $\chi(x,y)$. However, the proof remains same and so will not be repeated here.

Owing to the above result, it is straightforward to obtain the two-scale limit of (71) using the arguments indicated in Theorem 2.9.

*Theorem 3.4. Let $f^{\varepsilon} \rightharpoonup f$ weakly in $L^2(\Omega; L^2_{\#}(KY))$. Then the solution $u_1^{\varepsilon,K}$ of (71) has the following behaviour*:

(i) $X^{\varepsilon} u_1^{\varepsilon,K} \rightharpoonup u_1^K$ *weakly in* $H_0^1(\Omega)$.

(ii) $\widetilde{\nabla} u_1^{\varepsilon,K}$, *the gradient of* $u_1^{\varepsilon,K}$ *extended by zero outside* $\Omega^{\varepsilon}$, *converges to* $\chi_{KY^*}(y)(\nabla u_1^K(x) + \nabla_y \hat{u}_1^K(x,y))$ *strongly in the sense of two-scales. In particular, we have*

$$\lim_{\varepsilon\to 0}\int_{\Omega^{\varepsilon}} |\nabla u_1^{K,\Omega}|^2\,\mathrm{d}x = \frac{1}{|KY|}\int_{\Omega\times KY^*} |\nabla u_1^K + \nabla_y \hat{u}_1^K|^2\,\mathrm{d}x\,\mathrm{d}y$$

(iii) *The couple* $[u_1^K, \hat{u}_1^K] \in H_0^1(\Omega) \times L^2(\Omega; H^1(KY^*)/\mathbb{R})$ *is characterized as the solution of the following system*:

$$\begin{aligned} -\Delta_y \hat{u}_1^K(x,y) &= 0 \quad \text{in } \Omega \times KY^* \\ -\mathrm{div}_x \int_{KY^*} (\nabla u_1^K(x) + \nabla_y \hat{u}_1^K(x,y))\,\mathrm{d}y &= \int_{KY^*} f(x,y)\,\mathrm{d}y \quad \text{in } \Omega \\ [\nabla u_1^K(x) + \nabla_y \hat{u}_1^K(x,y)] \cdot \mathbf{n}_y &= 0 \quad \text{on } \Omega \times \partial H_j, \ \forall j \\ u_1^K(x) &= 0 \quad \text{on } \Gamma \\ y \in KY^* &\to \hat{u}_1^K(x,y) \text{ is } KY^*\text{-periodic} \end{aligned} \tag{72}$$

A weak formulation of (72) is useful and it is as follows:

$$\int_{\Omega\times KY^*} (\nabla u_1^K(x) + \nabla_y \hat{u}_1^K(x,y)) \cdot (\nabla v(x) + \nabla_y \hat{v}(x,y))\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega\times KY^*} f(x,y) v(x)\,\mathrm{d}x\,\mathrm{d}y \tag{73}$$

for all $v \in H_0^1(\Omega)$ and $\hat{v} \in L^2(\Omega; H_\#^1(KY^*)/\mathbb{R})$.

It is clear from (73) that we have existence and uniqueness of a solution to (72) via Lax–Milgram Lemma. It is in the next paragraph that we analyse the structure of this solution. To finish the present paragraph, let us draw the following consequence of the strong convergence of $\widetilde{\nabla u_1^{\varepsilon,K}}$ in the sense of two scales.

*Corollary 3.5. For any sequence* $\{\phi^\varepsilon\} \subset L^2(\Omega)^N$ *converging weakly in the sense of two-scales to* $\phi_0(x,y) \in L^2(\Omega \times KY)^N$, *we have*

$$\widetilde{\nabla u_1^{\varepsilon,K}} \cdot \phi^\varepsilon \rightharpoonup \int_{KY^*} (\nabla u_1^K(x) + \nabla_y \hat{u}_1^K(x,y)) \cdot \phi_0(x,y)\,\mathrm{d}y$$

*in the space* $L^1(\Omega)$ *provided with the weak topology* $\sigma(L^1, L^\infty)$.

### 3.4. *Structure of the homogenized problem* (72)

In this paragraph, we study the structure of the solution $(u_1^K, \hat{u}_1^K)$ of problem (72) obtained as the two-scale limit of problem (71). Even though system (72) looks deceptively similar to (42), the structure of the solution is very different. Indeed, the periodic boundary condition in (72) does not allow us to easily separate the variables $x$ and $y$ via the introduction of test functions such as $w_m$ (cf. (23), (43) and (44)). That is where the crucial difference between the approach followed in Section 2 and the one to be proposed here appears. Since $\hat{u}_1^K$ is $KY^*$-periodic, the idea is to decompose it in terms of Bloch waves to be introduced below. This justifies the title of this section. This method which has been applied to a class of problems in the book [1] will be briefly taken up here. Working in Bloch space has its own advantages; we will be able to separate the variables $x$ and $y$. The basic result underlying this method is the following result on Bloch decomposition according to which any function that is $KY^*$-periodic can be decomposed into functions which satisfy the so-called $(\mathrm{e}^{2\pi i j/K}; Y^*)$ periodicity condition. More generally, let us introduce

 

*Definition 3.6. Let $\theta \in \mathbb{R}^N$. A function $w \in L^2_{loc}(\mathbb{R}^N)$ is said to be $(e^{2\pi i\theta}; Y)$-periodic, or more simply $(\theta; Y)$-periodic if*

$$w(y+j) = e^{2\pi i\theta\cdot j} w(y) \quad \forall j \in \mathbb{Z}^N$$

*alternatively, if $e^{2\pi i\theta\cdot y} w(y)$ is $Y$-periodic.*

Such a class of functions is denoted by $L^2_\#(e^{2\pi i\theta}; Y)$. Observe that this space coincides with the class of periodic functions $L^2_\#(Y)$ when $\theta = 0$. Analogous to $H^1_\#(Y)$, we define

$$H^1_\#(e^{2\pi i\theta}; Y) = \{w \in H^1_{loc}(\mathbb{R}^N) \mid w \text{ is } (e^{2\pi i\theta}; Y)\text{-periodic}\}$$

With these notations, let us cite the result from [1].

*Lemma 3.7. We have the orthogonal decompositions*

$$L^2_\#(KY^*) = \bigoplus_{0 \leqslant j \leqslant K-1} L^2_\#(e^{2\pi i j/K}; Y^*)$$

$$H^1_\#(KY^*) = \bigoplus_{0 \leqslant j \leqslant K-1} H^1_\#(e^{2\pi i j/K}; Y^*)$$

*More precisely, given $w \in L^2_\#(KY^*)$ (resp. $H^1_\#(KY^*)$), there exist unique $\{w_j\} \subset L^2_\#(Y^*)$ (resp. $H^1_\#(Y^*)$ such that*

$$w(y) = \sum_{0 \leqslant j \leqslant K-1} w_j(y) e^{2\pi i j/K\cdot y}$$

*Further the following Parseval's equality holds:*

$$\frac{1}{K^N}\int_{KY^*} |w(y)|^2\,dy = \sum_{0 \leqslant j \leqslant K-1} \int_{Y^*} |w_j(y)|^2\,dy$$

The interest in the above result is obvious. We will be able to reduce $KY^*$-periodic problems such as (72) to problems on $Y^*$. Indeed, treating $x \in \Omega$ as a parameter, we decompose

$$\hat{u}_1^K(x,y) = \sum_{0 \leqslant j \leqslant K-1} \hat{u}_1^{K,j}(x,y) e^{2\pi i j/K\cdot y} \tag{74}$$

$$f(x,y) = \sum_{0 \leqslant j \leqslant K-1} f^j(x,y) e^{2\pi i j/K\cdot y} \tag{75}$$

where $\hat{u}^{K,j}(x,\cdot) \in H^1_\#(Y^*)$, $f^j(x,\cdot) \in L^2_\#(Y^*)$. Since the above decomposition is orthogonal, the system decouples into sub-systems for various $j$-components. Another observation is that when we substitute (74), (75) into (72), the terms corresponding to $j \neq 0$ have integrals zero. Thus we can rewrite (72) in the Bloch space separating the components corresponding to $j = 0$ and $\neq 0$ as follows:

$$\begin{aligned}
-\Delta_y \hat{u}_1^{K,0}(x,y) &= 0 \quad \text{in } \Omega \times Y^* \\
-\mathrm{div}_x \int_{Y^*} (\nabla u_1^K(x) + \nabla_y \hat{u}_1^{K,0}(x,y))\,dy &= \int_{Y^*} f^0(x,y)\,dy \quad \text{in } \Omega \\
[\nabla u_1^K(x) + \nabla_y u_1^{K,0}(x,y)] \cdot \mathbf{n}_y &= 0 \quad \text{on } \Omega \times \partial H \\
u_1^K(x) &= 0 \quad \text{on } \Gamma \\
y \in Y^* \to \hat{u}_1^{K,0}(x,y) &\text{ is } Y^*\text{-periodic}
\end{aligned} \tag{76}$$

$$
\begin{aligned}
&-\Delta_y(\hat{u}_1^{K,j}(x,y)e^{2\pi i\frac{j}{K}\cdot y})=0 \quad \text{in } \Omega\times Y^* \\
&\nabla_y(\hat{u}_1^{K,j}(x,y)\mathrm{e}^{2\pi \mathrm{i} j/K\cdot y})\cdot \mathbf{n}_y=0 \quad \text{on } \Omega\times\partial H \\
&y\in Y^*\to \hat{u}_1^{K,j}(x,y) \text{ is } Y^*\text{-periodic}
\end{aligned}
\tag{77}
$$

Following our usual practice (see (42), (43) and (44)), we can separate the variables $x$ and $y$ in (76). We can assert that $u_1^K$ solves the homogenized equation:

$$
\begin{aligned}
&-\mathrm{div}\,(A\nabla u_1^K(x)) = \int_{Y^*} f^0(x,y)\,\mathrm{d}y = \frac{1}{|KY|}\int_{KY^*} f(x,y)\,\mathrm{d}y \quad \text{in } \Omega \\
&u_1^K(x)=0 \quad \text{on } \Gamma
\end{aligned}
\tag{78}
$$

Further $\hat{u}_1^{K,0}$ can be expressed as

$$
\hat{u}_1^{K,0}(x,y)=\sum_{m=1}^{N} w_m(y)\frac{\partial u_1^K}{\partial x_m}(x)
$$

where, we recall, $w_m$ are solutions of the cell problems (23).

On the other hand, multiplying (77) by $\hat{u}_1^{K,j}(x,y)\mathrm{e}^{2\pi\mathrm{i}(j/K)\cdot y}$ and integrating by parts, we deduce easily that

$$
\hat{u}_1^{K,j}(x,y)\equiv 0 \quad \text{if } j\neq 0 \tag{79}
$$

Thus (74) becomes

$$
\hat{u}_1^K(x,y)=\hat{u}_1^{K,0}(x,y)=\sum_{m=1}^{N} w_m(y)\frac{\partial u_1^K}{\partial x_m}(x)
$$

Thus, in the above description, the parameter $K$ appears only at one place and it is in the right side of equation (78). We realize that this dependence is very weak. The crucial thing is to observe from (79) that the oscillations are reduced to that of system (42), (43) and (44) which corresponds to the case $K=1$. This is somewhat surprising: even though, we give a forcing in (71) which admits $KY$-oscillations, the gradient of the solution $u_1^{\varepsilon,K}$ oscillates only on $Y$-scale. These later oscillations, are already captured and described by our framework in Section 2. So, it does not seem to be necessary to consider the state space $L^2(\Omega; L^2_\#(KY))$ for the fluid potential with $K\geqslant 2$. (However, as will be seen in the next paragraph, the state space $S=(L^2(\Omega)^N)^{K^N}$ is needed to describe tube vibrations.) Reason for this phenomenon may be that the speed of propagation in the fluid region is $c_0$ which is independent of $\varepsilon$ and hence is of $O(1)$ as $\varepsilon\to 0$. Any disturbance created in the fluid region will travel fast and will not be felt in the $\varepsilon$-scale. (The case $K=1$ seems to be an exception to this interpretation). If, however, $c_0=O(\varepsilon)$, then there will be further interaction between fluid and the tubes and the whole scenario will change. This will be investigated in a future work.

### 3.5. *Homogenization of problem* (67)

As the next step in passing to the limit in the bilinear form associated with $T^{\varepsilon,K}$, let us study the asymptotic behaviour of (67). We will rewrite it with the data $\{\mathbf{s}_j^\varepsilon(x)\}\in S=(L^2(\Omega)^N)^{K^N}$ which varies with $\varepsilon$

$$\begin{aligned} -\Delta u_2^{\varepsilon,K} &= 0 \quad \text{in } \Omega^\varepsilon \\ \frac{\partial u_2^{\varepsilon,K}}{\partial n} &= P_K^\varepsilon\{\mathbf{s}_j^\varepsilon(x)\}\cdot\mathbf{n} \quad \text{on } \Gamma_i^\varepsilon, \quad i\in Q^\varepsilon \\ u_2^{\varepsilon,K} &= 0 \quad \text{on } \Gamma \end{aligned} \tag{80}$$

Two-scale analysis of the above system has been carried out in [2]. We briefly recall their results in our notation. The main point is to note that, unlike (66), the system (80) exhibits oscillations on $KY$-scale. These are described by the following two-scale system: Find $[u_2^K, \hat{u}_2^K]\in H_0^1(\Omega)\times L^2(\Omega; H^1(KY^*)/\mathbb{R})$ satisfying

$$\begin{aligned} -\Delta_y \hat{u}_2^K(x,y) &= 0 \quad \text{in } \Omega\times KY^* \\ -\text{div}_x \int_{KY^*}(\nabla u_2^K(x)+\nabla_y \hat{u}_2^K(x,y))\,\mathrm{d}y &= |H|\sum_j \text{div}_x \mathbf{s}_j(x) \quad \text{in } \Omega \\ [\nabla u_2^K(x)+\nabla_y \hat{u}_2^K(x,y)-\mathbf{s}_j(x)]\cdot\mathbf{n}_y &= 0 \quad \text{on } \Omega\times\partial H_j, \forall j \\ u_2^K(x) &= 0 \quad \text{on } \Gamma \\ y\in KY^* \to \hat{u}_2^K(x,y) &\text{ is } KY^*\text{-periodic} \end{aligned} \tag{81}$$

The above problem admits a variational formulation which can be written as follows:

$$\begin{aligned} &\int_{\Omega\times KY^*}(\nabla u_2^K(x)+\nabla_y \hat{u}_2^K(x,y))\cdot(\nabla\phi(x)+\nabla_y\hat{\phi}(x,y))\,\mathrm{d}x\,\mathrm{d}y \\ &= |H|\int_\Omega \phi(x)\sum_{0\leqslant j\leqslant K-1}\text{div}_x\mathbf{s}_j(x)\,\mathrm{d}x + \int_\Omega \sum_{0\leqslant j\leqslant K-1}\mathbf{s}_j(x)\cdot\int_{\partial H_j}\hat{\phi}(x,y)\mathbf{n}_y\,\mathrm{d}\gamma\,\mathrm{d}x \end{aligned} \tag{82}$$

for all $[\phi,\hat{\phi}]\in H_0^1(\Omega)\times L^2(\Omega; H_\#^1(KY^*)/\mathbb{R})$.

One of the main results proved in [2] is the following:

*Theorem 3.8. Assume the data in* (80) *satisfies* $\{\mathbf{s}_j^\varepsilon(x)\}\rightharpoonup\{\mathbf{s}_j(x)\}$ *in S weak. Then the solution* $u_2^{\varepsilon,K}$ *has the following behaviour*:

(i) $X^\varepsilon u_2^{\varepsilon,K}\rightharpoonup u_2^K$ *weakly in* $H_0^1(\Omega)$.

(ii) $\tilde{\nabla} u_2^{\varepsilon,K}$, *the gradient of* $u_2^{\varepsilon,K}$ *extended by zero outside* $\Omega^\varepsilon$, *converges to* $\chi_{KY^*}(y)(\nabla u_2^K(x)+\nabla_y \hat{u}_2^K(x,y))$ *weakly in the sense of two-scales.*

(iii) *If* $\{\mathbf{s}_j^\varepsilon(x)\}\to\{\mathbf{s}_j(x)\}$ *strongly in S, then* $\tilde{\nabla} u_2^{\varepsilon,K}$ *converges to* $\chi_{KY^*}(y)(\nabla u_2^K(x)+\nabla_y \hat{u}_2^K(x,y))$ *strongly in the sense of two-scales. In particular, we have*

$$\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon}|\nabla u_2^{\varepsilon,K}(x)|^2\,\mathrm{d}x = \frac{1}{|KY|}\int_{\Omega\times KY^*}|\nabla u_2^K+\nabla_y \hat{u}_2^K|^2\,\mathrm{d}x\,\mathrm{d}y$$

*Corollary 3.9. If* $\{\mathbf{s}_j^\varepsilon(x)\}\to\{\mathbf{s}_j(x)\}$ *strongly in S, then for any sequence* $\{\phi^\varepsilon\}\rightharpoonup\phi_0(x,y)\in L^2(\Omega\times KY)^N$ *weakly in the sense of two-scales, the following holds:*

$$\tilde{\nabla} u_2^{\varepsilon,K}\cdot\phi^\varepsilon \rightharpoonup \int_{KY^*}(\nabla u_2^K(x)+\nabla_y \hat{u}_2^K(x,y))\cdot\phi_0(x,y)\,\mathrm{d}y$$

*in the space* $(L^1(\Omega),\sigma(L^1,L^\infty))$.

*3.6. Structure of the homogenized problem (81)*

Following [2], we bring out, in this paragraph, the main feature of the solution of (81). The essential characteristic is the presence of $KY$-oscillations which were absent in the problem (82). To obtain them we need to separate the variables $x$ and $y$ in (81). Though the system under consideration resembles very much (48), the technique applied to (48) (see Section 2.5) does not generalize to the present context. Indeed, the test functions $w_m(y)$, introduced in (23), which are very useful in describing the oscillations in $Y$-scale are not enough to describe $KY$-oscillations.

To overcome this difficulty, the idea is the same as in Section 3.4 and it consists of transforming system (81) from the physical space to Bloch space. It will then be seen that the variables $x$ and $y$ can be separated. Further, the method naturally introduces new test functions ($w_m^{K,j}$ to be defined below) required to describe the new oscillations.

To this end, let us apply Lemma 3.7 and decompose the solution $\hat{u}_2^K(x,y)$ of (81) (which is $KY^*$-periodic) in the form

$$\hat{u}_2^K(x,y) = \sum_{0\leqslant j\leqslant K-1} \hat{u}_2^{K,j}(x,y)\mathrm{e}^{2\pi\mathrm{i}(j/K)\cdot y} \tag{83}$$

where $\hat{u}_2^{K,j} \in L^2(\Omega; H^1_\#(Y^*)/\mathbb{R})$. The task before us is to identify the coefficients $\hat{u}_2^{K,j}$. The canonical idea is to expand the data $\{\mathbf{s}_j(x)\}$ in the same fashion as in (83) and substitute it in (81) to obtain equations for $\hat{u}_2^{K,j}$. This technique is successful because the decomposition in Lemma 3.7 is given in terms of orthogonal subspaces, each one of them is left invariant under differential operators.

Since $\{\mathbf{s}_j(x)\}$ is independent of $y$, it is natural to consider constant vectors $\mathbf{s}_j \in \mathbb{C}^N$, for $0\leqslant j \leqslant K-1$. Associated with this set is the piece-wise constant function:

$$\mathbf{s}(y) = \sum_{0\leqslant j\leqslant K-1} \mathbf{s}_j \chi_{Y_j}(y), \quad y\in KY$$

Applying Lemma 3.7 to this function, we arrive at the following discrete Bloch decomposition.

*Lemma 3.10. Given* $\mathbf{s}_j \in \mathbb{C}^N$, $0\leqslant j\leqslant K-1$, *there is a unique set of vectors* $\mathbf{t}_{j'}\in\mathbb{C}^N$, $0\leqslant j'\leqslant K-1$ *such that*

$$\sum_{0\leqslant j\leqslant K-1} \mathbf{s}_j\chi_{Y_j}(y) = \sum_{0\leqslant j'\leqslant K-1} \mathbf{t}_{j'}\mathrm{e}^{2\pi\mathrm{i}(j'/K)\cdot E(y)} \quad \forall y\in KY$$

$$\frac{1}{K^N}\sum_{0\leqslant j\leqslant K-1} |\mathbf{s}_j|^2 = \sum_{0\leqslant j'\leqslant K-1} |\mathbf{t}_{j'}|^2 \quad \textit{(Parseval's identity)}$$

*(Here, $E(y)$ denotes the integer part of $y$). Indeed vectors $\mathbf{t}_{j'}$ are explicitly given by*

$$\mathbf{t}_{j'} = \frac{1}{K^N}\sum_{0\leqslant j\leqslant K-1} \mathbf{s}_j\mathrm{e}^{-2\pi\mathrm{i}(j/K)\cdot j'}$$

The above result defines a unitary isomorphism from $(\mathbb{C}^N)^{K^N}$ into itself, defined by the map $\{\mathbf{s}_j\}\to\{\mathbf{t}_{j'}\}$. It is denoted by $\mathscr{B}$ (Bloch transform) and it will be used later to describe the limit operator.

For the moment, denoting the image of $\{\mathbf{s}_j(x)\}$ under $\mathscr{B}$ by $\{\mathbf{t}_{j'}(x)]$, we obtain the following systems for $[u_2^K, \hat{u}_2^{K,0}]$ and $\hat{u}_2^{K,j}$, $j \neq 0$, respectively.

$$
\begin{aligned}
&-\Delta_y \hat{u}_2^{K,0}(x,y) = 0 \quad \text{in } \Omega \times Y^* \\
&-\mathrm{div}_x \int_\Omega (\nabla u_2^K(x) + \nabla_y \hat{u}_2^{K,0}(x,y))\,\mathrm{d}y = |H|\,\mathrm{div}\,\mathbf{t}_0(x) \quad \text{in } \Omega \\
&[\nabla u_2^K(x) + \nabla_y \hat{u}_2^{K,0}(x,y) - \mathbf{t}_0(x)] \cdot \mathbf{n}_y = 0 \quad \text{on } \Omega \times \partial H \\
&u_2^K(x) = 0 \quad \text{on } \Gamma \\
&y \in Y^* \to \hat{u}_2^{K,0}(x,y) \text{ is } Y^*\text{-periodic.}
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
&-\Delta_y \{\hat{u}_2^{K,j}(x,y) e^{2\pi i (j/K)\cdot y}\} \quad \text{in } \Omega \times Y^* \\
&\nabla_y \{\hat{u}_2^{K,j}(x,y) e^{2\pi i (j/K)\cdot y}\} \cdot \mathbf{n}_y = \mathbf{t}_j(x) \cdot \mathbf{n}_y \quad \text{on } \Omega \times \partial H \\
&y \in Y^* \to \hat{u}_2^{K,j}(x,y) \text{ is } Y^*\text{-periodic}
\end{aligned}
\tag{85}
$$

The advantage of working in Bloch space is now clear. The above systems are posed on the single cell $Y^*$ and consequently they are easily resolved. Indeed, system (84) is identical to (86) and so $u_2^K$ is the solution of the following homogenized system (cf. (88)):

$$
\begin{aligned}
-\mathrm{div}\,(A \nabla u_2^K) &= \mathrm{div}((I - A)\mathbf{t}_0) \quad \text{in } \Omega \\
u_2^K &= 0 \quad \text{on } \Gamma
\end{aligned}
\tag{86}
$$

Further, $\hat{u}_2^{K,0}$ is expressed as (cf. (89)):

$$
\hat{u}_2^{K,0}(x,y) = \sum_{m=1}^{N} w_m(y) \left( \frac{\partial u_2^K}{\partial x_m}(x) - t_{0m}(x) \right)
$$

where $w_m(y)$ are the test functions defined by the problem (23) and $t_{0m}(x)$ is the $m$th component of the vector $\mathbf{t}_0(x)$.

To solve (85), we are naturally led to consider, for $0 \neq j \leqslant K-1$ and $m = 1, 2, \ldots, N$,

$$
\begin{aligned}
-\Delta_y w_m^{K,j} &= 0 \quad \text{in } Y^* \\
\frac{\partial w_m^{K,j}}{\partial n} &= n_m \quad \text{on } \partial H \\
w_m^{K,j}(y) \text{ is } &\left( \frac{j}{K}; Y^* \right)\text{-periodic}
\end{aligned}
\tag{87}
$$

These are parametrized cell problems which generalize (23) with a newly introduced periodic condition. These are solved in the same manner as (23).

Thanks to these functions, we obtain

$$
\hat{u}_2^{K,j}(x,y) e^{2\pi i (j/k)\cdot y} = \sum_{m=1}^{N} t_{jm}(x) w_m^{K,j}(y), \quad j \neq 0
$$

 

where $t_{jm}(x)$, $m=1,2,\ldots,N$ are the components of $\mathbf{t}_j(x)$. Finally, we are in a position to give a complete resolution of (81). We announce it in the form of the following

*Theorem 3.11. Let* $\{\mathbf{t}_j(x)\}$ *be the discrete Bloch transform of* $\{\mathbf{s}_j(x)\}$ *given by the Bloch isomorphism* $\mathscr{B}$. *Then the solution* $[u_2^K, \hat{u}_2^K]$ *of* (81) *is characterized as follows*:

(i) $u_2^K$ *is the solution of* (86).

(ii) $\hat{u}_2^K(x,y) = \sum_{m=1}^N w_m(y)\{\frac{\partial u_2^K}{\partial x_m}(x) - t_{0m}(x)\} + \sum_{m=1}^N \sum_{j\neq 0} t_{jm}(x) w_m^{K,j}(y)$.

We conclude this paragraph by introducing a family of parametrized cell problems which include (87) as a special case. For $\theta \in \mathbb{R}^N$, let $w_m^\theta$, $m=1,2,\ldots,N$ be the solution of

$$\begin{aligned} -\Delta_y w_m^\theta &= 0 \quad \text{in } Y^* \\ \frac{\partial w_m^\theta}{\partial n} &= n_m \quad \text{on } \partial H \\ w_m^\theta & \text{ is } (\theta; Y^*)\text{-periodic} \end{aligned} \tag{88}$$

*3.7. Description of the microscopic limit operator*

After sufficient amount of preparation carried out in earlier paragraphs, let us turn our attention to the central issue of Section 3. Recall that we have introduced the operator $T^{\varepsilon,K}$ in (62) whose spectrum coincides with Helmholtz spectrum. Thus our study about the behaviour of the Helmholtz spectrum is reduced to the asymptotics of $\sigma(T^{\varepsilon,K})$ as $\varepsilon \to 0$. Since $T^{\varepsilon,K}$ is defined in terms of the solution $u^{\varepsilon,K}$ of problem (63), the first step is to analyze the behaviour of $u^{\varepsilon,K}$ as $\varepsilon \to 0$. In the preceding paragraph, we have not only obtained the limit of (63) but also described the structure of the limit in detail. The next step is to use this information to define the (weak) limit operator $T^K$. To this end, it is necessary to pass the limit in the quadratic form associated with $T^{\varepsilon,K}$. Since weak convergence is not sufficient to assert anything worthwhile on the behaviour of the spectrum, the third step is to establish the strong convergence $T^{\varepsilon,K} \to T^K$. In the present paragraph, we carry out the last two steps in one stroke by passing to the limit in the bilinear form defined by $T^{\varepsilon,K}$. More precisely, we take $f, f'^{\varepsilon} \in L^2(\Omega; L^2_\#(KY))$ and $\{\mathbf{s}_j(x)\}, \{\mathbf{s}_j'^{\varepsilon}(x)\} \in S$ such that

$$\begin{aligned} f'^{\varepsilon} &\rightharpoonup f' \quad \text{weakly in } L^2(\Omega; L^2_\#(KY)) \\ \{\mathbf{s}_j'^{\varepsilon}(x)\} &\rightharpoonup \{\mathbf{s}_j'(x)\} \quad \text{weakly in } S \end{aligned} \tag{89}$$

and consider the bilinear form

$$\langle T^{\varepsilon,K}[f, \{\mathbf{s}_j(x)\}], [f'^{\varepsilon}, \{\mathbf{s}_j'^{\varepsilon}(x)\}]\rangle$$

Our goal here is to pass to the limit in this scalar product and express the limit in the form

$$\langle T^K[f, \{\mathbf{s}_j(x)\}], [f', \{\mathbf{s}_j'(x)\}]\rangle$$

for a suitable linear operator $T^K$.

To achieve this, we denote by $u^{\varepsilon,K}$ and $u'^{\varepsilon,K}$, the solutions of (63) corresponding to data $[f, \{\mathbf{s}_j(x)\}]$ and $[f'^{\varepsilon}, \{\mathbf{s}_j'^{\varepsilon}(x)\}]$ respectively and use the expressions (68)–(70) for the bilinear form. Using the results established in earlier sections, we see that we can pass to the limit in each individual term and thereby compute the (strong) limits of $T_{ij}^{\varepsilon,K}$ $\forall i,j=1,2$. The main point we wish to highlight in the above process is the following: We need to pass to the limit in the integrals involving products of gradients of $u^{\varepsilon,K}$ and $u'^{\varepsilon,K}$. Since the gradient of $u^{\varepsilon,K}$ converges strongly in two-scales, there is absolutely no problem in the passage to the limit.

For instance, let us start with $T_{11}^{\varepsilon,K}$. Taking $\phi^{\varepsilon}(x)=\widetilde{\nabla}u_1^{\varepsilon,K}(x)$ and applying Corollary 3.5, we can pass to the limit in (69) and we obtain

$$\frac{1}{|KY|}\int_{\Omega\times KY}(T_{11}^{\varepsilon,K}f)f'^{\varepsilon}\,\mathrm{d}x\,\mathrm{d}y\to\frac{1}{c_0^2}\int_{\Omega\times KY^*}(\nabla u_1^K(x)+\nabla_y\hat{u}_1^K(x,y))\cdot(\nabla u_1'^K(x)+\nabla_y\hat{u}_1'^K(x,y))\,\mathrm{d}x\,\mathrm{d}y$$

which is equal, by (73), to

$$\frac{1}{c_0^2}\int_{\Omega\times KY^*}f'(x,y)u_1^K(x)\,\mathrm{d}x\,\mathrm{d}y$$

Thus, if we define the operator $T_{11}^K$ by

$$\begin{aligned}T_{11}^K &: L^2(\Omega;L^2_\#(KY))\to L^2(\Omega;L^2_\#(KY))\\ T_{11}^K f &= \frac{K^N}{c_0^2}u_1^K(x)\chi_{KY^*}(y)\end{aligned}\tag{90}$$

where $u_1^K$ is the solution of (78), then the above steps demonstrate that

$$T_{11}^{\varepsilon,K}\to T_{11}^K \quad \text{strongly}$$

Using similar arguments, we can find out the strong limits of $T_{12}^{\varepsilon,K}$, $T_{21}^{\varepsilon,K}$ and $T_{22}^{\varepsilon,K}$. Without entering into detail, we give below the expressions for the corresponding limit operators.

$$\begin{aligned}T_{12}^K &: S\to L^2(\Omega;L^2_\#(KY))\\ T_{12}^K\{\mathbf{s}_j(x)\} &= \frac{K^N}{c_0^2}u_2^K(x)\chi_{KY*}(y)\end{aligned}\tag{91}$$

where $u_2^K$ is the solution of (86).

$$\begin{aligned}T_{21}^K &: L^2(\Omega;L^2_\#(KY))\to S\\ T_{21}^K f &= \frac{\rho}{k}\{(A-I)\nabla u_1^K(x)\}_{0\leqslant j\leqslant K-1}\end{aligned}\tag{92}$$

where $u_1^K$ is the solution (78). We observe that the components of the above vector are independent of $j$.

In the computation of the limit of $T_{22}^{\varepsilon,K}$, we use the duality relation (61) and the fact that

$$E_K^{\varepsilon}P_K^{\varepsilon}\to I \quad \text{strongly on } S$$

the proof of which can be found in [2]. Indeed, these informations are needed to assert that

$$\varepsilon^N P_K^{\varepsilon}\{\mathbf{s}_j(x)\}\cdot P_K^{\varepsilon}\{\mathbf{s}_j'^{\varepsilon}(x)\}\to\int_{\Omega}\{\mathbf{s}_j(x)\}\cdot\{\mathbf{s}_j'(x)\}\,\mathrm{d}x$$

 

It is now easy to compute the limit of $T_{22}^{\varepsilon,K}$ and it is as follows:

$$T_{22}^K : S \to S$$
$$T_{22}^K f = \frac{\rho}{k} S^K + \frac{m}{k} I \tag{93}$$

where $S^K$ is an operator introduced in [2]:

$$S^K : S \to S$$
$$S^K\{\mathbf{s}_j(x)\} = \left\{ -|H|\nabla u_2^K(x) + \int_{\partial H_j} \hat{u}_2^K(x,y)\mathbf{n}_y \, \mathrm{d}\gamma \right\}_{0 \leqslant j \leqslant K-1} \tag{94}$$

where $[u_2^K, \hat{u}_2^K]$ is the solution of (81). We note that the above vector is a sum of two vectors, the first of which has components independent of $j$.

Finally, the limit of $T^{\varepsilon,K}$ is expressed in the form of a matrix of operators:

$$T^K : L^2(\Omega; L^2_\#(KY)) \times S \to L^2(\Omega; L^2_\#(KY)) \times S$$
$$T^K = \begin{bmatrix} T_{11}^K & T_{12}^K \\ T_{21}^K & T_{22}^K \end{bmatrix} \tag{95}$$

This operator is referred to as the *microscopic limit operator* associated with the Helmholtz model.

In order to bring out its structure more clearly, we make a linear change of variables in the space $S$ given by the Bloch isomorphism $\mathscr{B}$. We have

$$\mathscr{B}\{\mathbf{s}_j(x)\} = \{\mathbf{t}_j(x)\}$$

We obtain below the expression of $T^K$ in the variables $\{\mathbf{t}_j(x)\}$ instead of $\{\mathbf{s}_j(x)\}$. First of all, $S^K$ is transformed to $\mathscr{B}S^K\mathscr{B}^{-1}$. The structure of the later has been well described in [2] and we recall their result below.

*Theorem 3.12. The operators $\mathscr{B}S^K\mathscr{B}^{-1}$ acting on $(L^2(\Omega)^N)^{K^N}$ decomposes into*

$$\mathscr{B}S^K\mathscr{B}^{-1} = \operatorname{diag}\{S_j^K, 0 \leqslant j \leqslant K-1\}, \quad S_j^K \in \mathscr{L}(L^2(\Omega)^N).$$

*For $j=0$, the operator $S_0^K$ is given by*

$$S_0^K \mathbf{t}_0 = (A - I)\nabla u_2^K - (A - \theta I)\mathbf{t}_0 \quad \forall \mathbf{t}_0$$

*where $u_2^K$ is the solution of* (86).

*For $j \neq 0$, the operator $S_j^K$ is given by a numerical matrix in the sense that*

$$S_j^K \mathbf{t}_j(x) = A^{K,j}\mathbf{t}_j(x)$$

*where the matrix $A^{K,j}$ is defined by*

$$\bar{A}_{mm'}^{K,j} = \int_{Y*} \nabla w_m^{K,j} \cdot \nabla \bar{w}_{m'}^{K,j} \, \mathrm{d}y$$

*$w_m^{K,j}$ being the solution of* (87).

 

Next, we turn our attention to $T_{12}^K$. This operator is transformed to $T_{12}^K\mathscr{B}^{-1}$. Since $T_{12}^K$ involves only $u_2^K$ (see (91)), which in turn depends only on $\mathbf{t}_0(x)$, we see that $T_{12}^K\mathscr{B}^{-1}$ depends only on $\mathbf{t}_0$ and independent of $\mathbf{t}_j(x)$, $j \neq 0$.

With regard to $T_{21}^K$, we see that this is transformed to $\mathscr{B}T_{21}^K$. When $\mathbf{s}_j = \mathbf{s}$, independent of $j$, we have from Lemma 3.10, that

$$\mathscr{B}\{\mathbf{s},\dots,\mathbf{s}\} = \{\mathbf{s},\mathbf{0},\mathbf{0}\dots\mathbf{0}\}.$$

Using the above property, we get

$$\mathscr{B}T_{21}^K f = \frac{\rho}{k}\{(A-I)\nabla u_1^K(x),\mathbf{0},\dots,\mathbf{0}\}$$

where $u_1^K$ is the solution of (78).

In conclusion therefore, the operator $T^K$ is transformed to the following operator:

$$\widetilde{T}^K : L^2(\Omega; L^2_\#(KY)) \times S \to L^2(\Omega; L^2_\#(KY)) \times S$$
$$\widetilde{T}^K = \begin{bmatrix} T_{11}^K & T_{12}^K\mathscr{B}^{-1} \\ \mathscr{B}T_{21}^K & \mathscr{B}T_{22}^K\mathscr{B}^{-1} \end{bmatrix} \tag{96}$$

with

$$T_{11}^K f = \frac{K^N}{c_0^2} u_1^K(x)\chi_{KY^*}(y)$$

$$T_{12}^K\mathscr{B}^{-1}\{\mathbf{t}_j(x)\} = \frac{K^N}{c_0^2} u_2^K(x)\chi_{KY^*}(y)$$

hence dependent only on $\mathbf{t}_0(x)$ and

$$\mathscr{B}T_{21}^K f = \frac{\rho}{k}\{(A-I)\nabla u_1^K(x),\mathbf{0},\dots,\mathbf{0}\}$$

$$\mathscr{B}T_{22}^K\mathscr{B}^{-1} = \operatorname{diag}\left\{\frac{\rho}{k}S_j^K, 0 \leqslant j \leqslant K-1\right\} + \frac{m}{k}I$$

Final remarks on the structure of the transformed operator $\widetilde{T}^K$ are as follows: Apart from the usual homogenized matrix $A$, the generalized homogenized matrices $A^{K,j}$ (defined in Theorem 3.12) appear. Further, we need to compute $u_1^K$ and $u_2^K$ which are solutions of the homogenized problems (78) and (86) respectively. The dependence on the parameter $K$ mainly appears in the transformation $\mathscr{B} = \mathscr{B}^K$ and in the definition of the test functions $w_m^{K,j}$. The oscillations represented by these new test functions contribute through the matrices $A^{K,j}$ which appear in rather simple algebraic fashion in $\widetilde{T}^K$. This is in contrast to the contributions of the test functions $w_m$ through the homogenized matrix $A$ which appear in terms of the solutions of differential equations.

Let us now summarize our main arguments in the form of the following.

*Theorem 3.13. The operators $T^{\varepsilon,K}$ defined in* (62) *converge strongly as $\varepsilon \to 0$ to the operator $T^K$ defined in* (95) (*or equivalently to $\widetilde{T}^K$ of* (96)).

Since the spectrum is lower semi-continuous with respect to the strong convergence of the operators (see Proposition 2.1.11 of [2]), it follows that:

*Corollary 3.14. Given* $\lambda \in \sigma(\widetilde{T}^K) = \sigma(T^K)$*, there exists a sequence* $\lambda^n \in \sigma(T^{\varepsilon_n, K})$ *such that* $\lambda^n \to \lambda$*. In other words, we have* $\sigma(\widetilde{T}^K) = \sigma(T^K) \subseteq \sigma_\infty$.

Since the above conclusion is true for all $K \in \mathbb{N}$, we deduce that

$$\bigcup_{K=1}^{\infty} \sigma(T^K) \subseteq \sigma_\infty$$

This is the main assertion of this section which shows that we are able to characterize a large part of the limit spectrum $\sigma_\infty$ in terms of the operators $T^K$ or $\widetilde{T}^K$. Their definitions (unlike that of $T^\varepsilon$) do not involve any inhomogeneities and their spectra are straightforward to compute.

Our next result gives an important subset of $\sigma(\widetilde{T}^K)$ and hence contained in $\sigma_\infty$.

*Theorem 3.15. Let us consider the matrix* $A^{K,j}$ *defined in Theorem* 3.12. *Then we have*

$$\bigcup_{\substack{0 \leqslant j \leqslant K-1 \\ j \neq 0}} \sigma\left(\frac{\rho}{k} A^{K,j} + \frac{m}{k} I\right)$$

*is a subset of the set of eigenvalues of* $\widetilde{T}^K$ *and hence contained in* $\sigma_\infty$.

*Proof.* Indeed, let us consider the eigenvalue equation associated with $\widetilde{T}^K$:

$$\begin{aligned} &\widetilde{T}^K[f, \{\mathbf{t}_j(x)\}] = \lambda[f, \{\mathbf{t}_j(x)\}] \\ &[f, \{\mathbf{t}_j(x)\}] \neq [0, \mathbf{0}], \quad [f, \{\mathbf{t}_j(x)\}] \in L^2(\Omega; L^2_\#(KY)) \times S \end{aligned} \tag{97}$$

Using the definition of $\widetilde{T}^K$, the above system is found to be equivalent to the following three equations:

$$\frac{K^N}{c_0^2}[u_1^K(x)\chi_{KY^*}(y) + u_2^K(x)\chi_{KY^*}(y)] = \lambda f(x, y) \tag{98}$$

$$\frac{\rho}{k}[(A - I)(\nabla u_1^K(x) + \nabla u_2^K(x)) - (A - \theta I)\mathbf{t}_0(x)] + \frac{m}{k}\mathbf{t}_0(x) = \lambda \mathbf{t}_0(x) \tag{99}$$

$$\frac{\rho}{k} A^{K,j}\mathbf{t}_j(x) + \frac{m}{k}\mathbf{t}_j(x) = \lambda \mathbf{t}_j(x) \quad 0 \leqslant j \leqslant K - 1, \ j \neq 0$$

The above formulation clearly shows that the each one of the variables $[f, \mathbf{t}_0(x)]$, $t_j(x)$, $0 \leqslant j \leqslant K-1$, $j \neq 0$ is decoupled from others. To complete the proof, it is enough to note that for each $0 \leqslant j \leqslant K-1$, $j \neq 0$, $\mathbf{t}_j(x)$ (when it is non-zero) is an eigenvector associated with the matrix $(\rho/k)A^{K,j} + (m/k)I$ with eigenvalue $\lambda$. □

*Remark 3.16.* Before passing on to our next point, let us make a couple of observations on (97). Firstly, if $\lambda \neq 0$, then it follows from (98) that $f \in L^2(\Omega; L^2_\#(Y))$ and so

$$\int_{KY^*} f(x, y)\,\mathrm{d}y = K^N \int_{Y^*} f(x, y)\,\mathrm{d}y$$

 

With this information, equation (78) becomes

$$-\operatorname{div}(A\nabla u_1^K) = \int_{Y^*} f(x,y)\,\mathrm{d}y \quad \text{in } \Omega$$
$$u_1^K = 0 \quad \text{on } \Gamma \tag{100}$$

In particular, $u_1^K$ is independent of $K$.

Secondly, if $\mathbf{t}_0(x) \equiv 0$, then it follows from (86) that $u_2^K \equiv 0$. Equation (99) then implies that $u_1^K \equiv 0$. It now follows from (98) that $f \equiv 0$ provided $\lambda \neq 0$. Thus $\mathbf{t}_0(x)$, in a sense, determines uniquely $f$ in the equation (97).

We will now refine the above result. Indeed, for $\theta \in [0,1[^N$ introduce the matrix $A^\theta = (A^\theta_{mm'})$ by

$$\bar{A}^\theta_{mm'} = \int_{Y^*} \nabla w^\theta_m \cdot \nabla \bar{w}^\theta_{m'}\,\mathrm{d}y$$

where $w^\theta_m$ are defined by the cell problems (88). Let us remark that when $\theta = j/K$, $0 \leqslant j \leqslant K-1$, the matrix $A^\theta$ coincides with $A^{K,j}$. Since $\sigma_\infty$ is closed, it is an easy consequence of Theorem 3.15 that

$$\sigma\left(\frac{\rho}{k}A^\theta + \frac{m}{k}I\right) \subseteq \sigma_\infty \quad \forall \theta \in ]0,1[^N$$

Denoting by $\{\lambda_p(\theta)\}_{p=1}^N$ the eigenvalues of $(\rho/k)A^\theta + (m/k)I$, we introduce

$$a_p = \inf_{\theta \in ]0,1[^N} \lambda_p(\theta), \quad b_p = \sup_{\theta \in ]0,1[^N} \lambda_p(\theta)$$

We recall from [2] that $\lambda_p(\cdot)$ is a bounded continuous function on $]0,1[^N$. We define

$$\sigma_{\text{Bloch}} = \bigcup_{p=1}^N [a_p, b_p]$$

Since $\sigma_\infty$ is closed, we easily reach the conclusion that

$$\sigma_{\text{Bloch}} \subseteq \sigma_\infty$$

We name the above part *micro-spectrum* or *Bloch spectrum.*

## 4. COMPLETENESS OF THE LIMIT SPECTRUM

Having described the two subsets (macro-part and micro-part) of the limit spectrum $\sigma_\infty$ in the last two sections, we are now in a position to characterize it completely. It turns out that these two components do not exhaust $\sigma_\infty$; there is another one called boundary layer part which corresponds to the limit sequence of eigenvalues whose associated eigenvectors concentrate near the boundary of the domain. It is important to realize that these sequences are captured neither by the homogenization method nor by the Bloch wave method since they do not take into account a possible interaction between the boundary and the network of the tubes. More precisely, the aim of this section is to establish the following completeness result:

$$\sigma_\infty = \sigma_{\text{homo}} \cup \sigma_{\text{Bloch}} \cup \sigma_{\text{bdry}}$$

where $\sigma_{\text{homo}}$ and $\sigma_{\text{Bloch}}$ were defined in preceding sections and $\sigma_{\text{bdry}}$ is defined as follows:

$$\begin{aligned} \sigma_{\text{bdry}} = \{\lambda \in \mathbb{R} \mid \exists\, \lambda^\varepsilon \in \mathbb{R},\ & \exists [f^\varepsilon, \mathbf{s}^\varepsilon] \in L^2(\Omega; L^2_\#(Y)) \times L^2(\Omega)^N \\ & \|[f^\varepsilon, \mathbf{s}^\varepsilon]\| = 1,\ f^\varepsilon \rightharpoonup 0 \text{ in } L^2(\Omega; L^2_\#(Y)) \text{ weak} \\ & \mathbf{s}^\varepsilon \rightharpoonup 0 \text{ in } L^2(\Omega)^N \text{ weak},\ \forall \omega \subset\subset \Omega,\ \|\mathbf{s}^\varepsilon\|_{L^2(\omega)} \to 0 \\ & \text{and } T^\varepsilon[f^\varepsilon, \mathbf{s}^\varepsilon] = \lambda^\varepsilon [f^\varepsilon, \mathbf{s}^\varepsilon]\} \end{aligned} \tag{101}$$

We refer to $\sigma_{\text{bdry}}$ as the *boundary layer spectrum*. We notice that as in the case of $\sigma_{\text{Bloch}}$, $\sigma_{\text{bdry}}$ is also defined in terms of the behaviour of the tube displacements near the boundary. Fluid vibrations do not play a crucial role here.

*Theorem 4.1. The limit spectrum in the case of Helmholtz model is made up of three parts: the homogenized, the Bloch and the boundary layer spectra, i.e.*

$$\sigma_\infty = \sigma_{\text{homo}} \cup \sigma_{\text{Bloch}} \cup \sigma_{\text{bdry}} \tag{102}$$

*Proof.* The results of the previous sections and the very definition of $\sigma_{\text{bdry}}$ imply that the right-hand side of (102) is a subset of $\sigma_\infty$. The proof of the reverse inclusion consists in passing to the limit in the spectral equation of the Helmholtz model:

$$T^\varepsilon[f^\varepsilon, \mathbf{s}^\varepsilon] = \lambda^\varepsilon [f^\varepsilon, \mathbf{s}^\varepsilon], \quad \|[f^\varepsilon, \mathbf{s}^\varepsilon]\| = 1 \tag{103}$$

Extracting a subsequence, we can assume that

$$\begin{aligned} f^\varepsilon &\rightharpoonup f \quad \text{in } L^2(\Omega; L^2_\#(Y)) \text{ weak} \\ \mathbf{s}^\varepsilon &\rightharpoonup \mathbf{s} \quad \text{in } L^2(\Omega)^N \text{ weak} \\ \lambda^\varepsilon &\to \lambda \in \sigma_\infty \end{aligned}$$

Several cases have to be considered. Since $0 \in \sigma_{\text{homo}}$, we can, without loss of generality, suppose that $\lambda \neq 0$.

*Case* (i). Assume $\mathbf{s} \neq \mathbf{0}$. Since $T^\varepsilon$ converges to $T$ strongly and $T^\varepsilon$ and $T$ are self-adjoint, we can easily pass to the limit in (103) and obtain

$$T[f, \mathbf{s}] = \lambda [f, \mathbf{s}] \tag{104}$$

Since $\mathbf{s} \neq \mathbf{0}$, this implies that $\lambda$ is an eigenvalue of $T$. In particular, $\lambda \in \sigma_{\text{homo}}$.

*Case* (ii). Assume $\mathbf{s} = \mathbf{0}$. Owing to the specific nature of the problem, we will now show that

$$f = 0 \tag{105}$$

Furthermore, we have strong convergence, i.e.

$$f^\varepsilon \to 0 \quad \text{in } L^2(\Omega; L^2_\#(Y)) \tag{106}$$

With $\mathbf{s}=\mathbf{0}$, the second component of equation (104) can be rewritten as (see (57))

$$(A-I)\nabla u_1=0$$

where $u_1$ is solution (41). It is well known from the classical estimates on the homogenized matrix that $A$ and $(I-A)$ are both positive definite matrices. As a consequence, we get $\nabla u_1=0$ and hence $u_1=0$. Now, applying the definition of $T_{11}$, we have

$$\lambda f=T_{11}f=\frac{1}{c_0^2}u_1(x)\chi_{Y*}(y)=0$$

whence $f=0$. This proves (105).

Next, we move on to prove the strong convergence (106). To this end, let us consider the first component of (103):

$$T_{11}^{\varepsilon}f^{\varepsilon}+T_{12}^{\varepsilon}\mathbf{s}^{\varepsilon}=\lambda^{\varepsilon}f^{\varepsilon}$$

Using the definitions of $T_{11}^{\varepsilon}$ and $T_{12}^{\varepsilon}$, the above equation can be recast as follows:

$$\frac{1}{c_0^2}E_{1,\Omega}^{\varepsilon}E_0^{\varepsilon}[u_1^{\varepsilon}+u_2^{\varepsilon}]=\lambda^{\varepsilon}f^{\varepsilon} \tag{107}$$

Since $f^{\varepsilon}\rightharpoonup 0$ in $L^2(\Omega;L^2_{\#}(Y))$ weak, owing to Theorem 2.9, we get $X^{\varepsilon}u_1^{\varepsilon}\rightharpoonup 0$ in $H_0^1(\Omega)$ weak. Since $\mathbf{s}^{\varepsilon}\rightharpoonup 0$ in $L^2(\Omega)^N$ weak, using Theorem 2.11, we get $X^{\varepsilon}u_2^{\varepsilon}\rightharpoonup 0$ in $H_0^1(\Omega)$ weak. If we use these convergence properties in (107) and the fact that $E_{1,\Omega}^{\varepsilon}$ is an isometry, we immediately get (106).

The above analysis shows that $f^{\varepsilon}$ does not oscillate at all. This suggests that we must consider subcases depending on the behaviour of $\mathbf{s}^{\varepsilon}$. Following two subcases arise according to whether the energy of $\mathbf{s}^{\varepsilon}$ concentrates near the boundary $\Gamma$ or not.

*Case* (ii)(a). In addition to $\mathbf{s}=\mathbf{0}$ and $f=0$, we suppose further that

$$\lim_{\varepsilon\to 0}\|\mathbf{s}^{\varepsilon}\|_{L^2(\omega)}=0 \quad \forall\omega\subset\subset\Omega$$

In this case, by the very definition of $\sigma_{\text{bdry}}$, we get $\lambda\in\sigma_{\text{bdry}}$.

*Case* (ii)(b). In addition to $\mathbf{s}=\mathbf{0}$, $f=0$ and (106), we suppose that there exists $\omega\subset\subset\Omega$ such that

$$\lim_{\varepsilon\to 0}\|\mathbf{s}^{\varepsilon}\|_{L^2(\omega)}=\alpha>0$$

Here, we conclude that $\lambda\in\sigma_{\text{Bloch}}$. Indeed, let us consider the second component of (103):

$$T_{22}^{\varepsilon}\mathbf{s}^{\varepsilon}=\lambda^{\varepsilon}\mathbf{s}^{\varepsilon}-T_{21}^{\varepsilon}f^{\varepsilon} \tag{108}$$

Since $T_{21}^{\varepsilon}$ is bounded, it follows that $T_{21}^{\varepsilon}f^{\varepsilon}\to 0$ in $L^2(\Omega)^N$. Thus the last term can be neglected in the convergence analysis. The operator $T_{22}^{\varepsilon}$ acts on tube displacements only and thus the study of (108) is done in [2]. Applying their analysis (see step 3 of the proof of Theorem 3.2.9), we conclude that $\lambda\in\sigma_{\text{Bloch}}$. This completes the proof of our theorem. □

## 5. FINAL CONCLUSION

In this article, we described the asymptotic behaviour of the vibration frequencies of a periodic tube-bundle immersed in a slightly compressible fluid when the number of tubes is large (or equivalently, the period is small). Indeed, as the period goes to zero, an asymptotic analysis of the spectrum of such a coupled structure was performed with the help of a new method, the so-called Bloch-wave homogenization method which is a blend of two-scale convergence, Bloch wave decomposition, and classical homogenization techniques. Our main result proves that the limiting spectrum is made of three parts; each one having a very clear engineering meaning: the macroscopic or homogenized spectrum, the microscopic or Bloch spectrum, and the boundary-layer spectrum.

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### REFERENCES

1. Conca, C., Planchard, J., Vanninathan, M. Fluids and Periodic Structures, no. 38 in Research in Applied Mathematics. Wiley: New York, 1995.
2. Allaire, G., Conca, C. Bloch wave homogenization for a spectral problem in fluid-solid structures. *Archive for Rational Mechanics and Analysis* 1996; **135**:197–257.
3. Allaire, G., Conca, C. Boundary layers in the homogenization of a spectral problem in fluid-solid structures. *SIAM Journal of Mathematical Analysis* 1989; **29**:343–379.
4. Kesavan, S. Homogenization of elliptic eigenvalue problems I and II. *Applied Mathematical Optimization* 1979; **5**: 153–167 and 197–216.
5. Vanninathan, M. Homogenization of eigenvalue problems in perforate domains. *Proceedings of the Indian Academy of Sciences, Mathematical Sciences* 1981; **90**:239–271.
6. Bensoussan, A., Lions, J.L., Papanicolaou, G. *Asymptotic Analysis for Periodic Structures*. North-Holland: Amsterdam, 1978.
7. Allaire, G., Conca, C. Analyse asymptotique spectrale de l'equation des ondes. Complétude du spectre de Bloch. *Comptes Rendus de l'Academie des Sciences Serie l, Mathématique, Bordas, Paris* 1995; **t321**:557–562.
8. Allaire, G., Conca, C. Analyse asymptotique spectrale de l'equation des ondes. Homogénésation par ondes de Bloch. *Comptes Rendus de l'Academie des Sciences Serie l, Mathématique, Bordas, Paris* 1995; **t321**:293–298.
9. Cioranescu, D., Saint Jean Paulin, J. Homogenization in open sets with holes. *Journal of Mathematical Analysis and Applications* 1979; **71**:590–607.
10. Allaire, G. Homogenization and two-scale convergence. *SIAM Journal of Mathematical Analysis* 1992; **23**: 1482–1518.

 