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Homogenization of the Navier-Stokes Equations in Open Sets Perforated with Tiny Holes II: Non-Critical Sizes of the Holes for a Volume Distribution and a Surface Distribution of Holes

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Abstract

This paper is devoted to the homogenization of the Stokes or Navier-Stokes equations with a Dirichlet boundary condition in a domain containing many tiny solid obstacles, periodically distributed in each direction of the axes. For obstacles of critical size it was established in Part I that the limit problem is described by a law of Brinkman type. Here we prove that for smaller obstacles, the limit problem reduces to the Stokes or Navier-Stokes equations, and for larger obstacles, to Darcy's law. We also apply the abstract framework of Part I to the case of a domain containing tiny obstacles, periodically distributed on a surface. (For example, in three dimensions, consider obstacles of size ε^2, located at the nodes of a regular plane mesh of period ε.) This provides a mathematical model for fluid flows through mixing grids, based on a special form of the Brinkman law in which the additional term is concentrated on the plane of the grid.

Contents

Introduction

This two-part paper is devoted to the homogenization of the Stokes or Navier-Stokes equations, with a Dirichlet boundary condition, in open sets perforated with tiny holes. The ultimate purpose is to derive effective equations for the study of viscous fluid flows in a domain containing many tiny obstacles (mathematically represented by holes perforating a given open set). Throughout this paper we consider the Stokes equations (S_ε) in an open set Ω_ε obtained by removing from a given open set Ω a collection of holes $(T_i^\varepsilon)_{1\leq i\leq N(\varepsilon)}$:

$$(S_\varepsilon)\qquad \begin{cases} \text{Find } (u_\varepsilon, p_\varepsilon)\in[H_0^1(\Omega_\varepsilon)]^N\times[L^2(\Omega_\varepsilon)/\mathbb{R}] \text{ such that} \\ \nabla p_\varepsilon - \Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ \qquad \nabla\cdot u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon. \end{cases}$$

In the first section of Part I an abstract framework of hypotheses on the holes (T_i^ε), was introduced following an idea of D. CIORANESCU & F. MURAT [9]. Under those hypotheses we established that the homogenized problem is described by a Brinkman-type law, and we proved the convergence of the homogenization process, as well as some other results related to the correctors. The second section of Part I dealt with the verification of those hypotheses in the case of a volume distribution of holes having a so-called critical size. This verification led to the proof that in this case the homogenization of the Stokes equations yields a Brinkman-type law.

Part II includes the third and the fourth sections of this paper. In the third, we investigate all the other possible sizes of the holes, and we prove that for smaller sizes the homogenized problem is a Stokes system, and for larger sizes, Darcy's law. Moreover, our study illuminates the name "critical" given to the size introduced in the second section. More precisely, we consider identical holes T_i^ε periodically distributed in each direction of the axes with period 2ε, each hole being similar to the same model hole T, rescaled to the size a_ε. We define a ratio σ_ε between the current size of the holes and the critical one.

$$\sigma_\varepsilon = \left(\frac{\varepsilon^N}{a_\varepsilon^{N-2}}\right)^{1/2} \text{ for } N\geqq 3, \qquad \sigma_\varepsilon = \varepsilon\left|\log\left(\frac{a_\varepsilon}{\varepsilon}\right)\right|^{1/2} \text{ for } N=2.$$

Let $(u_\varepsilon, p_\varepsilon)$ be the unique solution of the Stokes system (S_ε). Let $\tilde{u}_\varepsilon$ be the extension of the velocity by 0 in $\Omega - \Omega_\varepsilon$. Let P_ε be the extension of the pressure p_ε defined by

$$P_\varepsilon = p_\varepsilon \text{ in } \Omega_\varepsilon \quad \text{and} \quad P_\varepsilon = \frac{1}{|C_i^\varepsilon|}\int_{C_i^\varepsilon} p_\varepsilon \quad \text{in each hole } T_i^\varepsilon$$

where C_i^ε is a "control" volume around the hole T_i^ε defined as the part outside T_i^ε of the ball of radius ε with same center as T_i^ε. Then we prove the

Theorem. *According to the scaling of the hole size there are three different limit flow regimes:*

(i) *If* $\lim_{\varepsilon\to 0}\sigma_\varepsilon = +\infty$ *(so that the holes are small, see Theorem 3.3.1), then* $(\tilde{u}_\varepsilon, P_\varepsilon)$ *converges strongly to* (u, p) *in* $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, *where* (u, p) *is the unique solution of the Stokes problem*

$$\nabla p - \Delta u = f \text{ in } \Omega, \quad \nabla \cdot u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

(ii) *If* $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \sigma > 0$ *(so that the holes have critical size, see Part I), then* $(\tilde{u}_\varepsilon, P_\varepsilon)$ *converges weakly to* (u, p) *in* $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, *where* (u, p) *is the unique solution of the Brinkman-type law*

$$\nabla p - \Delta u + \frac{1}{\sigma^2} M_0 u = f \text{ in } \Omega, \quad \nabla \cdot u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

(iii) *If* $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$ *(so that the holes are large, see Theorem 3.4.4 and Propositions 3.4.8, 3.4.11, and 3.4.12), then* $\left(\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2}, P_\varepsilon\right)$ *converges strongly to* (u, p) *in* $[L^2(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, *where* (u, p) *is the unique solution of Darcy's law*

$$u = M_0^{-1}(f - \nabla p) \text{ in } \Omega, \quad \nabla \cdot u = 0 \text{ in } \Omega, \quad u \cdot n = 0 \text{ on } \partial\Omega.$$

Moreover, if $N = 2$, *then* $M_0 = \pi\, Id$, *whatever the shape of the model hole* T, *and if* $N \geqq 3$, *then* ${}^t e_i M_0 e_k = \frac{1}{2^N} \int_{\mathbb{R}^N - T} \nabla w_k : \nabla w_i$ *where, for* $1 \leqq k \leqq N$, e_k *is the* k^{th} *unit basis vector in* $\mathbb{R}^N$, *and* w_k *is the solution of the following Stokes system*

$$\nabla q_k - \Delta w_k = 0 \text{ in } \mathbb{R}^N - T, \quad \nabla \cdot w_k = 0 \text{ in } \mathbb{R}^N - T,$$
$$w_k = 0 \text{ on } \partial T, \quad w_k = e_k \text{ at infinity}.$$

In the fourth section we consider a different geometric situation, namely a surface distribution of the holes. For simplicity, we assume that this surface is a hyperplane H that intersects the open set Ω. More precisely, we consider identical holes T_i^ε, the centers of which are periodically distributed in each direction of the axes of H with period 2ε, each hole being similar to the same model hole T, rescaled at size a_ε (see Figure 4). Note that it is the centers of the holes that are located on the hyperplane H; the holes themselves are closed subsets of Ω that are not necessarily included in H. Typically, the appropriate size a_ε of the holes is ε^2 for $N = 3$, and $e^{-1/\varepsilon}$ for $N = 2$. It is worth noticing that this size a_ε, critical for a surface distribution, is larger than the critical size for a volume distribution, because the number of the holes is smaller, roughly $1/\varepsilon^{N-1}$ instead of $1/\varepsilon^N$. The abstract framework introduced in Part I must be modified slightly to reflect the weaker estimate satisfied by the extension of the pressure in this case. We shall prove (see Theorem 4.1.3) the following

Theorem. *Let the holes be distributed in a hyperplane* H *and have a size* a_ε *such that*

$$\lim_{\varepsilon\to 0} \frac{a}{\varepsilon^{(N-1)/(N-2)}} = C_0 \text{ for } N \geqq 3 \qquad \text{or} \qquad \lim_{\varepsilon\to 0} -\varepsilon \log(a_\varepsilon) = C_0 \text{ for } N = 2$$

where C_0 *is a strictly positive constant* $(0 < C_0 < +\infty)$.

Let $(u_\varepsilon, p_\varepsilon)$ *be the unique solution of the Stokes system* (S_ε). *Let* $\tilde{u}_\varepsilon$ *be the extension of the velocity by* 0 *in* $\Omega - \Omega_\varepsilon$. *Let* P_ε *be the extension of the pressure* p_ε *defined by*

$$P_\varepsilon = p_\varepsilon \text{ in } \Omega_\varepsilon \quad \text{and} \quad P_\varepsilon = \frac{1}{|C_i^\varepsilon|} \int_{C_i^\varepsilon} p_\varepsilon \text{ in each hole } T_i^\varepsilon,$$

where C_i^ε *is the same control volume around* T_i^ε *as defined in the previous theorem. Then* $(\tilde{u}_\varepsilon, P_\varepsilon)$ *converges weakly to* (u, p) *in* $[H_0^1(\Omega)]^N \times [L^{q'}(\Omega)/\mathbb{R}]$, *where* $1 \leqq q' < N/(N-1) \leqq 2$, *and* (u, p) *is the unique solution of the following Brinkman law:*

$$\text{Find } (u, p) \in [H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \text{ such that}$$

$$\nabla p - \Delta u + Mu = f \quad \text{in } \Omega,$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega.$$

Moreover, the matrix M *is concentrated on the hyperplane* H *(i.e., equal to* 0 *in* $\Omega - H$). *More precisely, let* δ_H *denote the measure defined as the unit mass concentrated on H. If* $N = 2$, *then* $M = \frac{2\pi}{C_0} Id\, \delta_H$, *whatever the shape of the model hole T, and if* $N \geqq 3$, *then* ${}^t e_i M e_k = \frac{C_0^{N-2}}{2^{N-1}} \int_{\mathbb{R}^N - T} \nabla w_k : \nabla w_i \, \delta_H$, *where* w_k *is the solution of the same Stokes problem in* $\mathbb{R}^N - T$ *as described in the previous theorem.*

This theorem provides an effective model for computing viscous fluid flows through porous walls, or mixing grids. For example, consider a mixing grid made of small vanes fixed at the nodes of a thin plane lattice (which is neglected). The matrix M (which may be non-diagonal in the three-dimensional case) models the mixing and slowing effect of the vanes. For works related to flows through grids, sieves, or porous walls, we refer to C. CONCA [10] and E. SANCHEZ-PALENCIA [26], [27].

Notation. Throughout this paper, C denotes various real positive constants independent of ε. The duality products between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and between $[H_0^1(\Omega)]^N$ and $[H^{-1}(\Omega)]^N$, are each denoted by $\langle \, , \rangle_{H^{-1}, H_0^1(\Omega)}$. $(e_k)_{1 \leqq k \leqq N}$ is the canonical basis of $\mathbb{R}^N$.

3. Non-Critical Sizes of the Holes for a Volume Distribution

3.1. Setting of the problem

As in Part I of this paper, we consider a volume distribution of the holes in a domain Ω, but the size of the holes will be specified in each subsection. Let Ω be a bounded connected open set in $\mathbb{R}^N$ $(N \geqq 2)$, with Lipschitz boundary $\partial\Omega$, Ω

being locally located on one side of its boundary. The set Ω is covered with a regular mesh of size 2ε, each cell being a cube P_i^ε, identical to $(-\varepsilon\ +\varepsilon)^N$. At the center of each cube P_i^ε included in Ω we make a hole T_i^ε, each hole being similar to the same closed set T rescaled at size a_ε. We assume that T contains a small open ball B_α (with radius $\alpha > 0$), and is strictly included in the open ball B_1 of unit radius. We also assume that $B_1 - T$ is a connected open set, locally located on one side of its Lipschitz boundary. The open set Ω_ε is obtained by removing from Ω all the holes $(T_i^\varepsilon)_{1\leq i\leq N(\varepsilon)}$ (where the number of holes $N(\varepsilon)$ is equal to $|\Omega|/(2\varepsilon)^N[1 + o(1)]$). Because only the cells entirely included in Ω are perforated, it follows that no hole meets the boundary $\partial\Omega$. Thus Ω_ε is also a bounded connected open set, locally located on one side of its Lipschitz boundary $\partial\Omega_\varepsilon$ (see Figure 1 in Part I). Thus

$$\Omega_\varepsilon = \Omega - \bigcup_{i=1}^{N(\varepsilon)} T_i^\varepsilon. \tag{3.1.1}$$

The flow of an incompressible viscous fluid in the domain Ω_ε under the action of an exterior force $f \in [L^2(\Omega)]^N$, with a no-slip (Dirichlet) boundary condition, is described by the following Stokes problem, where u_ε is the velocity, and p_ε the pressure of the fluid (the viscosity and density of the fluid have been set equal to 1).

Find $(u_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)/\mathbb{R}]$ such that

$$\begin{aligned} \nabla p_\varepsilon - \Delta u_\varepsilon &= f \quad \text{in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon. \end{aligned} \tag{3.1.2}$$

Throughout this paper, the size of the holes is smaller than the size of the mesh, *i.e.*,

$$\lim_{\varepsilon\to 0} \frac{a_\varepsilon}{\varepsilon} = 0 \quad \text{or equivalently} \quad 0 \leqq a_\varepsilon \ll \varepsilon. \tag{3.1.3}$$

The case of the hole size exactly of order ε (so that $\lim_{\varepsilon\to 0} a_\varepsilon/\varepsilon > 0$) has been extensively studied by the two-scale method (see [16], [20], [25], and [28]). Here, the situation is completely different because the holes are much smaller than the period, as expressed by assumption (3.1.3). In particular, the celebrated two-scale method is useless.

In the first part of this paper we introduced a so-called critical size of the holes (2.1.1). Now, we define a ratio σ_ε between the actual size of the holes and the critical size:

$$\sigma_\varepsilon = \left(\frac{\varepsilon^N}{a_\varepsilon^{N-2}}\right)^{1/2} \quad \text{for } N \geqq 3 \qquad \sigma_\varepsilon = \varepsilon \left|\log\left(\frac{a_\varepsilon}{\varepsilon}\right)\right|^{1/2} \quad \text{for } N = 2. \tag{3.1.4}$$

To be precise, if the limit of σ_ε, as ε tends to zero, is strictly positive and finite, then the hole size is called critical. In that case we already know from Part I that the homogenized system is a Brinkman law. The goal of this section is to investigate all the other sizes. For smaller sizes (for which $\lim_{\varepsilon\to 0} \sigma_\varepsilon = +\infty$) we show that the limit problem is a Stokes system, while for larger sizes (for which $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$) it is a Darcy law.

Remark 3.1 1. For the same geometry, the homogenization of the Laplace equation involves the same critical size (see [9]). The investigation of all other sizes, for the Laplace equation, has been addressed by H. KACIMI in her thesis [14].

3.2. Critical size: Brinkman's law

Let us give a very brief summary of Part I. First, we establish the main results of convergence for the homogenization process using an abstract framework of hypotheses on the holes. Second, we verify these hypotheses in the case of a volume distribution of the holes, for holes of critical size. Let us recall

Hypotheses (H1)–(H6). We assume that the holes T_i^ε are such that there exist functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leqq k \leqq N}$ and a linear map R_ε such that

(H1) $w_k^\varepsilon \in [H^1(\Omega)]^N, \quad q_k^\varepsilon \in L^2(\Omega)$.

(H2) $\nabla \cdot w_k^\varepsilon = 0 \quad$ in $\Omega \quad$ and $w_k^\varepsilon = 0$ on the holes T_i^ε.

(H3) $w_k^\varepsilon \rightharpoonup e_k$ in $[H^1(\Omega)]^N$ weakly, $q_k^\varepsilon \rightharpoonup 0$ in $L^2(\Omega)/\mathbb{R}$ weakly.

(H4) $\mu_k \in [W^{-1,\infty}(\Omega)]^N$.

(H5) For each sequence v_ε, for each v such that

$$v_\varepsilon \rightharpoonup v \text{ in } [H^1(\Omega)]^N \text{ weakly}, \quad v_\varepsilon = 0 \text{ on the holes } T_i^\varepsilon,$$

and for each $\phi \in D(\Omega)$ we have

$$\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \to \langle \mu_k \ \phi v \rangle_{H^{-1}, H_0^1(\Omega)}.$$

(H6)
$$\begin{cases} R_\varepsilon \in L([H_0^1(\Omega)]^N; [H_0^1(\Omega_\varepsilon)]^N), \\ u \in [H_0^1(\Omega_\varepsilon)]^N \text{ implies that } R_\varepsilon \tilde{u} = u \text{ in } \Omega_\varepsilon, \\ \nabla \cdot u = 0 \text{ in } \Omega \text{ implies that } \nabla \cdot (R_\varepsilon u) = 0 \text{ in } \Omega_\varepsilon, \\ \|R_\varepsilon u\|_{H_0^1(\Omega_\varepsilon)} \leqq C \|u\|_{H_0^1(\Omega)} \text{ and } C \text{ does not depend on } \varepsilon. \end{cases}$$

Combining Theorem 1.1.8 and Propositions 2.1.2, 2.1.4, and 2.1.6, we obtain

Theorem 3.2.1. *Let the hole size be critical, i.e., let*

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma > 0 \quad \textit{and} \quad \sigma < +\infty. \tag{3.2.1}$$

Let $(u_\varepsilon, p_\varepsilon)$ be the unique solution of (3.1.2). *Let $\tilde{u}_\varepsilon$ be the extension by* 0 *in the holes (T_i^ε) of the velocity u_ε. Let $P_\varepsilon(p_\varepsilon)$ be the extension of the pressure p_ε defined by*

$$P_\varepsilon(p_\varepsilon) = p_\varepsilon \text{ in } \Omega_\varepsilon \quad \text{and} \quad P_\varepsilon(p_\varepsilon) = \frac{1}{|C_i^\varepsilon|} \int_{C_i^\varepsilon} p_\varepsilon \text{ in each hole } T_i^\varepsilon,$$

where C_i^ε is a "control" volume around the hole T_i^ε defined as the part outside T_i^ε of the ball of radius ε with same center as T_i^ε. Then $(\tilde{u}_\varepsilon, P_\varepsilon(p_\varepsilon))$ converges weakly to

(u, p) in $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, where (u, p) is the unique solution of the following Brinkman equations:

Find $(u, p) \in [H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$ such that

$$\nabla p - \Delta u + \frac{1}{\sigma^2} M_0 u = f \quad \text{in } \Omega, \qquad (3.2.2)$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega.$$

If $N = 2$, then $M_0 = \pi\, Id$, whatever the shape of the model hole T; if $N \geqq 3$, then ${}^t e_i M_0 e_k = \frac{1}{2^N} \int_{\mathbb{R}^N - T} \nabla w_k : \nabla w_i$ where w_k is the solution of the following problem Stokes equations:

$$\begin{aligned} \nabla q_k - \Delta w_k &= 0 \quad \text{in } \mathbb{R}^N - T, \\ \nabla \cdot w_k &= 0 \quad \text{in } \mathbb{R}^N - T, \\ w_k &= 0 \quad \text{on } \partial T, \\ w_k &= e_k \quad \text{at infinity}. \end{aligned} \qquad (3.2.3)$$

Remark 3.2.2. We point out a slight change in our notation. In the first part of this paper we defined the critical size of the holes by (2.1.1), *i.e.*,

$$\lim_{\varepsilon \to 0} \frac{a_\varepsilon}{\varepsilon^{N/(N-2)}} = C_0 \text{ for } N \geqq 3, \qquad \lim_{\varepsilon \to 0} -\varepsilon^2 \log(a_\varepsilon) = C_0 \text{ for } N = 2,$$

where C_0 is a strictly positive constant $(0 < C_0 < +\infty)$. Actually (2.1.1) is exactly equivalent to definition (3.2.1) if the constants C_0 and σ are related by

$$C_0 = \sigma^{\frac{-2}{N-2}} \text{ for } N \geqq 3, \qquad C_0 = \sigma^2 \text{ for } N = 2. \qquad (3.2.4)$$

Furthermore, we change the name of the matrix appearing in Brinkman's law. In order to make explicit how this matrix depends on the rescaled size of the holes (namely C_0 or σ) we use a new notation M_0. The matrix M_0 does not depend on C_0 or σ, and is related to notation M used in the first part by

$$M = \frac{1}{\sigma^2} M_0. \qquad (3.2.5)$$

That allows us to greatly simplify the presentation of this section, and to emphasize the continuous transition from one limit regime to another.

Remark 3.2.3. Other results, including correctors and error estimates, are proved in Part I. Let us mention that, when the space dimension is $N = 2$ or 3, Theorem 3.2.1 can be easily generalized to the Navier-Stokese quations (see Remark 1.1.10). In our framework the non-linear convective term in the Navier-Stokes equations turns out to be a compact perturbation of the Stokes equations, so the corresponding homogenized system is simply a Brinkman-type problem including a non-linear convective term, without any change in the matrix M_0.

3.3. Smaller holes: Stokes equations

We now assume that the size of the holes is smaller than the critical size, *i.e.*,

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon = +\infty. \tag{3.3.1}$$

In other words,

$$a_\varepsilon \ll \varepsilon^{\frac{N}{N-2}} \text{ for } N \geqq 3, \quad a_\varepsilon = e^{\frac{-1}{C_\varepsilon}} \quad \text{and} \quad C_\varepsilon \ll \varepsilon^2 \text{ for } N = 2.$$

Then, using the abstract framework of Part I, we prove

Theorem 3.3.1. *Let the hole size satisfy* (3.3.1). *Let* $(u_\varepsilon, p_\varepsilon)$ *be the unique solution of the Stokes problem* (3.1.2). *Let* $\tilde{u}_\varepsilon$ *be the extension by* 0 *in the holes* (T_i^ε) *of the velocity* u_ε. *Let* $P_\varepsilon(p_\varepsilon)$ *be the extension of the pressure* p_ε *defined by*

$$P_\varepsilon(p_\varepsilon) = p_\varepsilon \text{ in } \Omega_\varepsilon \quad \text{and} \quad P_\varepsilon(p_\varepsilon) = \frac{1}{|C_i^\varepsilon|} \int_{C_i^\varepsilon} p_\varepsilon \text{ in each hole } T_i^\varepsilon,$$

where C_i^ε *is a "control" volume around the hole* T_i^ε *defined as the part outside* T_i^ε *of the ball of radius* ε *with same center as* T_i^ε. *Then* $(\tilde{u}_\varepsilon, P_\varepsilon(p_\varepsilon))$ *converges strongly to* (u, p) *in* $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, *where* (u, p) *is the unique solution of the following Stokes problem*

$$\text{Find } (u, p) \in [H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \text{ such that}$$
$$\begin{aligned} \nabla p - \Delta u &= f \quad \text{in } \Omega, \\ \nabla \cdot u &= 0 \quad \text{in } \Omega, \end{aligned} \tag{3.3.2}$$

Remark 3.3.2. Theorem 3.3.1 expresses the fact that obstacles that are too small cannot significantly slow down the fluid flow. Thus, nothing happens in the limit: the Stokes flow is unperturbed. When the space dimension is $N = 2$ or 3, Theorem 3.3.1 can be generalized to the Navier-Stokes equations, as previously done in Remark 3.2.3.

Proof. This proof follows the pattern of Part I of this paper. In particular, we use the abstract framework introduced in the first section. For this purpose, we first have to check Hypotheses (H1)–(H6). Next we show that, in the present situation, the matrix M is equal to zero. (In light of (3.2.5), this result is not surprising because $\sigma = +\infty$.) Finally we show that the weak convergence of the solutions $(\tilde{u}_\varepsilon, P_\varepsilon(p_\varepsilon))$, ensured by Theorem 1.1.8, is indeed strong.

We construct a linear map R_ε and functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leqq k \leqq N}$ exactly as we did in subsections 2.2 and 2.3, replacing everywhere the critical hole size by the current smaller one. Then, it easy to see that they fulfill Hypotheses (H1)–(H6). Moreover, an easy but tedious computation accounting for (3.3.1) yields

$$w_k^\varepsilon \to 0 \text{ in } [H^1(\Omega)]^N \text{ strongly}, \quad q_k^\varepsilon \to 0 \text{ in } L^2(\Omega)/\mathbb{R} \text{ strongly}. \tag{3.3.3}$$

Because Hypotheses (H1)–(H6) are satisfied, all the results of the abstract framework hold. But from (3.3.3) and Remark 1.1.3 we deduce that $M \equiv 0$ in the present situation. Thus the homogenized equations are Stokes equations.

Furthermore, Theorem 1.1.8 yields the convergence of the energy

$$\int_\Omega |\nabla \tilde{u}_\varepsilon|^2 = \int_\Omega f \cdot \tilde{u}_\varepsilon \to \int_\Omega f \cdot u = \int_\Omega |\nabla u|^2 + \langle Mu, u \rangle_{H^{-1}, H_0^1(\Omega)}. \tag{3.3.4}$$

Because M is identically equal to zero, (3.3.4) is equivalent to

$$\int_\Omega |\nabla \tilde{u}_\varepsilon|^2 \to \int_\Omega |\nabla u|^2.$$

This means that $\tilde{u}_\varepsilon$ converges strongly to u in $[H_0^1(\Omega)]^N$. Now it remains to prove the strong convergence of the pressure. For any sequence v_ε that converges weakly to v in $[H_0^1(\Omega)]^N$ we recall Definition (1.1.8) of the extension $P_\varepsilon(p_\varepsilon)$:

$$\langle \nabla [P_\varepsilon(p_\varepsilon)]\ v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} = \langle \nabla p_\varepsilon, R_\varepsilon v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega_\varepsilon)} .$$

Introducing Stokes equations (3.1.2) into this equation and integrating the result by parts give

$$\langle \nabla [P_\varepsilon(p_\varepsilon)], v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} = \int_{\Omega_\varepsilon} f \cdot R_\varepsilon v_\varepsilon - \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla (R_\varepsilon v_\varepsilon). \tag{3.3.5}$$

According to the explicit construction of the operator R_ε (see Section 2.2) it turns out that both the sequences v_ε and $R_\varepsilon v_\varepsilon$ converge weakly to the same limit v in $[H_0^1(\Omega)]^N$. Then, because of the strong convergence of $\tilde{u}_\varepsilon$ in $[H_0^1(\Omega)]^N$, we deduce from (3.3.5) that

$$\lim_{\varepsilon \to 0} \langle \nabla [P_\varepsilon(p_\varepsilon)], v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} = \int_\Omega f \cdot v - \int_\Omega \nabla u \cdot \nabla v. \tag{3.3.6}$$

Introducing the homogenized Stokes equation (3.3.2) into (3.3.6) leads to

$$\nabla [P_\varepsilon(p_\varepsilon)] \to \nabla p \quad \text{in } [H^{-1}(\Omega)]^N.$$

Thanks to Lemma 1.1.5, we obtain the desired result, *i.e.*, $P_\varepsilon(p_\varepsilon)$ converges strongly to p in $L^2(\Omega)/\mathbb{R}$. Q.E.D.

3.4. Larger holes: Darcy's law

We now assume that the size of the holes exceeds the critical size, *i.e.*,

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0. \tag{3.4.1}$$

In other words,

$$\varepsilon^{\frac{N}{N-2}} \ll a_\varepsilon \quad \text{for } N \geqq 3, \qquad a_\varepsilon = e^{\frac{-1}{C_\varepsilon}} \quad \text{and} \quad \varepsilon^2 \ll C_\varepsilon \quad \text{for } N = 2.$$

However, the size a_ε is still smaller than the inter-hole distance ε, so that (3.4.1) yields

$$\varepsilon \ll \sigma_\varepsilon \ll 1. \tag{3.4.2}$$

This case is somewhat more complicated than the former one, and some modifications of Hypotheses (H1)–(H6) are required in order to carry out the pattern of Part I. The structure of this subsection is the following. First we establish a Poincaré inequality in Ω_ε with a sharp constant (Lemma 3.4.1). Second, we introduce the modified Hypotheses (H1*)–(H6*). Third, in this abstract framework of hypotheses we establish the convergence of the homogenization process (Theorem 3.4.4, Propositions 3.4.3 and 3.4.6). Fourth, we check that Hypotheses (H1*)–(H6*) hold in the present geometrical situation (Propositions 3.4.8, 3.4.9, 3.4.10). The reader should be aware that the present subsection includes both the abstract framework, and its verification. Two distinct sections were used for this purpose in Part I.

Lemma 3.4.1. *There exists a constant C that does not depend on ε such that*

$$\|u\|_{L^2(\Omega_\varepsilon)} \leqq C\sigma_\varepsilon \|\nabla u\|_{L^2(\Omega_\varepsilon)}$$

for any $u \in H_0^1(\Omega_\varepsilon)$, *where* σ_ε *is defined in* (3.1.4).

Proof. (This lemma has also been proved by H. KACIMI [14].) Let $u \in D(\Omega_\varepsilon)$. We extend u continuously by 0 in each hole T_i^ε. Denoting by $B_i'^\varepsilon$ the ball circumscribed in the cube P_i^ε, we have

$$\|u\|^2_{L^2(\Omega_\varepsilon)} \leqq \sum_{i=1}^{N(\varepsilon)} \|u\|^2_{L^2(B_i'^\varepsilon)} \leqq (2N+1)\, \|u\|^2_{L^2(\Omega_\varepsilon)}.$$

Let r be the distance between the center of $B_i'^\varepsilon$ and a point $x \in B_i'^\varepsilon$. As the model hole T contains a small ball B_α (see Section 3.1), each hole T_i^ε also contains a small ball $B_i^{\alpha a_\varepsilon}$ of radius αa_ε. Thus $u(r = \alpha a_\varepsilon) = 0$, and

$$u(x) = \int_{\alpha a_\varepsilon}^{r} \frac{\partial u}{\partial r}(x + (t - r)\, e_r)\, dt.$$

Then

$$\|u\|^2_{L^2(B_i'^\varepsilon)} \leqq C \int_{\alpha a_\varepsilon}^{2\varepsilon} \left[\int_{\alpha a_\varepsilon}^{r} \frac{\partial u}{\partial r}(x + (t - r)\, e_r)\, dt \right]^2 r^{N-1}\, dr.$$

But the Schwarz inequality gives

$$\left[\int_{\alpha a_\varepsilon}^{r} \frac{\partial u}{\partial r}(x + (t - r)\, e_r)\, dt \right]^2 \leqq \left[\int_{\alpha a_\varepsilon}^{r} \left[\frac{\partial u}{\partial r}(x + (t - r)\, e_r) \right]^2 t^{N-1}\, dt \right] \left[\int_{\alpha a_\varepsilon}^{r} \frac{dt}{t^{N-1}} \right].$$

Thus

$$\|u\|^2_{L^2(B_i'^\varepsilon)} \leqq C \int_{\alpha a_\varepsilon}^{2\varepsilon} \left[\int_{\alpha a_\varepsilon}^{2\varepsilon} \left[\frac{\partial u}{\partial r}(x + (t - r)\, e_r) \right]^2 t^{N-1}\, dt \right] \left[\int_{\alpha a_\varepsilon}^{2\varepsilon} \frac{dt}{t^{N-1}} \right] r^{N-1}\, dr$$

$$\leqq C\varepsilon^N \|\nabla u\|^2_{L^2(B_i'^\varepsilon)} \left[\int_{\alpha a_\varepsilon}^{2\varepsilon} \frac{dt}{t^{N-1}} \right] \leqq C\sigma_\varepsilon^2\, \|\nabla u\|^2_{L^2(B_i'^\varepsilon)}.$$

Summing the above estimates from $i = 1$ to $N(\varepsilon)$ leads to the desired result. Q.E.D.

Modified hypotheses (H1*)–(H6*) We assume that the holes T_i^ε are such that there exist functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leq k \leq N}$ and a linear map R_ε such that

(H1*) $w_k^\varepsilon \in [H^1(\Omega)]^N, \quad q_k^\varepsilon \in L^2(\Omega).$

(H2*) $\nabla \cdot w_k^\varepsilon = 0$ in Ω and $w_k^\varepsilon = 0$ on the holes T_i^ε.

(H3*) $w_k^\varepsilon \rightharpoonup e_k$ in $[L^2(\Omega)]^N$ weakly, $\sigma_\varepsilon \|\nabla w_k^\varepsilon\|_{L^2(\Omega)} \leq C$, $\sigma_\varepsilon \|q_k^\varepsilon\|_{L^2(\Omega)} \leq C$ where the constant C does not depend on ε.

(H4*) $\mu_k \in [L^\infty(\Omega)]^N$.

(H5*) For each sequence $v_\varepsilon \in [H^1(\Omega)]^N$, for each $v \in [L^2(\Omega)]^N$ such that

$$v_\varepsilon \rightharpoonup v \text{ in } [L^2(\Omega)]^N \text{ weakly,}$$

$$\|\nabla v_\varepsilon\|_{L^2(\Omega)} \leq C/\sigma_\varepsilon \quad \text{where } C \text{ does not depend on } \varepsilon,$$

$$v_\varepsilon = 0 \quad \text{on the holes } T_i^\varepsilon,$$

and for each $\phi \in D(\Omega)$ the following limit holds

$$\sigma_\varepsilon^2 \langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi v_\varepsilon \rangle_{H^{-1},H_0^1(\Omega)} \to \int_\Omega \phi \mu_k \cdot v.$$

$$\text{(H6*)} \quad \begin{cases} R_\varepsilon \in L([H_0^1(\Omega_\varepsilon)]^N; [H_0^1(\Omega_\varepsilon)]^N), \\ u \in [H_0^1(\Omega_\varepsilon)]^N \quad \text{implies that } R_\varepsilon \tilde{u} = u \text{ in } \Omega_\varepsilon, \\ \nabla \cdot u = 0 \text{ in } \Omega \quad \text{implies that } \nabla \cdot (R_\varepsilon u) = 0 \text{ in } \Omega_\varepsilon, \\ \|\nabla (R_\varepsilon u)\|_{L^2(\Omega_\varepsilon)} \leq C \left[\|\nabla u\|_{L^2(\Omega)} + \dfrac{1}{\sigma_\varepsilon} \|u\|_{L^2(\Omega)} \right] \quad \text{and} \\ C \text{ does not depend on } \varepsilon. \end{cases}$$

Remark 3.4.2. The modified hypotheses (H1*)–(H6*) are very close to those introduced in Part I, and have the same physical and mathematical meaning (*i.e.*, as viscous layers in the vicinity of the holes, and test functions in the energy method). Moreover, for a given family of holes $(T_i^\varepsilon)_{1 \leq i \leq N(\varepsilon)}$ the functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leq k \leq N}$ that satisfy Hypotheses (H1*)–(H5*) are "quasi-unique" (see Section IV.1 in [1] for more details).

Proposition 3.4.3. *Let* $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leq k \leq N}$ *be functions that satisfy the modified hypotheses* (H1*)–(H5*). *Let* M_0 *be the matrix with columns* $(\mu_k)_{1 \leq k \leq N}$ *and entries* $(\mu_k^i)_{1 \leq k,i \leq N}$ *defined by* $\mu_k^i = \mu_k \cdot e_i$. *Then for each* $\phi \in D(\Omega)$,

$$\langle \mu_k^i, \phi \rangle_{D',D(\Omega)} = \lim_{\varepsilon \to 0} \sigma_\varepsilon^2 \int_\Omega \phi \, \nabla w_k^\varepsilon : \nabla w_i^\varepsilon. \tag{3.4.3}$$

In particular, M_0 *is a symmetric and positive-definite matrix in the following sense:*

$$\langle M_0 \Phi, \Phi \rangle_{D',D(\Omega)} \geq C^{-1} \|\Phi\|_{L^2(\Omega)} \geq 0 \quad \textit{for each } \Phi \in [D(\Omega)]^N \tag{3.4.4}$$

where C is the constant in Poincaré's inequality (see Lemma 3.4.1).

Proof. Taking $v_\varepsilon = w_i^\varepsilon$ and $v = e_i$ in Hypothesis (H5*), and integrating the limit by parts gives

$$\sigma_\varepsilon^2 \langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi w_i^\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} = -\sigma_\varepsilon^2 \int_\Omega q_k^\varepsilon w_i^\varepsilon \cdot \nabla \phi + \sigma_\varepsilon^2 \int_\Omega \nabla w_k^\varepsilon : w_i^\varepsilon \nabla \phi + \sigma_\varepsilon^2 \int_\Omega \phi \nabla w_k^\varepsilon : \nabla w_i^\varepsilon \to \int_\Omega \phi \mu_k \cdot e_i. \tag{3.4.5}$$

From (H3*) it follows that

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon^2 \int_\Omega \nabla w_k^\varepsilon : w_i^\varepsilon \nabla \phi = 0, \qquad \lim_{\varepsilon \to 0} \sigma_\varepsilon^2 \int_\Omega q_k^\varepsilon w_i^\varepsilon \cdot \nabla \phi = 0.$$

Thus (3.4.3) is deduced from (3.4.5). Moreover M_0 is a symmetric matrix, since it is the limit of a sequence of symmetric matrices $(\nabla w_k^\varepsilon : \nabla w_i^\varepsilon)_{1 \leq i,k \leq N}$. On the other hand, for each $\Phi \in [D(\Omega)]^N$ one has $\sum_{k=1}^N \phi_k w_k^\varepsilon \in [H_0^1(\Omega_\varepsilon)]^N$. The Poincaré inequality implies that

$$\left\| \sum_{k=1}^N \phi_k w_k^\varepsilon \right\|_{L^2(\Omega_\varepsilon)}^2 \leq C^2 \sigma_\varepsilon^2 \left[\left\| \sum_{k=1}^N \phi_k \nabla w_k^\varepsilon \right\|_{L^2(\Omega_\varepsilon)}^2 + \left\| \sum_{k=1}^N \nabla \phi_k \, w_k^\varepsilon \right\|_{L^2(\Omega_\varepsilon)}^2 \right]. \tag{3.4.6}$$

From (H3*) we deduce that

$$\lim_{\varepsilon \to 0} \left\| \sum_{k=1}^N \phi_k w_k^\varepsilon \right\|_{L^2(\Omega_\varepsilon)}^2 \geq \|\Phi\|_{L^2(\Omega)}^2, \qquad \lim_{\varepsilon \to 0} \sigma_\varepsilon^2 \left\| \sum_{k=1}^N \nabla \phi_k \, w_k^\varepsilon \right\|_{L^2(\Omega_\varepsilon)}^2 = 0.$$

Then, using (3.4.3) we obtain

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon^2 \int_\Omega \left| \sum_{k=1}^N \phi_k \nabla w_k^\varepsilon \right|^2 = \langle M_0 \Phi, \Phi \rangle_{D', D(\Omega)}.$$

We pass to the limit in (3.4.6) and obtain (3.4.4). Q.E.D.

Now, we are able to prove the main theorem of this section, which corresponds to Proposition 1.1.4 and Theorem 1.1.8, established in the case of a critical size of the holes.

Theorem 3.4.4. *Let the hole size satisfy* (3.4.1), *and let Hypotheses* (H1*)–(H6*) *hold. Denote by* M_0 *the matrix defined in Proposition 3.4.3. Let* $(u_\varepsilon, p_\varepsilon)$ *be the unique solution of the Stokes system* (3.1.2). *Let* $\tilde{u}_\varepsilon$ *be the extension of the velocity by* 0 *in* $\Omega - \Omega_\varepsilon$. *Let* $P_\varepsilon(p_\varepsilon)$ *be a function defined by*

$$\langle \nabla P_\varepsilon(p_\varepsilon), v \rangle_{H^{-1}, H_0^1(\Omega)} = \langle \nabla p_\varepsilon, R_\varepsilon v \rangle_{H^{-1}, H_0^1(\Omega_\varepsilon)} \qquad \textit{for each } v \in [H_0^1(\Omega)]^N.$$

Then $P_\varepsilon(p_\varepsilon)$ *is an extension of the the pressure (i.e.,* $P_\varepsilon(p_\varepsilon) \equiv p_\varepsilon$ *in* Ω_ε*) such that*

$$\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} \rightharpoonup u \text{ in } [L^2(\Omega)]^N \text{ weakly}, \qquad P_\varepsilon(p_\varepsilon) \to p \text{ in } L^2(\Omega)/\mathbb{R} \text{ strongly}$$

where (u, p) *is the unique solution of Darcy's law*:

$$\begin{aligned} &\textit{Find } (u,p) \in [L^2(\Omega)]^N \times [H^1(\Omega)/\mathbb{R}] \textit{ such that} \\ &u = M_0^{-1}(f - \nabla p) \quad \textit{in } \Omega, \\ &\nabla \cdot u = 0 \quad \textit{in } \Omega, \\ &u \cdot n = 0 \quad \textit{on } \partial\Omega. \end{aligned} \tag{3.4.7}$$

Remark 3.4.5. Note the that matrix M_0^{-1}, which appears in Darcy's law (3.4.7), is the same for all values of the hole size. On the other hand, it is totally different from that usually obtained by the two-scale method (see, *e.g.*, [25]), when the holes have a size ε of the same order of magnitude as the period.

Proof. This proof is divided into two steps. First, we obtain *a priori* estimates for the solution of the Stokes equations (3.1.2). Then, we pass to the limit in those equations with the help of the energy method introduced by L. TARTAR [29].
Step 1. Multiplying the momentum equation in (3.1.2) by u_ε and integrating the product by parts give

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{\Omega_\varepsilon} f \cdot u_\varepsilon .$$

The Poincaré inequality furnished by Lemma 3.4.1 yields

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqq C\sigma_\varepsilon \|f\|_{L^2(\Omega_\varepsilon)}, \qquad \|\tilde{u}_\varepsilon\|_{L^2(\Omega)} \leqq C\sigma_\varepsilon^2 \|f\|_{L^2(\Omega)}.$$

There exists some $u \in [L^2(\Omega)]^N$ and a subsequence of $\tilde{u}_\varepsilon$, still denoted by $\tilde{u}_\varepsilon$, such that $\dfrac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2}$ converges weakly to u in $[L^2(\Omega)]^N$. Now, using Hypothesis (H6*), we construct an extension of the pressure (following an idea of L. TARTAR [28]). Let $F_\varepsilon \in [H^{-1}(\Omega)]^N$ be defined by

$$\langle F_\varepsilon, v\rangle_{H^{-1},H_0^1(\Omega)} = \langle \nabla p_\varepsilon, R_\varepsilon v\rangle_{H^{-1},H_0^1(\Omega_\varepsilon)} \qquad \text{for each } v \in [H_0^1(\Omega)]^N \tag{3.4.8}$$

where R_ε is the linear operator involved in Hypothesis (H6*). Replacing ∇p_ε by $f + \Delta u_\varepsilon$ in (3.4.8) leads to

$$\langle F_\varepsilon, v\rangle_{H^{-1},H_0^1(\Omega)} = \int_{\Omega_\varepsilon} f \cdot R_\varepsilon v - \int_{\varepsilon} \nabla u_\varepsilon : \nabla(R_\varepsilon v). \tag{3.4.9}$$

Thanks to the estimate of R_ε in (H6*), and to the Poincaré inequality, we deduce from (3.4.9) that

$$\|F_\varepsilon\|_{H^{-1}(\Omega)} \leqq C \|f\|_{L^2(\Omega)}. \tag{3.4.10}$$

Then, arguing as in Proposition 1.1.4, we conclude that F_ε is the gradient of a function $P_\varepsilon(p_\varepsilon) \in L^2(\Omega)$ that is equal to p_ε in Ω_ε. Moreover, from estimate (3.4.10) we obtain

$$\|P_\varepsilon(p_\varepsilon)\|_{L^2(\Omega)/\mathbb{R}} \leqq C \|f\|_{L^2(\Omega)}.$$

Let v_ε be a sequence that converges weakly to 0 in $[H_0^1(\Omega)]^N$. In order to prove the strong convergence of $P_\varepsilon(p_\varepsilon)$ in $L^2(\Omega)/\mathbb{R}$, we take $v = v_\varepsilon$ in formula (3.4.9), and we obtain the estimate

$$\begin{aligned} |\langle \nabla P_\varepsilon(p_\varepsilon), v_\varepsilon \rangle_{H^{-1},H_0^1(\Omega)}| &\leqq C \|f\|_{L^2(\Omega)} [\sigma_\varepsilon \|\nabla(R_\varepsilon v_\varepsilon)\|_{L^2(\Omega_\varepsilon)} + \|R_\varepsilon v_\varepsilon\|_{L^2(\Omega_\varepsilon)}] \\ &\leqq C \|f\|_{L^2(\Omega)} [\sigma_\varepsilon \|\nabla v_\varepsilon\|_{L^2(\Omega)} + \|v_\varepsilon\|_{L^2(\Omega)}] \qquad (3.4.11) \end{aligned}$$

because of (H6*) and the Poincaré inequality. According to the Rellich Theorem we have

$$\|v_\varepsilon\|_{L^2(\Omega)} \to 0.$$

On the other hand, the scaling (3.4.2) of the holes implies that σ_ε converges to 0. Thus (3.4.11) leads to

$$\langle \nabla P_\varepsilon(p_\varepsilon), v_\varepsilon \rangle_{H^{-1},H_0^1(\Omega)} \to 0.$$

There exists $p \in L^2(\Omega)$ such that a subsequence $P_\varepsilon(p_\varepsilon)$ converges strongly to p in $L^2(\Omega)/\mathbb{R}$.

Step 2. Now we apply the energy method: For any $\phi \in D(\Omega)$, we introduce in the variational formulation (1.1.2) the test functions $(\phi w_k^\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N$ and $(\phi q_k^\varepsilon) \in L^2(\Omega_\varepsilon)/\mathbb{R}$. We obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla u_\varepsilon : \nabla(\phi w_k^\varepsilon) - \int_{\Omega_\varepsilon} p_\varepsilon \nabla \cdot (\phi w_k^\varepsilon) &= \int_{\Omega_\varepsilon} f \cdot (\phi w_k^\varepsilon), \\ \int_{\Omega_\varepsilon} (\phi q_k^\varepsilon) \nabla \cdot u_\varepsilon &= 0. \end{aligned} \qquad (3.4.12)$$

The proof is very similar to that of Theorem 1.1.8. From (3.4.12) we arrive at

$$\begin{aligned} \sigma_\varepsilon^2 \left\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi \frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} \right\rangle_{H^{-1},H_0^1(\Omega)} + \int_\Omega q_k^\varepsilon \tilde{u}_\varepsilon \cdot \nabla\phi - \int_\Omega \tilde{u}_\varepsilon \nabla\phi : \nabla w_k^\varepsilon \\ + \int_\Omega \nabla \tilde{u}_\varepsilon : w_k^\varepsilon \nabla\phi - \int_\Omega P_\varepsilon(p_\varepsilon) w_k^\varepsilon \cdot \nabla\phi = \int_\Omega \phi f \cdot w_k^\varepsilon. \qquad (3.4.13) \end{aligned}$$

Because $\dfrac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2}$ fulfills the assumptions of Hypothesis (H5*), we have

$$\sigma_\varepsilon^2 \left\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi \frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} \right\rangle_{H^{-1},H_0^1(\Omega)} \to \int_\Omega \phi \mu_k \cdot u.$$

Using (H3*) and the *a priori* estimates of the solution $(u_\varepsilon, p_\varepsilon)$, we pass to the limit in the other terms of (3.4.13):

$$\int_\Omega \phi \mu_k \cdot u - \int_\Omega p e_k \cdot \nabla\phi = \int_\Omega f \cdot e_k.$$

From Proposition 3.4.3 we know that the matrix M_0 is invertible, so that

$$u = M_0^{-1}(f - \nabla p) \quad \text{in } \Omega. \qquad (3.4.14)$$

On the other hand,

$$\tilde{u}_\varepsilon \in [H_0^1(\Omega)]^N, \quad \frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} \rightharpoonup u \text{ in } [L^2(\Omega)]^N \text{ weakly}, \quad \nabla \cdot \tilde{u}_\varepsilon = 0 \text{ in } \Omega. \tag{3.4.15}$$

We deduce from (3.4.15) (see, *e.g.*, [25]) that

$$\nabla \cdot u = 0 \quad \text{in } \Omega, \qquad u \cdot n = 0 \quad \text{on } \partial\Omega. \tag{3.4.16}$$

A regrouping of (3.4.14) and (3.4.16) leads to Darcy's law, which has a unique solution because of (H4*). More precisely,

(H4*) implies that $M_0 \in [L^\infty(\Omega)]^{N^2}$, which implies that ${}^t\xi M_0^{-1}\xi \geqq \|M_0\|_{L^\infty(\Omega)} |\xi|^2$.

Because of uniqueness, all the subsequences of $(\tilde{u}_\varepsilon, P_\varepsilon(u_\varepsilon))$ converge to the same limit. Thus the entire sequence converges. Q.E.D.

Now, we give some corrector results which improve the convergence of the velocity in Theorem 3.3.4.

Proposition 3.4.6. *Let the hole size satisfy* (3.4.1) *and let Hypotheses* (H1*)–(H5*) *be satisfied. For each sequence z_ε such that*

$$z_\varepsilon \rightharpoonup z \textit{ in } [L^2(\Omega)]^N \textit{ weakly}, \quad (\sigma_\varepsilon \nabla z_\varepsilon) \textit{ is bounded in } [L^2(\Omega)]^{N^2}, \tag{3.4.17}$$

$$\sigma_\varepsilon \nabla \cdot z_\varepsilon \textit{ converges strongly in } L^2(\Omega), \quad z_\varepsilon = 0 \textit{ on the holes } T_i^\varepsilon$$

it follows that

$$\liminf_{\varepsilon \to 0} \sigma_\varepsilon^2 \int_\Omega |\nabla z_\varepsilon|^2 \geqq \int_\Omega {}^t z M_0 z. \tag{3.4.18}$$

The proof is identical to that of Proposition 1.2.1, and is therefore omitted.

Proposition 3.4.7. *Let the hole size satisfy* (3.4.1) *and let Hypotheses* (H1*)–(H5*) *be satisfied. Let the sequence w_k^ε be bounded in $[L^\infty(\Omega)]^N$ and converge strongly to e_k in $[L^2(\Omega)]^N$. Then, for each sequence z_ε such that*

$$z_\varepsilon \rightharpoonup z \textit{ in } [L^2(\Omega)]^N \textit{ weakly}, \quad (\sigma_\varepsilon \nabla z_\varepsilon) \textit{ is bounded in } [L^2(\Omega)]^{N^2},$$

$$\sigma_\varepsilon \nabla \cdot z_\varepsilon \textit{ converges strongly in } L^2(\Omega), \quad z_\varepsilon = 0 \textit{ on the holes } T_i^\varepsilon,$$

$$\liminf_{\varepsilon \to 0} \sigma_\varepsilon^2 \int_\Omega |\nabla z_\varepsilon|^2 = \int_\Omega {}^t z M_0 z$$

it follows that

$$z_\varepsilon \to z \quad \textit{in } [L^2(\Omega)]^N \textit{ strongly}.$$

Proof. We follow the lines of the proof of Proposition 1.2.2. For any $\Phi \in [D(\Omega)]^N$, we obtain

$$\liminf_{\varepsilon \to 0} \int_\Omega \sigma_\varepsilon^2 |\nabla(z_\varepsilon - W_\varepsilon \Phi)|^2 = \int_\Omega {}^t(z - \Phi)\, M_0(z - \Phi). \tag{3.4.19}$$

Then, the Poincaré inequality yields

$$\liminf_{\varepsilon \to 0} \int_\Omega |z_\varepsilon - W_\varepsilon \Phi|^2 \leqq C \|M_0\|_{L^\infty(\Omega)} \|z - \Phi\|^2_{L^2(\Omega)}. \tag{3.4.20}$$

Without any further assumptions, we can merely deduce from (3.4.20) that $(z_\varepsilon - W_\varepsilon z)$ converges strongly to 0 in $[L^1(\Omega)]^N$. If we assume that the sequence w_k^ε is bounded in $[L^\infty(\Omega)]^N$, then we get

$$(z_\varepsilon - W_\varepsilon z) \to 0 \quad \text{in } [L^2(\Omega)]^N. \tag{3.4.21}$$

Moreover, the strong convergence of w_k^ε in $[L^2(\Omega)]^N$ and the Lebesgue Dominated Convergence Theorem yield the strong convergence of $W_\varepsilon z$ to z in $[L^2(\Omega)]^N$. Thus, from (3.4.21) we obtain

$$z_\varepsilon \to z \quad \text{strongly in } [L^2(\Omega)]^N. \quad \text{Q.E.D.}$$

Proposition 3.4.8. *Let the hole size satisfy* (3.4.1) *and let Hypotheses* (H1*)–(H6*) *be satisfied. Let the sequence w_k^ε be bounded in $[L^\infty(\Omega)]^N$ and converge strongly to e_k in $[L^2(\Omega)]^N$. Then the convergence of the velocity given by Theorem 3.4.4 can be improved*:

$$\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} \to u \quad \text{in } [L^2(\Omega)]^N \text{ strongly}.$$

Proof. We easily check assumptions (3.4.17) for the sequence $\tilde{u}_\varepsilon/\sigma_\varepsilon^2$, and we remark that

$$\sigma_\varepsilon^2 \int_\Omega \left|\nabla\left(\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2}\right)\right|^2 = \int_\Omega f \cdot \frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} \to \int_\Omega f \cdot u = \int_\Omega (M_0 u + \nabla p) \cdot u = \int_\Omega {}^t u M_0 u.$$

Hence the result follows from Proposition 3.4.7. Q.E.D.

Remark 3.4.9. Theorem 3.4.4 and Proposition 3.4.8 can be generalized to the Navier-Stokes equations, when the space dimension N is equal to 2 or 3. In this case we obtain the same homogenized system (3.4.7), because the non-linear term disappears when ε tends to zero $\left(\textit{i.e.,} \int_\Omega (\tilde{u}_\varepsilon \cdot \nabla \tilde{u}_\varepsilon) \cdot \phi w_k^\varepsilon \to 0\right)$.

Now it remains to verify Hypotheses (H1*)–(H6*). Roughly speaking, we proceed as in the first part of this paper.

Proposition 3.4.10. *Let the hole size satisfy* (3.4.1) *so that it is larger than the critical size. Then there exists a linear map R_ε that satisfies Hypothesis* (H6*) *such that the associated extension of the pressure is constant inside each hole, specifically*

$$P_\varepsilon(p_\varepsilon) = p_\varepsilon \text{ in } \Omega_\varepsilon \quad \text{and} \quad P_\varepsilon(p_\varepsilon) = \frac{1}{|C_i^\varepsilon|} \int_{C_i^\varepsilon} p_\varepsilon \text{ in each hole } T_i^\varepsilon,$$

where C_i^ε is a control volume defined as the part outside T_i^ε of the ball of radius ε and same center as T_i^ε.

Proof. We construct R_ε as in Section 2.2. First, we recall the following decomposition of each cube P_i^ε entirely included in Ω

$$\overline{P}_i^\varepsilon = T_i^\varepsilon \cup \overline{C}_i^\varepsilon \cup \overline{K}_i^\varepsilon \quad \text{with} \quad K_i^\varepsilon = P_i^\varepsilon - \overline{B}_i^\varepsilon \tag{3.4.22}$$

where T_i^ε is the hole, C_i^ε is the control volume, and K_i^ε is the remainder, *i.e.*, the corners of P_i^ε (see Figure 2). Let $u \in [H_0^1(\Omega)]^N$. For each cube P_i^ε entirely included in Ω, we know (*cf.* Lemma 2.2.1) that the following Stokes problem has a unique solution, which depends linearly on u:

$$\text{Find } (v_i^\varepsilon, q_i^\varepsilon) \in [H^1(C_i^\varepsilon)]^N \times [L^2(C_i^\varepsilon)/\mathbb{R}] \text{ such that}$$

$$\nabla q_i^\varepsilon - \Delta v_i^\varepsilon = -\Delta u \quad \text{in } C_i^\varepsilon,$$

$$\nabla \cdot v_i^\varepsilon = \nabla \cdot u + \frac{1}{|C_i^\varepsilon|} \int_{T_i^\varepsilon} \nabla \cdot u \quad \text{in } C_i^\varepsilon,$$

$$v_i^\varepsilon = u \quad \text{on } \partial C_i^\varepsilon - \partial T_i^\varepsilon,$$

$$v_i^\varepsilon = 0 \quad \text{on } \partial T_i^\varepsilon.$$

Then we define $R_\varepsilon u$ by:

For each cube P_i^ε entirely included in Ω,

$$R_\varepsilon u = u \text{ in } K_i^\varepsilon = P_i^\varepsilon - B_i^\varepsilon, \quad R_\varepsilon u = v_i^\varepsilon \text{ in } C_i^\varepsilon, \; R_\varepsilon u = 0 \text{ in } T_i^\varepsilon.$$

For each cube P_i^ε that meets $\partial\Omega$,

$$R_\varepsilon u = u \quad \text{in } P_i^\varepsilon \cap \Omega.$$

As in Proposition 2.2.2 we easily check that Hypothesis (H6*) holds for such an operator R_ε. The only difficulty is to obtain the estimate of $R_\varepsilon u$. Lemmas 2.2.3 and 2.2.4 lead to the following estimate of v_i^ε:

$$\|\nabla v_i^\varepsilon\|_{L^2(C_i^\varepsilon)}^2 \leqq C \left[\|\nabla u\|_{L^2(C_i^\varepsilon \cup T_i^\varepsilon)}^2 + \frac{K_\eta^2}{\varepsilon^2} \|u\|_{L^2(C_i^\varepsilon \cup T_i^\varepsilon)}^2 \right], \tag{3.4.23}$$

with $\eta = \dfrac{a_\varepsilon}{\varepsilon}$, which implies $\dfrac{K_\eta^2}{\varepsilon^2} = \dfrac{1}{\sigma_\varepsilon^2}$. Then, summing estimates (3.4.23) for all the cubes P_i^ε, we obtain the desired result:

$$\|\nabla(R_\varepsilon u)\|_{L^2(\Omega_\varepsilon)} \leqq C \left[\|\nabla u\|_{L^2(\Omega)} + \frac{1}{\sigma_\varepsilon} \|u\|_{L^2(\Omega)} \right]. \quad \text{Q.E.D.}$$

Proposition 3.4.11. *For* $N = 2$. *Let the hole size exceed the critical size, so that*

$$\lim_{\varepsilon \to 0} \varepsilon \left| \log \left(\frac{a_\varepsilon}{\varepsilon} \right) \right|^{1/2} = 0.$$

Then there exist functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leqq k \leqq 2}$ *that satisfy Hypotheses* (H1*)–(H5*) (*and also the assumptions of Proposition 3.4.8, so that* w_k^ε *is bounded in* $[L^\infty(\Omega)]^N$

and compact in $[L^2(\Omega)]^N$). *Furthermore, for any shape or size of the model hole T, the matrix* M_0 *defined in Proposition 3.4.3 is given by* $M_0 = \pi\, I d$.

Before stating an equivalent proposition for $N \geqq 3$, we recall the so-called local problem (2.1.5), introduced in the first part of this paper. Let $N \geqq 3$. For $k \in \{1, \ldots, N\}$, the local problem is

$$\begin{aligned} &\text{Find } (q_k, w_k) \text{ such that} \\ &\|q_k\|_{L^2(\mathbb{R}^N - T)} < +\infty \quad \text{and} \quad \|\nabla w_k\|_{L^2(\mathbb{R}^N - T)} < +\infty, \\ &\nabla q_k - \Delta w_k = 0 \quad \text{in } \mathbb{R}^N - T, \\ &\nabla \cdot w_k = 0 \quad \text{in } \mathbb{R}^N - T, \\ &w_k = 0 \quad \text{on } \partial T, \\ &w_k = e_k \quad \text{at infinity}. \end{aligned} \tag{3.2.3}$$

We proved in the appendix of Part I that system (3.2.3) is well posed. We still denote by F_k the drag force applied on T by the above Stokes flow, *i.e.*, $F_k = \int_{\partial T} \left(\frac{\partial w_k}{\partial n} - q_k n \right)$. It turns out that the system (3.2.3) is also the local problem for the present case of holes having a size larger than the critical size.

Proposition 3.4.12. *For* $N \geqq 3$, *let the hole size be larger than the critical size, so that*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^N}{a_\varepsilon^{N-2}} = 0.$$

Then there are functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leqq k \leqq N}$, *constructed from the solutions* $(w_k, q_k)_{1 \leqq k \leqq N}$ *of the local problem* (3.2.3) *that satisfy Hypotheses* (H1*)–(H5*)) *and also the assumptions of Propositions 3.4.8, i.e.,* w_k^ε *is bounded in* $[L^\infty(\Omega)]^N$ *and compact in* $[L^2(\Omega)]^N$).

Furthermore, the matrix M_0 *defined in Proposition 3.4.3 is given by the following formulae*

$$M_0 e_k = \mu_k = \frac{1}{2^N} F_k \quad \textit{for each } k \in \{1, 2, \ldots, N\}$$

or, equivalently

$${}^t\xi M_0 \xi = \frac{1}{2^N} \inf_{w \in E} \|\nabla w\|^2_{L^2(\mathbb{R}^N - T)}$$

with

$$E = \{w \in [H^1_{\mathrm{loc}}(\mathbb{R}^N)]^N / w = 0 \ \textit{in} \ T, \ \nabla \cdot w = 0 \ \textit{in} \ \mathbb{R}^N - T, \ w = \xi \ \textit{at infinity}\}.$$

Remark 3.4.13. We emphasize that the matrix M_0 is the same for all the sizes of the holes that satisfy (3.4.1), and is equal to the one appearing in Brinkman's law (3.2.2). From Propositions 3.4.10–3.4.12, we now know that Hypotheses (H1*)–(H6*) are satisfied by some functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leqq k \leqq N}$ and some map R_ε:

for any value $N \geqq 2$. Of course, because of that, the Convergence Theorem 3.3.4, and the Corrector Theorem 3.4.8 hold true.

As in Part I, we give error estimates for the velocity and the pressure. The main difference with Theorem 2.1.9 is the absence of correctors (recall that the velocity and the pressure converges strongly without correctors).

Theorem 3.4.14. *Let the hole size be larger than the critical size, so that* (3.4.2) *holds. Then, the following bounds hold for the errors*

$$\left\| \frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} - u \right\|_{L^2(\Omega)} \leqq C\left(\frac{\varepsilon}{\sigma_\varepsilon} + \sigma_\varepsilon\right) \|u\|_{W^{2,\infty}(\Omega)},$$

$$\|p_\varepsilon - p\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leqq C\left(\frac{\varepsilon}{\sigma_\varepsilon} + \sigma_\varepsilon\right) \|u\|_{W^{2,\infty}(\Omega)}. \tag{3.4.32}$$

Proof of Proposition 3.4.11 ($N = 2$). As in Subsection 2.3, for $k = 1, 2$ we define functions $(w_k^\varepsilon, q_k^\varepsilon) \in [H^1(P_i^\varepsilon)]^2 \times L^2(P_i^\varepsilon)$, with $\int_{P_i^\varepsilon} q_k^\varepsilon = 0$, by

For each cube P_i^ε which meets $\partial\Omega$:

$$w_k^\varepsilon = e_k, \quad q_k^\varepsilon = 0 \quad \text{in } P_i^\varepsilon \cap \Omega.$$

For each cube P_i^ε entirely included in Ω:

$$\begin{aligned} w_k^\varepsilon = e_k, &\quad q_k^\varepsilon = 0 \quad \text{in } K_i^\varepsilon, \\ \nabla q_k^\varepsilon - \Delta w_k^\varepsilon = 0, &\quad \nabla \cdot w_k^\varepsilon = 0 \text{ in } C_i^\varepsilon, \\ w_k^\varepsilon = 0, &\quad q_k^\varepsilon = 0 \quad \text{in } T_i^\varepsilon. \end{aligned} \tag{3.4.24}$$

We compare these functions with the same ones obtained when the model hole T is the unit ball. As $T \subset B_1$, let us define for each cube P_i^ε a ball $B_i^{a_\varepsilon}$ of radius a_ε that strictly contains the hole T_i^ε (see Figure 2). Now, for $k = 1, 2$ we define functions $(w_{0k}^\varepsilon, q_{0k}^\varepsilon)$ by (3.4.24) in which T_i^ε is replaced by $B_i^{a_\varepsilon}$. Denoting by r_i and e_r^i the radial co-ordinate and unit vector in each $C_i^\varepsilon - B_i^{a_\varepsilon}$, we can compute $(w_{0k}^\varepsilon, q_{0k}^\varepsilon)_{1 \leqq k \leqq 2}$ by

$$w_{0k}^\varepsilon = x_k r_i f(r_i)\, e_r^i + g(r_i)\, e_k, \quad q_{0k}^\varepsilon = x_k h(r_i) \quad \text{for } r_i \in [a_\varepsilon; \varepsilon]$$

with

$$f(r_i) = \frac{1}{r_i^2}\left(A + \frac{B}{r_i^2}\right) + C,$$

$$g(r_i) = -A \log r_i - \frac{B}{2r_i^2} - \tfrac{3}{2} C r_i^2 + D,$$

$$h(r_i) = \frac{2A}{r_i^2} - 4C,$$

$$A = -\frac{\varepsilon^2}{\sigma_\varepsilon^2}[1 + o(1)], \qquad B = \frac{\varepsilon^2 a_\varepsilon^2}{\sigma_\varepsilon^2}[1 + o(1)],$$

$$C = \frac{1}{\sigma_\varepsilon^2}[1 + o(1)], \qquad D = 1 - \frac{\varepsilon^2 \log \varepsilon}{\sigma_\varepsilon^2}[1 + o(1)].$$

An easy but tedious computation gives

$$\|q_{0k}^{\varepsilon}\|_{L^2(\Omega)} \leqq \frac{C}{\sigma_\varepsilon}, \quad \|\nabla w_{0k}^{\varepsilon}\|_{L^2(\Omega)} \leqq \frac{C}{\sigma_\varepsilon},$$

$$\|w_{0k}^{\varepsilon} - e_k\|_{L^2(\Omega)} \leqq C\frac{\varepsilon^2}{\sigma_\varepsilon^2}, \quad \left(\frac{\partial w_{0k}^{\varepsilon}}{\partial r_i} - q_{0k}^{\varepsilon} e_r^i\right)\delta_i^{a_\varepsilon} = \frac{2\varepsilon^2}{a_\varepsilon \sigma_\varepsilon^2}[1 + o(1)]\, e_k\, \delta_i^{a_\varepsilon} \tag{3.4.25}$$

where $\delta_i^{a_\varepsilon}$ is the measure defined as the unit mass concentrated on the sphere $\partial B_i^{a_\varepsilon}$, *i.e.*,

$$\langle \delta_i^{a_\varepsilon}, \phi\rangle_{D',D(\mathbb{R}^N)} = \int_{\partial B_i^{a_\varepsilon}} \phi(s)\, ds \quad \text{for any } \phi \in D(\mathbb{R}^N).$$

Then, for $k = 1, 2$ we define the "difference" functions $(w_k'^{\varepsilon}, q_k'^{\varepsilon})$ by

$$w_k'^{\varepsilon} = w_k^{\varepsilon} - w_{0k}^{\varepsilon} \in [H_0^1(\Omega)]^2, \quad q_k'^{\varepsilon} = q_k^{\varepsilon} - q_{0k}^{\varepsilon} \in L^2(\Omega).$$

They satisfy

$$\left\{\begin{array}{l} \nabla q_k'^{\varepsilon} - \Delta w_k'^{\varepsilon} = \left(\dfrac{\partial w_{0k}^{\varepsilon}}{\partial r_i} - q_{0k}^{\varepsilon} e_r^i\right)\delta_i^{a_\varepsilon} \\ \nabla \cdot w_k'^{\varepsilon} = 0, \end{array}\right\} \quad \text{in each control volume } C_i^{\varepsilon}. \tag{3.4.26}$$

$$\left\{\begin{array}{l} w_k'^{\varepsilon} = 0 \\ q_k'^{\varepsilon} = 0 \end{array}\right\} \quad \text{elsewhere in } \Omega - \bigcup_{i=1}^{N(\varepsilon)} C_i^{\varepsilon}.$$

Now, arguing as in Lemma 2.3.1, we show that the difference functions $(w_k'^{\varepsilon}, q_k'^{\varepsilon})$ are "negligible". Thus, as far as the verification of Hypotheses (H1*)–(H5*) is concerned, there are almost no differences between the case of spherical holes and the general case of arbitrary holes.

From (3.4.26) we deduce that

$$\int_{C_i^{\varepsilon}} |\nabla w_k'^{\varepsilon}|^2 = \int_{\partial B_i^{a_\varepsilon}} \left(\frac{\partial w_{0k}^{\varepsilon}}{\partial r_i} - q_{0k}^{\varepsilon} e_r^i\right) \cdot w_k'^{\varepsilon} = \frac{2\varepsilon^2}{a_\varepsilon \sigma_\varepsilon^2}[1 + o(1)] \int_{\partial B_i^{a_\varepsilon}} e_k \cdot w_k'^{\varepsilon} \tag{3.4.27}$$

in each C_i^{ε}. Recall the trace estimate (2.3.11) obtained in Lemma 2.3.1:

$$\left|\int_{\partial B_i^{a_\varepsilon}} e_k \cdot w_k'^{\varepsilon}\right| \leqq C a_\varepsilon \|\nabla w_k'^{\varepsilon}\|_{L^2(B_i^{a_\varepsilon} - T_i^{\varepsilon})}.$$

Thus from (3.4.27) we deduce that

$$\|\nabla w_k'^{\varepsilon}\|_{L^2(C_i^{\varepsilon})} \leqq C\frac{\varepsilon^2}{\sigma_\varepsilon^2}.$$

With the help of Lemma 2.2.4 we obtain an equivalent inequality for $q_k'^{\varepsilon}$. Hence

$$\|q_k'^{\varepsilon}\|^2_{L^2(\Omega)} = \frac{|\Omega|}{(2\varepsilon)^2}[1 + o(1)]\, \|q_k'^{\varepsilon}\|^2_{L^2(C_i^{\varepsilon})} \leqq C\frac{\varepsilon^2}{\sigma_\varepsilon^4},$$

$$\|\nabla w_k'^{\varepsilon}\|^2_{L^2(\Omega)} = \frac{|\Omega|}{(2\varepsilon)^2}[1 + o(1)]\, \|\nabla w_k'^{\varepsilon}\|^2_{L^2(C_i^{\varepsilon})} \leqq C\frac{\varepsilon^2}{\sigma_\varepsilon^4}. \tag{3.4.28}$$

Moreover, because $w_k'^{\varepsilon} \in [H_0^1(C_i^\varepsilon)]^2$, the Poincaré inequality in C_i^ε leads to

$$\|w_k'^{\varepsilon}\|_{L^2(C_i^\varepsilon)} \leqq C\varepsilon \|\nabla w_k'^{\varepsilon}\|_{L^2(C_i^\varepsilon)}.$$

Thus

$$\|w_k'^{\varepsilon}\|_{L^2(\Omega)} \leqq C\frac{\varepsilon^2}{\sigma_\varepsilon^2}. \tag{3.4.29}$$

Finally, as the functions $(w_k^\varepsilon, q_k^\varepsilon)$ are equal to the sums of $(w_{0k}^\varepsilon, q_{0k}^\varepsilon)$ and $(w_k'^{\varepsilon}, q_k'^{\varepsilon})$, we check that Hypotheses (H1*)–(H3*) are satisfied, by regrouping (3.4.25), (3.4.28), and (3.4.29).

In order to verify that (H4*) and (H5*) also hold, we decompose $(\nabla q_k^\varepsilon - \Delta w_k^\varepsilon)$ thus:

$$\nabla q_k^\varepsilon - \Delta w_k^\varepsilon = \mu_{0k}^\varepsilon + \mu_k'^{\varepsilon} - \gamma_k^\varepsilon,$$

with

$$\mu_{0k}^\varepsilon = \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i\right)\delta_i^\varepsilon, \qquad \mu_k'^{\varepsilon} = \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k'^{\varepsilon}}{\partial r_i} - q_k'^{\varepsilon} e_r^i\right)\delta_i^\varepsilon,$$

$$\gamma_k^\varepsilon = \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial n_i} - q_k^\varepsilon n_i\right)\delta T_i^\varepsilon,$$

where δ_i^ε and $\delta_{T_i^\varepsilon}$ are the unit masses concentrated on the sphere ∂B_i^ε and on the hole boundary ∂T_i^ε, and where n_i is the unit exterior normal to T_i^ε. Now, for any $\phi \in D(\Omega)$, any sequence $v_\varepsilon \in [H^1(\Omega)]^N$, and any function $v \in [L^2(\Omega)]^N$ such that

$$v_\varepsilon \rightharpoonup v \quad \text{in } [L^2(\Omega)]^N \text{ weakly},$$

$$\|\nabla v_\varepsilon\|_{L^2(\Omega)} \leqq \frac{C}{\sigma_\varepsilon} \quad \text{where } C \text{ does not depend on } \varepsilon, \tag{3.4.30}$$

$$v_\varepsilon = 0 \quad \text{on the holes } T_i^\varepsilon,$$

we seek the limit of $\sigma_\varepsilon^2\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi v_\varepsilon\rangle_{H^{-1},H_0^1(\Omega)}$ as ε tends to zero. First, because v is equal to 0 on the holes, we have

$$\sigma_\varepsilon^2\langle \gamma_k^\varepsilon, \phi v_\varepsilon\rangle_{H^{-1},H_0^1(\Omega)} = 0.$$

Second, arguing as in Lemma 2.3.3, we introduce the map R_ε, defined in Proposition 3.4.7, which satisfies (H6*), not in Ω_ε, but in $\Omega - \bigcup_{i=1}^{N(\varepsilon)} B_i^{a_\varepsilon}$; using it we obtain

$$\begin{aligned}\sigma_\varepsilon^2\langle \mu_k'^{\varepsilon}, \phi v_\varepsilon\rangle_{H^{-1},H_0^1(\Omega)} &= \sigma_\varepsilon^2\langle \mu_k'^{\varepsilon}, \phi R_\varepsilon v_\varepsilon\rangle_{H^{-1},H_0^1(\Omega)} \\ &= -\int_\Omega q_k'^{\varepsilon}\,\nabla\cdot(\phi R_\varepsilon v_\varepsilon) + \int_\Omega \nabla w_k'^{\varepsilon}\cdot\nabla(\phi R_\varepsilon v_\varepsilon) \to 0.\end{aligned}$$

Third, in each cell P_i^ε, we compute

$$\sigma_\varepsilon^2\left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i\right)\delta_i^\varepsilon = 2\varepsilon[-e_k + 4(e_k\cdot e_r^i)\,e_r^i]\,[1 + o(1)]\,\delta_i^\varepsilon,$$

where $o(1)$ is a sequence of real numbers (not depending on x) that tends to zero as ε does. To verify (H5*), we must prove that

$$\sigma_\varepsilon^2 \langle \mu_{0k}^\varepsilon, \phi v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \to \int_\Omega \phi \mu_k \cdot v. \tag{3.4.32}$$

By Lemma 2.3.4 we know that

$$\sigma_\varepsilon^2 \mu_{0k}^\varepsilon = \sum_{i=1}^{N(\varepsilon)} 2\varepsilon[-e_k + 4(e_k \cdot e_r^i)\, e_r^i]\ [1 + o(1)]\, \delta_i^\varepsilon \to \pi e_k \quad \text{in } [H^{-1}(\Omega)]^2 \text{ strongly.}$$

But this result is inadequate here, since the sequence v_ε satisfying (3.4.30) does not converge weakly in $[H^1(\Omega)]^2$. Nevertheless, we are still able to pass to the limit in (3.4.32) by using the full power of the proof of Lemma 2.3.4.

Let us define a P_i^ε-periodic function $z_\varepsilon \in [H^1(\Omega)]^N$ by

$$z_\varepsilon = -(r_i^2 - \varepsilon^2)\, e_k + 8 x_k r_i \left(\frac{r_i}{\varepsilon} - 1\right) e_r^i \quad \text{in each ball } B_i^\varepsilon, \quad z_\varepsilon = 0 \text{ elsewhere.}$$

Then

$$\sigma_\varepsilon^2 \mu_{0k}^\varepsilon = -\Delta z_\varepsilon + m_\varepsilon \tag{3.4.33}$$

where m_ε is a P_i^ε-periodic function such that

$$m_\varepsilon = 4\left(4\frac{r_i}{\varepsilon} - 5\right) e_k + 40\frac{r_i}{\varepsilon}(e_k \cdot e_r^i)\, e_r^i \quad \text{in each ball } B_i^\varepsilon, \quad m_\varepsilon = 0 \text{ elsewhere.}$$

An easy computation shows that

$$\|\nabla z_\varepsilon\|_{L^\infty(\Omega)} \leqq C\varepsilon, \qquad \|m_\varepsilon\|_{L^\infty(\Omega)} \leqq C, \qquad \int_{P_i^\varepsilon} m_\varepsilon = 4\pi\varepsilon^2 e_k.$$

Thus, m_ε converges to its average πe_k in $[L^\infty(\Omega)]^N$ in the weak star topology. From (3.4.33) we get

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon^2 \langle \mu_{0k}^\varepsilon, \phi v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} = \lim_{\varepsilon \to 0} \langle m_\varepsilon, \phi v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)}.$$

By applying Lemma 3.4.15 below we complete the verification of (H5*). By the way, we obtain $\mu_k = \pi e_k$, so that (H4*) also holds true. Q.E.D.

Lemma 3.4.15. *Let m_ε be a P_i^ε-periodic sequence in $L^\infty(\Omega)$ that converges to its average m in $L^\infty(\Omega)$ in the weak star topology. Then*

$$\langle m_\varepsilon, \phi_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \to \int_\Omega m\phi = m \int_\Omega \phi$$

for each sequence $\phi_\varepsilon \in H^1(\Omega)$ and for each $\phi \in L^2(\Omega)$ such that

$$\phi_\varepsilon \rightharpoonup \phi \quad \textit{in } L^2(\Omega) \textit{ weakly},$$

$$\|\nabla \phi_\varepsilon\|_{L^2[(\Omega)} \leqq \frac{C}{\sigma_\varepsilon} \quad \textit{where } C \textit{ does not depend on } \varepsilon.$$

The proof of this lemma requires only elementary arguments, and is left to the reader (see Lemma IV.2.3 in [1] if necessary).

Proof of Proposition 3.4.12 ($N \geqq 3$). As in Part I we use a decomposition of P_i^ε into smaller subdomains which differs from the one used in the two-dimensional case. We set

$$\overline{P}_i^\varepsilon = T_i^\varepsilon \cup \overline{C}_i'^\varepsilon \cup \overline{D}_i^\varepsilon \cup \overline{K}_i^\varepsilon \tag{3.4.34}$$

where $C_i'^\varepsilon$ is the open ball of radius $\varepsilon/2$ centered in P_i^ε and perforated by T_i^ε, D_i^ε is equal to B_i^ε perforated by $\overline{C}_i'^\varepsilon \cup T_i^\varepsilon$, and K_i^ε is the remainder, *i.e.*, the corners of P_i^ε (see Figure 3). As in Section 2.3, we define functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leqq k \leqq N} \in [H^1(P_i^\varepsilon)]^N \times L^2(P_i^\varepsilon)$ with $\int_{D_i^\varepsilon} q_k^\varepsilon = 0$ by

$$w_k^\varepsilon = e_k, \quad q_k^\varepsilon = 0 \quad \text{in } P_i^\varepsilon \cap \Omega$$

for each cube P_i^ε which meets $\partial\Omega$, and by

$$\begin{Bmatrix} w_k^\varepsilon = e_k \\ q_k^\varepsilon = 0 \end{Bmatrix} \text{ in } K_i^\varepsilon, \qquad \begin{Bmatrix} \nabla q_k^\varepsilon - \Delta w_k^\varepsilon = 0 \\ \nabla \cdot w_k^\varepsilon = 0 \end{Bmatrix} \text{ in } D_i^\varepsilon,$$

$$\begin{Bmatrix} w_k^\varepsilon = w_k\left(\dfrac{x}{a_\varepsilon}\right) \\ q_k^\varepsilon = \dfrac{1}{a_\varepsilon} q_k\left(\dfrac{x}{a_\varepsilon}\right) \end{Bmatrix} \text{ in } C_i'^\varepsilon, \qquad \begin{Bmatrix} w_k^\varepsilon = 0 \\ q_k^\varepsilon = 0 \end{Bmatrix} \text{ in } T_i^\varepsilon,$$

for each cube P_i^ε entirely included in Ω, where (w_k, q_k) are the solutions of the local problem (3.2.3). Then, with the help of Lemma 2.3.5 (which furnishes asymptotic expansions of w_k and q_k), we readily obtain

$$\|\nabla w_k^\varepsilon\|_{L^2(\Omega)} \leqq \frac{C}{\sigma_\varepsilon}, \quad \|q_k^\varepsilon\|_{L^2(\Omega)} \leqq \frac{C}{\sigma_\varepsilon},$$

$$\|w_k^\varepsilon - e_k\|_{L^2(\Omega)} \leqq C \begin{cases} \left(\dfrac{\varepsilon}{\sigma_\varepsilon}\right)^2 & \text{for } N = 3, \\ \left|\log \dfrac{\varepsilon}{a_\varepsilon}\right|^{1/2} \left(\dfrac{\varepsilon}{\sigma_\varepsilon}\right)^2 & \text{for } N = 4, \\ \left(\dfrac{\varepsilon}{\sigma_\varepsilon}\right)^{\frac{N}{N-2}} & \text{for } N \geqq 5. \end{cases} \tag{3.4.35}$$

Obviously Hypotheses (H1*)–(H3*) are satisfied. In order to verify that (H4*) and (H5*) also hold, we decompose $(\nabla q_k^\varepsilon - \Delta w_k^\varepsilon)$ thus:

$$\begin{aligned} \nabla q_k^\varepsilon - \Delta w_k^\varepsilon = & \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i\right) \delta_i^{\varepsilon/2} + \nabla \cdot (\chi_\varepsilon(q_k^\varepsilon \, Id - \nabla w_k^\varepsilon)) \\ & - \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial n_i} - q_k^\varepsilon n_i\right) \delta_{T_i^\varepsilon}, \end{aligned} \tag{3.4.36}$$

where $\delta_i^{\varepsilon/2}$ and $\delta_{T_i^\varepsilon}$ are the unit masses concentrated on the sphere $\partial C_i'^\varepsilon \cap \partial D_i^\varepsilon$ and on the hole boundary T_i^ε, and where χ_ε is the characteristic function of $\bigcup_{i=1}^{N(\varepsilon)} D_i^\varepsilon$ (which equals to 1 on this set, and 0 elsewhere).

Now, for any $\phi \in D(\Omega)$, and any sequence $v_\varepsilon \in [H^1(\Omega)]^N$ and function $v \in [L^2(\Omega)]^N$ satisfying (3.4.30), we seek the limit of $\sigma_\varepsilon^2 \langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi v_\varepsilon \rangle_{H^{-1},H_0^1(\Omega)}$, as ε tends to zero. First, because v_ε is equal to zero on the holes, we have

$$\sigma_\varepsilon^2 \left\langle \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial n_i} - q_k^\varepsilon n_i \right) \delta_{T_i^\varepsilon}, \phi v_\varepsilon \right\rangle_{H^{-1},H_0^1(\Omega)} = 0.$$

Second, it is easy to compute

$$\|\nabla \cdot (\chi_\varepsilon(q_k^\varepsilon \, Id - \nabla w_k^\varepsilon))\|_{H^{-1}(\Omega)}^2 \leqq \int_\Omega \chi_\varepsilon (q_k^\varepsilon)^2 + \int_\Omega \chi_\varepsilon \, |\nabla w_k^\varepsilon|^2 \leqq C \frac{\varepsilon^2}{\sigma_\varepsilon^4}.$$

Using this estimate yields

$$\sigma_\varepsilon^2 \langle \nabla \cdot (\chi_\varepsilon(q_k^\varepsilon \, Id - \nabla w_k^\varepsilon)), \phi v_\varepsilon \rangle_{H^{-1},H_0^1(\Omega)} \to 0.$$

Third, we also compute

$$\sigma_\varepsilon^2 \left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i \right) (r_i = \varepsilon/2) = \frac{2^{N-2}}{S_N} [F_k + N(F_k \cdot e_r^i) \, e_r^i] \, \varepsilon + O(a_\varepsilon),$$

where $O(a_\varepsilon)$ is a function of x. Consequently, as in the proof of Lemma 2.3.7, we have to use the Comparison Lemma 2.3.8 (due to D. CIORANESCU & F. MURAT [9]). Moreover, as for $N = 2$, we also need Lemma 3.4.12, because the sequence v_ε is not bounded in $[H^1(\Omega)]^N$. Combining these two ingredients is a little technical, although not difficult. Finally we can still pass to the limit, and from (3.4.36) we obtain

$$\begin{aligned} \lim_{\varepsilon \to 0} \sigma_\varepsilon^2 \langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi v_\varepsilon \rangle_{H^{-1},H_0^1(\Omega)} &= \sigma_\varepsilon^2 \left\langle \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i \right) \delta_i^{\varepsilon/2}, \phi v_\varepsilon \right\rangle_{H^{-1},H_0^1(\Omega)} \\ &= \int_\Omega \phi v \cdot \frac{F_k}{2^N}. \end{aligned}$$

Hypothesis (H5*) is verified with $\mu_k = \frac{F_k}{2^N}$, which is a constant vector, so that (H4*) also holds. Q.E.D.

Proof of Theorem 3.4.14. We only give a very brief sketch of the proof, which follows the pattern of Proposition 1.2.5 and Theorem 2.1.9. The same arguments give inequalities similar to (1.2.38) and (1.2.42), namely

$$\|p - p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leqq C\sigma_\varepsilon \, \|\nabla r_\varepsilon\|_{L^2(\Omega)} + C \, \|u\|_{W^{2,\infty}(\Omega)} \left[\sigma_\varepsilon + \frac{\|\sigma_\varepsilon^2 M_\varepsilon - M\|_{H^{-1}(\Omega)}}{\sigma_\varepsilon} \right], \tag{3.4.37}$$

$$\frac{\sigma_\varepsilon^2 \|\nabla r_\varepsilon\|_{L^2(\Omega)}^2}{\|u\|_{W^{2,\infty}(\Omega)}^2} \leqq C \frac{\sigma_\varepsilon \|\nabla r_\varepsilon\|_{L^2(\Omega)}}{\|u\|_{W^{2,\infty}(\Omega)}} \left[\sigma_\varepsilon + \frac{\|\sigma_\varepsilon^2 M_\varepsilon - M\|_{H^{-1}(\Omega)}}{\sigma_\varepsilon} + \|Id - W_\varepsilon\|_{L^2(\Omega)}\right]$$

$$+ \|Id - W_\varepsilon\|_{L^2(\Omega)} \left[\sigma_\varepsilon + \frac{\|\sigma_\varepsilon^2 M_\varepsilon - M\|_{H^{-1}(\Omega)}}{\sigma_\varepsilon}\right] \tag{3.4.38}$$

with the usual notations of Part I: $r_\varepsilon = \tilde{u}_\varepsilon/\sigma_\varepsilon^2 - W_\varepsilon u$, and $M_\varepsilon e_k = \mu_k^\varepsilon$. From (3.4.25) and (3.4.29) for $N = 2$, and (3.4.35) for $N \geqq 3$, we obtain

$$\|Id - W_\varepsilon\|_{L^2(\Omega)} \leqq C \frac{\varepsilon}{\sigma_\varepsilon}. \tag{3.4.39}$$

Furthermore, using Lemma 2.4.3 for $\sigma_\varepsilon^2 \mu_k^\varepsilon$ (instead of μ_k^ε) leads to

$$\|\sigma_\varepsilon^2 M_\varepsilon - M\|_{H^{-1}(\Omega)} \leqq C\varepsilon. \tag{3.4.40}$$

Using these estimates and the Poincaré inequality for r_ε, we deduce from (3.4.38) that

$$\|r_\varepsilon\|_{L^2(\Omega)} \leqq C\left(\frac{\varepsilon}{\sigma_\varepsilon} + \sigma_\varepsilon\right) \|u\|_{W^{2,\infty}(\Omega)}.$$

Again using (3.4.39) leads to the desired result for $\tilde{u}_\varepsilon/\sigma_\varepsilon^2 - u$, which we substitute into (3.4.37) to complete the argument. Q.E.D.

4. Periodically Distributed Holes on a Surface

This fourth section is devoted to the verification of Hypotheses (H1)–(H6) (introduced in the first section) for a domain containing many tiny holes that are periodically distributed on a surface (repesented mathematically by a smooth $(N-1)$-dimensional manifold). For the sake of simplicity we assume that this surface is an hyperplane. More precisely, let Ω be a bounded connected open set in $\mathbb{R}^N$ ($N \geqq 2$), with Lipschitz boundary $\partial\Omega$, Ω being locally located on one side of its boundary. We assume that Ω has a non-empty intersection with the hyperplane $H = \{x \in \mathbb{R}^N / x_N = 0\}$. We define the open set H_ε to be a slice of Ω of thickness 2ε near H by $H_\varepsilon = \{x \in \Omega / |x_N| < \varepsilon\}$. The set H_ε is covered with a regular mesh of size 2ε, each cell being a cube P_i^ε, identical to $(-\varepsilon, +\varepsilon)^N$. At the center of each cube P_i^ε included in H_ε there is a hole T_i^ε, each of which is similar to the same closed set T rescaled to size a_ε. We assume that T is strictly included in the unit open ball B_1 and that $B_1 - T$ is a connected open set, locally located on one side of its Lipschitz boundary. Moreover, we assume that the size of the holes a_ε is critical for the surface distribution, *i.e.*,

$$\lim_{\varepsilon \to 0} \frac{a_\varepsilon}{\varepsilon^{\frac{N-1}{N-2}}} = C_0 \quad \text{for } N \geqq 3, \qquad \lim_{\varepsilon \to 0} -\varepsilon \log(a_\varepsilon) = C_0 \quad \text{for } N = 2 \tag{4.1.1}$$

where C_0 is a strictly positive constant ($0 < C_0 < +\infty$). Assumption (4.1.1) gives a unique and explicit scaling of the holes size for $N \geqq 3$, but not for the two-dimensional case. Actually, when $N = 2$, many different sizes of the holes satisfy (4.1.1) with the same constant C_0 (*e.g.*, $a_\varepsilon = \varepsilon^p \exp(-C_0/\varepsilon)$ for any $p \in \mathbb{R}$). In any case, assumption (4.1.1) is enough for the sequel, so we do not make more precise the scaling of the holes in two dimensions.

Elementary geometrical considerations give the number of holes $N(\varepsilon) = \frac{|H \cap \Omega|}{(2\varepsilon)^{N-1}}[1 + o(1)]$, where $|H \cap \Omega|$ is a measure in $\mathbb{R}^{N-1}$. Compared with the case of a volume distribution, the critical size is larger, but the number of holes is smaller.

In each cell P_i^ε we define B_i^ε as the open ball of radius ε included in P_i^ε. We also define a control volume $C_i^\varepsilon = B_i^\varepsilon - T_i^\varepsilon$ around each hole (see Figure 2). The open set $\Omega_\varepsilon = \Omega - \bigcup_{i=1}^{N(\varepsilon)} T_i^\varepsilon$ is obtained by removing from Ω all the holes $(T_i^\varepsilon)_{1 \le i \le N(\varepsilon)}$, and because we perforate only the cells entirely included in Ω, we are sure that no hole meets the boundary $\partial\Omega$. Then Ω_ε is also a bounded connected open set, locally located on one side of its Lipschitz boundary $\partial\Omega_\varepsilon$ (see Figure 4). Note that the centers of the holes are located on the hyperplane H although the holes T_i^ε are closed subsets of H_ε, not necessarily included in H.

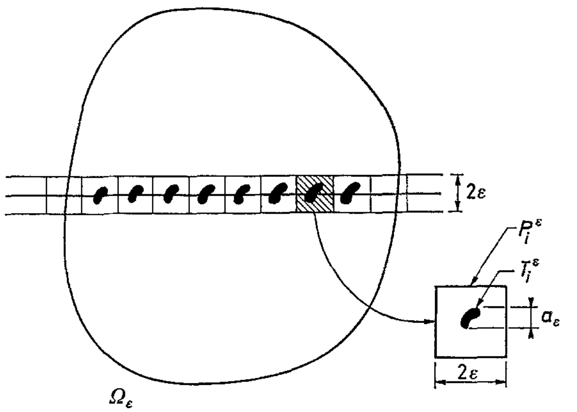

Fig. 4

As usual we consider the Stokes problem in Ω_ε:

$$\text{Find } (u_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)/\mathbb{R}] \text{ such that}$$
$$\begin{aligned} \nabla p_\varepsilon - \Delta u_\varepsilon &= f \quad \text{in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon. \end{aligned} \tag{4.1.2}$$

Because the distribution of holes is not uniform in the domain Ω, we expect a singular behavior of the solutions $(u_\varepsilon, p_\varepsilon)$ near the hyperplane H as ε tends to zero. In other words, in each part of Ω away from H, the sequence of solutions should converge "nicely", but in the vicinity of H the convergence should get worse. Actually, it turns out that because of this effect, the overall convergence of the pressure is weaker than previously. To reflect this fact, Hypothesis (H6), and consequently (H3), need slight changes. Thus, in the first subsection we give the modifications of the abstract framework, together with the main results. The second subsection is devoted to the verification of Hypotheses (H1) and (H6).

4.1. Modified abstract framework and main results

We assume that the holes T_i^ε are such that there exist functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leqq k \leqq N}$ and a linear map R_ε such that

(H1) $w_k^\varepsilon \in [H^1(\Omega)]^N, \quad q_k^\varepsilon \in L^2(\Omega)$,

(H2) $\nabla \cdot w_k^\varepsilon = 0$ in Ω and $w_k^\varepsilon = 0$ on the holes T_i^ε,

(H3) $w_k^\varepsilon \rightharpoonup e_k$ in $[H^1(\Omega)]^N$ weakly, $q_k^\varepsilon \rightharpoonup 0$ in $L^2(\Omega)/\mathbb{R}$ weakly,

$w_k^\varepsilon \to e_k$ in $[L^q(\Omega)]^N$ strongly, for some $q > N$,

(H4) $\mu_k \in [W^{-1,\infty}(\Omega)]^N$,

(H5) $\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \to \langle \mu_k, \phi v \rangle_{H^{-1}, H_0^1(\Omega)}$

for each sequence v_ε, for each v such that

$$v_\varepsilon \rightharpoonup v \text{ in } [H^1(\Omega)]^N \text{ weakly}, \quad v_\varepsilon = 0 \text{ on the holes } T_i^\varepsilon$$

and for each $\phi \in D(\Omega)$,

$$\text{(H6)} \quad \begin{cases} R_\varepsilon \in L([H_0^1(\Omega) \cap L^\infty(\Omega)]^N; [H_0^1(\Omega_\varepsilon)]^N), \\ \text{If } u \in [H_0^1(\Omega_\varepsilon)]^N, \text{ then } R_\varepsilon \tilde{u} = u \text{ in } \Omega_\varepsilon, \\ \text{If } \nabla \cdot u = 0 \text{ in } \Omega, \text{ then } \nabla \cdot (R_\varepsilon u) = 0 \text{ in } \Omega_\varepsilon, \\ \|R_\varepsilon u\|_{H_0^1(\Omega_\varepsilon)} \leqq C[\|u\|_{H_0^1(\Omega)} + \|u\|]_{L^\infty(\Omega)} \text{ and } C \text{ does not depend on } \varepsilon. \end{cases}$$

Remark 4.1.1. This Hypothesis (H6) is somewhat weaker than that for a volume distribution of the holes, because R_ε operates in $[H_0^1(\Omega) \cap L^\infty(\Omega)]^N$, which contains smoother functions than $[H_0^1(\Omega)]^N$. Hypothesis (H3) is stronger than that for a volume distribution, because the functions $(w_k^\varepsilon)_{1 \leqq k \leqq N}$ should converge strongly to $(e_k)_{1 \leqq k \leqq N}$ in $[L^q(\Omega)]^N$ for some $q > N$. Combining these two modifications permits us still to prove the convergence of the homogenization process, with some slight changes in the proof of the convergence of the pressure, due to the weaker form of (H6). Roughly speaking, all the results of the abstract framework (introduced in Part I) hold, provided we change the $L^2(\Omega)$-estimate of the pressure by a $L^{q'}(\Omega)$-estimate, with $q' < N/(N-1)$ (see Chapter III in [1] for details).

Proposition 4.1.2. *Let* $q > N$ *and* $1 < q' < \dfrac{N}{N-1}$ *be such that* $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. *If there exists a linear map* R_ε *satisfying* (H6), *then the operator* P_ε *defined by*

$$\langle \nabla [P_\varepsilon(q_\varepsilon)], u \rangle_{W^{-1,q'}, W_0^{1,q}(\Omega)} = \langle \nabla q_\varepsilon, R_\varepsilon u \rangle_{H^{-1}, H_0^1(\Omega_\varepsilon)} \quad \textit{for each } u \in [W_0^{1,q}(\Omega)]^N \tag{4.1.3}$$

is a linear continuous extension map from $L^2(\Omega_\varepsilon)/\mathbb{R}$ *into* $L^{q'}(\Omega)/\mathbb{R}$ *such that*

(i) $P_\varepsilon(q_\varepsilon) = q_\varepsilon \quad \text{in } L^2(\Omega_\varepsilon)/\mathbb{R}$,

(ii) $\|P_\varepsilon(q_\varepsilon)\|_{L^{q'}(\Omega)/\mathbb{R}} \leqq C \, \|q_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}}$,

(iii) $\|\nabla[P_\varepsilon(q_\varepsilon)]\|_{W^{-1,q'}(\Omega)} \leqq C \, \|\nabla q_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)}$

for each $q_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$ *where* C *is a constant which does not depend on* q_ε *or* ε.

Proof. This proof is similar to that of Proposition 1.1.4; we only point out that $W_0^{1,q}(\Omega)$ is continuously embedded in $H_0^1(\Omega) \cap L^\infty(\Omega)$ because $q > N$. Thus formula (4.1.3) is meaningful since u belongs to the domain of R_ε. Note that the extension operator P_ε, defined here, is weaker than that in Proposition 1.1.4, because $1 < q' < 2$ implies that $L^2(\Omega)$ is strictly included in $L^{q'}(\Omega)$. Q.E.D.

Theorem 4.1.3. *Let Hypotheses* (H1)–(H6) *be satisfied, and let* $(u_\varepsilon, p_\varepsilon)$ *be the unique solution of the Stokes equations* (4.1.2). *Let* $\tilde{u}_\varepsilon$ *be the extension of the velocity* u_ε *by* 0 *in the holes* T_i^ε. *Let* P_ε *be the extension operator defined in Proposition 4.1.2. Then, for any value of* q' *such that* $1 < q' < N/(N-1)$, $(\tilde{u}_\varepsilon, P_\varepsilon(u_\varepsilon))$ *converges weakly to* (u, p) *in* $[H_0^1(\Omega)]^N \times [L^{q'}(\Omega)/\mathbb{R}]$, *where* (u, p) *is the unique solution of the following Brinkman law:*

$$\begin{aligned} &\text{Find } (u,p) \in [H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \text{ such that} \\ &\nabla p - \Delta u + Mu = f \quad \text{in } \Omega, \\ &\nabla \cdot u = 0 \quad \text{in } \Omega \end{aligned} \tag{4.1.4}$$

where M *is the matrix defined by its columns* $Me_k = \mu_k$.

Proof. This proof is similar to that of Theorem 1.1.8: The only change comes from the weaker estimate on the pressure. Indeed, Proposition 4.1.2 yields

$$P_\varepsilon(p_\varepsilon) \rightharpoonup p \quad \text{in } L^{q'}(\Omega)/\mathbb{R} \text{ weakly, with } 1 < q' < \frac{N}{N-1}.$$

In order to pass to the limit in the variational formulation (1.1.15) under Hypotheses (H1)–(H6), we point out that (H3) implies that w_k^ε converges strongly to e_k in $[L^q(\Omega)]^N$. Because we can choose q and q' such that $\frac{1}{q} + \frac{1}{q'} = 1$, we have

$$\int_\Omega P_\varepsilon(p_\varepsilon) \, w_k^\varepsilon \cdot \nabla\phi \rightarrow \int_\Omega p e_k \cdot \nabla\phi.$$

Therefore the proof can proceed exactly as for Theorem 1.1.8. Q.E.D.

Now we give some results which make explicit the extension of the pressure and the matrix M. Their proofs may be found in Section 4.2.

Proposition 4.1.4. *Let the hole size satisfy* (4.1.1). *Then there exists a linear map* R_ε *that satisfies Hypothesis* (H6), *such that the associated extension of the pressure* *is a constant inside each hole*:

$$P_\varepsilon(p_\varepsilon) = p_\varepsilon \text{ in } \Omega_\varepsilon \quad \text{and} \quad P_\varepsilon(p_\varepsilon) = \frac{1}{|C_i^\varepsilon|} \int_{C_i^\varepsilon} p_\varepsilon \text{ in each hole } T_i^\varepsilon, \tag{4.1.5}$$

where C_i^ε is a control volume defined as the part outside T_i^ε of the ball of radius ε and same center as T_i^ε.

Proposition 4.1.5. *For $N = 2$, let the hole size satisfy* (4.1.1), *i.e.*,

$$\lim_{\varepsilon \to 0} - \varepsilon \log(a_\varepsilon) = C_0 > 0.$$

Then there exist functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leqq k \leqq 2}$ that satisfy Hypotheses (H1)–(H5). *Moreover,*

$$M = \frac{2\pi}{C_0} Id\, \delta_H, \tag{4.1.6}$$

whatever the shape and size of the model hole, where δ_H denotes the measure defined as the unit mass concentrated on the hyperplane H, i.e.,

$$\langle \delta_H, \phi \rangle_{D', D(\mathbb{R}^N)} = \int_H \phi(s)\, ds \quad \text{for any } \phi \in D(\mathbb{R}^N).$$

Before stating an equivalent proposition for $N \geqq 3$, we recall that $(w_k, q_k)_{1 \leqq k \leqq N}$ are the solutions of the so-called local problem (3.2.3).

Proposition 4.1.6. *For $N \geqq 3$, let the hole size satisfy* (4.1.1):

$$\lim_{\varepsilon \to 0} \frac{a_\varepsilon}{\varepsilon^{\frac{N-1}{N-2}}} = C_0 > 0.$$

Then there are functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leqq k \leqq N}$, constructed from solution (w_k, q_k) of the local problem, that satisfy Hypotheses (H1)–(H5). *Moreover, the matrix M is given by*

$${}^t e_i M e_k = \frac{C_0^{N-2}}{2^{N-1}} \left(\int_{\mathbb{R}^N - T} \nabla w_k : \nabla w_i \right) \delta_H \tag{4.1.7}$$

where δ_H denotes the measure defined as the unit mass concentrated on the hyperplane H.

Remark 4.1.7. Up to a factor of 2, the value of the matrix M is the same for a volume or a surface distribution of the holes, but in the latter case we emphasize that the matrix M is concentrated on the hyperplane H, *i.e.*, $M = 0$ elsewhere in $\Omega - H$. When $N = 2$ or 3, Theorem 4.1.3 and Propositions 4.1.4–4.1.6 can be generalized to the case of the Navier-Stokes equations, as previously done for Theorem 3.2.1 (the nonlinear term is still a compact perturbation, see Remark 3.2.3), with the same functions satisfying Hypotheses (H1)–(H5), and therefore, with the same matrix M as for the Stokes equations.

Theorem 4.1.8. *Let Hypotheses* (H1)–(H6) *hold and let the solution* (u, p) *of the homogenized system* (4.1.4) *be smooth, say*

$$u \in [W^{1,N+\eta}(\Omega)]^N \quad \textit{for } \eta > 0. \tag{4.1.8}$$

Then the solution $(u_\varepsilon, p_\varepsilon)$ *of the Stokes system* (4.1.2) *satisfies*

$$\begin{aligned} &(\tilde{u}_\varepsilon - W_\varepsilon u) \to 0 \quad \textit{in } [H_0^1(\Omega)]^N \textit{ strongly,} \\ &P_\varepsilon(p_\varepsilon - p - u \cdot Q_\varepsilon) \to 0 \quad \textit{in } L^{q'}(\Omega)/\mathbb{R} \textit{ strongly, with } 1 < q' < \frac{N}{N-1}, \end{aligned} \tag{4.1.9}$$

where W_ε *is the matrix defined by its columns* $W_\varepsilon e_k = w_k^\varepsilon$, *and* Q_k *is the vector defined by its entries* $Q_\varepsilon \cdot e_k = q_k^\varepsilon$.

The proof is exactly the same as that of Theorems 1.2.3 and 1.2.4, provided we take into account the weaker estimate of the pressure. Note that (4.1.9) holds in the entire domain Ω. It turns out that on each part of Ω, below and above H (let us call them Ω^+ and Ω^-), the convergence of $(u_\varepsilon, p_\varepsilon)$ to its limit (u, p) is strong in $[H^1(\Omega^{+/-})]^N \times L^2(\Omega^{+/-})/\mathbb{R}$. This means that in Ω^+ and Ω^- the correctors are equal to zero (*i.e.*, $W_\varepsilon = Id$ and $Q_\varepsilon = 0$), and that the *weak* convergence of the solutions is concentrated on H, as is the matrix M.

Theorem 4.1.9. *Let the solution* (u, p) *of Brinkman's law* (4.1.4) *be smooth, say* $u \in [W^{1,\infty}(\Omega)]^N$. *Let* q' *be a real number such that* $1 < q' < \dfrac{N}{N-1}$. *Then there exists a positive constant* C *that depends only on* Ω, T, *and* q' *such that*

$$\begin{aligned} &\|\tilde{u}_\varepsilon - W_\varepsilon u\|_{H_0^1(\Omega)} \leqq C\varepsilon^{1/2} \|u\|_{W^{1,\infty}(\Omega)}, \\ &\|p_\varepsilon - p - u \cdot Q_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)/\mathbb{R}} \leqq C\varepsilon^{1/2} \|u\|_{W^{1,\infty}(\Omega)}. \end{aligned} \tag{4.1.10}$$

Remark 4.1.10. It is worth noticing that the error estimates (4.1.10) are weaker than those (2.1.9) obtained for a volume distribution of the holes. This is partly due to the weaker assumption on the smoothness of the homogenized solution u. Actually we can prove with standard regularity theorems that u belongs to $[W^{1,\infty}(\Omega)]^N$ if the boundary $\partial\Omega$ is smooth enough. But, because the term Mu in Brinkman's law is a measure concentrated in the hyperplane H (see (4.1.6) and (4.1.7)), the first derivatives of u are discontinuous across H if the force f is smooth. Therefore u cannot be smoother, and the present estimates (4.1.10), although weaker than (2.1.9), are optimal.

Remark 4.1.11. We assume that the holes T_i^ε are identical, but this condition can be weakened, as previously observed in Remark 2.1.10. In two dimensions, the holes may be entirely different from one another; provided that they have the required size, we still have the same results (in particular $M = 2\pi/C_0 \, Id \, \delta_H$). In other dimensions, the hole shape may vary smoothly without interfering with the convergence of the homogenization process (of course the matrix M is no longer constant in H).

4.2. Verification of Hypotheses (H1)—(H6)

This subsection is devoted to the proofs of the results stated in the previous subsection. Basically, we proceed exactly as in the case of a volume distribution of the holes, giving details only for the differences between the two cases.

Proof of Proposition 4.1.4. Let $u \in [H_0^1(\Omega)]^N$. For each cube P_i^ε entirely included in H_ε, we know (*cf.* Lemma 2.2.1) that the following Stokes problem has a unique solution which depends linearly on u.

Find $(v_i^\varepsilon, q_i^\varepsilon) \in [H^1(C_i^\varepsilon)]^N \times [L^2(C_i^\varepsilon)/\mathbb{R}]$ such that

$$\nabla q_i^\varepsilon - \Delta v_i^\varepsilon = -\Delta u \quad \text{in } C_i^\varepsilon,$$

$$\nabla \cdot v_i^\varepsilon = \nabla \cdot u + \frac{1}{|C_i^\varepsilon|} \int_{T_i^\varepsilon} \nabla \cdot u \quad \text{in } C_i^\varepsilon,$$

$$v_i^\varepsilon = u \quad \text{on } \partial C_i^\varepsilon - \partial T_i^\varepsilon,$$

$$v_i^\varepsilon = 0 \quad \text{on } \partial T_i^\varepsilon.$$

Then we define $R_\varepsilon u$ by

$$R_\varepsilon u = u \text{ in } K_i^\varepsilon = P_i^\varepsilon - B_i^\varepsilon, \quad R_\varepsilon u = v_i^\varepsilon \text{ in } C_i^\varepsilon, \quad R_\varepsilon u = 0 \text{ in } T_i^\varepsilon.$$

for each cube P_i^ε entirely included in H_ε,

$$R_\varepsilon u = u \quad \text{elsewhere in } \Omega - \bigcup_{i=1}^{N(\varepsilon)} P_i^\varepsilon.$$

As in the proof of Proposition 2.2.2 we easily check that Hypothesis (H6) holds for such an operator R_ε. The only difference comes from the estimate of $R_\varepsilon u$. Recall estimate (3.4.23):

$$\|\nabla v_i^\varepsilon\|_{L^2(C_i^\varepsilon)}^2 \leqq C\left[\|\nabla u\|_{L^2(C_i^\varepsilon \cup T_i^\varepsilon)}^2 + \frac{K_\eta^2}{\varepsilon^2}\|u\|_{L^2(C_i^\varepsilon \cup T_i^\varepsilon)}^2\right]. \tag{4.2.1}$$

Using the definition of K_η and recalling that $\eta = a_\varepsilon/\varepsilon$, where the size a_ε is given by (4.1.1), we get

$$\frac{K_\eta^2}{\varepsilon^2} \leqq \frac{C}{\varepsilon}.$$

Then, summing the estimates (4.2.1) for all the cubes P_i^ε, and recalling that $R_\varepsilon u = u$ in $\Omega - H_\varepsilon$, we obtain

$$\|\nabla(R_\varepsilon u)\|_{L^2(\Omega_\varepsilon)}^2 \leqq C\left[\|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon}\|u\|_{L^2(H_\varepsilon)}^2\right].$$

But $\|u\|_{L^2(H_\varepsilon)}^2 \leqq C\varepsilon \|u\|_{L^\infty(\Omega)}^2$. Thus we obtain the desired result

$$\|\nabla(R_\varepsilon u)\|_{L^2(\Omega_\varepsilon)} \leqq C[\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^\infty(\Omega)}].$$

For the proof of (4.1.5), we refer to Proposition 2.1.1. Q.E.D.

Proof of Proposition 4.1.5. In order to verify Hypotheses (H1)–(H5), we construct functions $(w_k^\varepsilon, q_k^\varepsilon)_{1\leqq k\leqq 2}$ exactly as we did in the case of a volume distribution. See (3.4.22) for the definitions of C_i^ε and K_i^ε. For $k = 1, 2$ we define functions $(w_k^\varepsilon, q_k^\varepsilon) \in [H^1(P_i^\varepsilon)]^2 \times L^2(P_i^\varepsilon)$, with $\int_{P_i^\varepsilon} q_k^\varepsilon = 0$, by

$$\begin{Bmatrix} w_k^\varepsilon = e_k \\ q_k^\varepsilon = 0 \end{Bmatrix} \text{ in } K_i^\varepsilon, \quad \begin{Bmatrix} \nabla q_k^\varepsilon - \Delta w_k^\varepsilon = 0 \\ \nabla \cdot w_k^\varepsilon = 0 \end{Bmatrix} \text{ in } C_i^\varepsilon, \quad \begin{Bmatrix} w_k^\varepsilon = 0 \\ q_k^\varepsilon = 0 \end{Bmatrix} \text{ in } T_i^\varepsilon, \tag{4.2.2}$$

for each cube P_i^ε entirely included in H_ε, and by

$$\begin{Bmatrix} w_k^\varepsilon = e_k \\ q_k^\varepsilon = 0 \end{Bmatrix} \quad \text{elsewhere in } \Omega - \bigcup_{i=1}^{N(\varepsilon)} P_i^\varepsilon.$$

We compare these functions with the same ones obtained when the model hole T is the unit ball. As $T \subset B_1$ let us define for each cube P_i^ε a ball $B_i^{a_\varepsilon}$ of radius a_ε that strictly contains the hole T_i^ε (see Figure 2 in Part I). Now, we define functions $(w_{0k}^\varepsilon, q_{0k}^\varepsilon)_{1\leqq k\leqq 2}$ by (4.2.2) in which T_i^ε is replaced by $B_i^{a_\varepsilon}$. Denoting by r_i and e_r^i the radial coordinate and unit vector in each $C_i^\varepsilon - B_i^{a_\varepsilon}$, we can compute $(w_{0k}^\varepsilon, q_{0k}^\varepsilon)_{1\leqq k\leqq 2}$:

$$w_{0k}^\varepsilon = x_k r_i f(r_i)\, e_r^i + g(r_i)\, e_k, \qquad q_{0k}^\varepsilon = x_k h(r_i) \qquad \text{for } r_i \in [a_\varepsilon, ; \varepsilon],$$

with

$$f(r_i) = \frac{1}{r_i^2}\left(A + \frac{B}{r_i^2}\right) + C, \quad g(r_i) = -A \log r_i - \frac{B}{2r_i^2} - \tfrac{3}{2} C r_i^2 + D,$$

$$h(r_i) = \frac{2A}{r_i^2} - 4C,$$

$$A = -\frac{\varepsilon}{C_0}[1 + o(1)], \qquad B = \frac{\varepsilon}{C_0} e^{-\frac{2C_0}{\varepsilon}} [1 + o(1)],$$

$$C = \frac{1}{\varepsilon C_0}[1 + o(1)], \qquad D = 1 - \frac{\varepsilon \log \varepsilon}{C_0}[1 + o(1)].$$

Taking into account the smaller number of holes $N(\varepsilon) = \dfrac{|H \cap \Omega|}{(2\varepsilon)^{N-1}}[1 + o(1)]$, we carry out a computation similar to that which gives (3.4.25) to obtain

$$\|q_{0k}^\varepsilon\|_{L^2(\Omega)} \leqq C, \qquad \|\nabla w_{0k}^\varepsilon\|_{L^2(\Omega)} \leqq C,$$

$$\|w_{0k}^\varepsilon - e_k\|_{L^q(\Omega)} \leqq C \varepsilon^{1/q}\, \varepsilon\, |\log \varepsilon| \qquad \text{for } 1 \leqq q < +\infty, \tag{4.2.3}$$

$$\left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i\right) \delta_i^{a_\varepsilon} = \frac{2\varepsilon}{C_0 a_\varepsilon}[1 + o(1)]\, e_k\, \delta_i^{a_\varepsilon}$$

where $\delta_i^{a_\varepsilon}$ is the measure defined as the unit mass concentrated on the sphere $\partial B_i^{a_\varepsilon}$. Then we define the "difference" functions $(w_k'^\varepsilon, q_k'^\varepsilon)_{1\leqq k\leqq 2}$ by

$$w_k'^\varepsilon = w_k^\varepsilon - w_{0k}^\varepsilon \in [H_0^1(\Omega)]^2, \qquad q_k'^\varepsilon = q_k^\varepsilon - q_{0k}^\varepsilon \in L^2(\Omega)$$

which satisfy

$$\left\{\begin{array}{l}\nabla q_k'^{\varepsilon} - \Delta w_k'^{\varepsilon} = \left(\dfrac{\partial w_{0k}^{\varepsilon}}{\partial r_i} - q_{0k}^{\varepsilon} e_r^i\right) \delta_i^{a_\varepsilon} \\ \nabla \cdot w_k'^{\varepsilon} = 0\end{array}\right\} \quad \text{in each control volume } C_i^{\varepsilon}, \tag{4.2.4}$$

$$\left\{\begin{array}{l} w_k'^{\varepsilon} = 0 \\ q_k'^{\varepsilon} = 0 \end{array}\right\} \quad \text{elsewhere in } \Omega - \bigcup_{i=1}^{N(\varepsilon)} C_i^{\varepsilon}.$$

From (4.2.3) and (4.2.4), as in Lemma 2.3.1, we obtain

$$\|q_k'^{\varepsilon}\|_{L^2(\Omega)} \leqq C\varepsilon, \quad \|\nabla w_k'^{\varepsilon}\|_{L^2(\Omega)} \leqq C\varepsilon, \quad \|w_k'^{\varepsilon}\|_{L^q(\Omega)} \leqq C\varepsilon \quad \text{for } 1 \leqq q < +\infty. \tag{4.2.5}$$

Regrouping (4.2.3) and (4.2.5) we check that Hypotheses (H1)–(H3) are satisfied by the functions $(w_k^\varepsilon, q_k^\varepsilon)_{1\leqq k\leqq 2}$ defined in (4.2.2).

In order to verify (H4) and (H5), we decompose $(\nabla q_k^\varepsilon - \Delta w_k^\varepsilon)$ thus:

$$\nabla q_k^\varepsilon - \Delta w_k^\varepsilon = \mu_{0k}^\varepsilon + \mu_k'^{\varepsilon} - \gamma_k^\varepsilon,$$

with

$$\mu_{0k}^\varepsilon = \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i\right) \delta_i^\varepsilon, \qquad \mu_k'^{\varepsilon} = \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k'^{\varepsilon}}{\partial r_i} - q_k'^{\varepsilon} e_r^i\right) \delta_i^\varepsilon,$$

$$\gamma_k^\varepsilon = \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial n_i} - q_k^\varepsilon n_i\right) \delta T_i^\varepsilon,$$

where δ_i^ε and $\delta_{T_i^\varepsilon}$ are the unit masses concentrated on the sphere ∂B_i^ε and on the hole boundary ∂T_i^ε, and where n_i is the unit exterior normal to T_i^ε. It is easy to see that $\gamma_k^\varepsilon \equiv 0$ in $[H^{-1}(\Omega_\varepsilon)]^2$, and that $\mu_k'^{\varepsilon}$ converges strongly to 0 in $[H^{-1}(\Omega)]^2$. On the other hand, we have

$$\left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i\right) \delta_i^\varepsilon = \frac{2}{C_0}[-e_k + 4(e_k \cdot e_r^i)\, e_r^i]\,[1 + o(1)]\, \delta_i^\varepsilon.$$

Then arguing as in Lemma 2.3.3 and using Lemma 4.2.1 below, we prove that μ_{0k}^ε converges strongly to $\mu_k = \dfrac{2\pi}{C_0} e_k\, \delta_H$ in $[H^{-1}(\Omega)]^2$. Finally, as is well known, the measure δ_H belongs to $W^{-1,\infty}(\Omega)$, so that Hypotheses (H4) and (H5) hold.

Q.E.D.

Lemma 4.2.1. *Let d be a fixed real number in $(0; 1]$. Let $\delta_i^{d\varepsilon}$ be the unit mass concentrated on the sphere $\partial B_i^{d\varepsilon}$. Let S_N denote the area of the unit sphere in $\mathbb{R}^N$. (Recall that the centers of the cubes P_i^ε are periodically distributed only on the hyperplane H.) For $N \geqq 2$ the following convergences hold:*

$$\sum_{i=1}^{N(\varepsilon)} \delta_i^{d\varepsilon} \to \frac{S_N\, d^{N-1}}{2^{N-1}}\, \delta_H \qquad \textit{in } H^{-1}(\Omega) \textit{ strongly},$$

$$\sum_{i=1}^{N(\varepsilon)} (e_k \cdot e_r^i)\, e_r^i\, \delta_i^{d\varepsilon} \to \frac{S_N\, d^{N-1}}{N\, 2^{N-1}}\, e_k\, \delta_H \qquad \textit{in } [H^{-1}(\Omega)]^N \textit{ strongly}.$$

The proof of Lemma 4.2.1 is very similar to that of Lemma 2.3.4 and is left to the reader (see [1], if necessary).

Proof of Proposition 4.1.6. As in the case of a volume distribution we use the decomposition (3.4.34) of each cube P_i^ε included in H_ε, and we define functions $(w_k^\varepsilon, q_k^\varepsilon)_{1\leq k\leq N} \in [H^1(P_i^\varepsilon)]^N \times L^2(P_i^\varepsilon)$ with $\int_{D_i^\varepsilon} q_k^\varepsilon = 0$ by

$$\begin{Bmatrix} w_k^\varepsilon = e_k \\ q_k^\varepsilon = 0 \end{Bmatrix} \text{ in } K_i^\varepsilon, \qquad \begin{Bmatrix} \nabla q_k^\varepsilon - \Delta w_k^\varepsilon = 0 \\ \nabla \cdot w_k^\varepsilon = 0 \end{Bmatrix} \text{ in } D_i^\varepsilon,$$

$$\begin{Bmatrix} w_k^\varepsilon = w_k\left(\dfrac{x}{a_\varepsilon}\right) \\ q_k^\varepsilon = \dfrac{1}{a_\varepsilon} q_k\left(\dfrac{x}{a_\varepsilon}\right) \end{Bmatrix} \text{ in } C_i'^\varepsilon, \qquad \begin{Bmatrix} w_k^\varepsilon = 0 \\ q_k^\varepsilon = 0 \end{Bmatrix} \text{ in } T_i^\varepsilon \tag{4.2.6}$$

for each cube P_i^ε entirely included in H_ε,

$$\begin{Bmatrix} w_k^\varepsilon = e_k \\ q_k^\varepsilon = 0 \end{Bmatrix} \text{ elsewhere in } \Omega - \bigcup_{i=1}^{N(\varepsilon)} P_i^\varepsilon.$$

where (w_k, q_k) are the solutions of the local Stokes problem (3.2.3). Then, with the help of Lemma 2.3.5 (which furnishes asymptotic expansions of w_k and q_k), we readily obtain

$$\begin{gathered} \|\nabla w_k^\varepsilon\|^2_{L^2(C_i'^\varepsilon)} \leqq a_\varepsilon^{N-2} \|\nabla w_k\|^2_{L^2(\mathbb{R}^N - T)} \leqq C\varepsilon^{N-1}, \\ \|q_k^\varepsilon\|^2_{L^2(C_i'^\varepsilon)} \leqq a_\varepsilon^{N-2} \|q_k\|^2_{L^2(\mathbb{R}^N - T)} \leqq C\varepsilon^{N-1}, \\ \|w_k^\varepsilon - e_k\|^q_{L^q(C_i^\varepsilon)} \leqq C\varepsilon^N \left(\frac{a_\varepsilon}{\varepsilon}\right)^{q(N-2)} \leqq C\varepsilon^{N+q} \quad \text{for } q > \frac{N}{N-2}, \\ w_k^\varepsilon = O(\varepsilon) \quad \text{and} \quad \nabla w_k^\varepsilon = O(1) \quad \text{on } \partial C_i'^\varepsilon \cap \partial D_i^\varepsilon. \end{gathered} \tag{4.2.7}$$

Then

$$\begin{gathered} \|\nabla w_k^\varepsilon\|_{L^2(\Omega)} \leqq C, \quad \|q_k^\varepsilon\|_{L^2(\Omega)} \leqq C, \\ \|w_k^\varepsilon - e_k\|_{L^q(\Omega)} \leqq C\varepsilon^{\frac{2(N-1)}{q(N-2)}} \quad \text{for } q > \frac{N}{N-2}. \end{gathered} \tag{4.2.8}$$

Obviously Hypotheses (H1)–(H3) are satisfied, and for the remaining (H4) and (H5) we decompose $(\nabla q_k^\varepsilon - \Delta w_k^\varepsilon)$ by

$$\begin{aligned} \nabla q_k^\varepsilon - \Delta w_k^\varepsilon = & \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i\right) \delta_i^{\varepsilon/2} + \nabla \cdot (\chi_\varepsilon(q_k^\varepsilon \, Id - \nabla w_k^\varepsilon)) \\ & - \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial n_i} - q_k^\varepsilon n_i\right) \delta_{T_i^\varepsilon}, \end{aligned} \tag{4.2.9}$$

where $\delta_i^{\varepsilon/2}$ and $\delta_{T_i^\varepsilon}$ are the unit masses concentrated on the sphere $\partial C_i'^\varepsilon \cap \partial D_i^\varepsilon$ and on the hole boundary T_i^ε, and where χ_ε is the characteristic function of

$\bigcup\limits_{i=1}^{N(\varepsilon)} D_i^\varepsilon$. It is easy to see that $\gamma_k^\varepsilon \equiv 0$ in $[H^{-1}(\Omega_\varepsilon)]^N$ and that $\nabla \cdot (\chi_\varepsilon(q_k^\varepsilon\, Id - \nabla w_k^\varepsilon))$ converges strongly to 0 in $[H^{-1}(\Omega)]^N$. On the other hand,

$$\left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i\right)\delta_i^{\varepsilon/2} = \frac{2^{N-2}C_0^{N-2}}{S_N}\,[F_k + N(F_k \cdot e_r^i)\, e_r^i] + O\left(\varepsilon^{\frac{1}{N-2}}\right), \quad (4.2.10)$$

where $O(\varepsilon^{1/(N-2)})$ is a function of x. Consequently, as in Lemma 2.3.7, we have to use the Comparison Lemma 2.3.8 (of D. CIORANESCU & F. MURAT [9]). Nevertheless, with the help of Lemma 4.2.1, we deduce from (4.2.10) that

$$\sum_{i=1}^{N(\varepsilon)}\left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i\right)\delta_i^{\varepsilon/2} \to \mu_k = \frac{C_0^{N-2}}{2^{N-1}}\,F_k\,\delta_H \quad \text{in } [H^{-1}(\Omega)]^N \text{ strongly.}$$

Thus Hypothesis (H5) holds. So does (H4), because the measure δ_H belongs to $W^{-1,\infty}(\Omega)$. Q.E.D.

Proof of Theorem 4.1.9. Because we assume that $u \in [W^{1,\infty}(\Omega)]^N$, instead of $u \in [W^{2,\infty}(\Omega)]^N$, we cannot use the results of Proposition 1.2.5. However, recall equalities (1.2.34) and (1.2.41), which are established in the proof of Proposition 1.2.5. Define $\alpha_\varepsilon = p_\varepsilon - p - u \cdot Q_\varepsilon$ and $r_\varepsilon = \tilde{u}_\varepsilon - W_\varepsilon u$. Let v_ε be any bounded sequence in $[W_0^{1,q}(\Omega)]^N$, with $q > N$. Then

$$\langle \nabla[P_\varepsilon(\alpha_\varepsilon)], v_\varepsilon\rangle_{H^{-1},H_0^1(\Omega)} = \int_\Omega (Id - W_\varepsilon)\nabla u : \nabla(R_\varepsilon v_\varepsilon) - \int_\Omega \nabla r_\varepsilon : \nabla(R_\varepsilon v_\varepsilon)$$

$$+ \int_\Omega \nabla u : (R_\varepsilon v_\varepsilon \cdot \nabla W_\varepsilon) - \int_\Omega Q_\varepsilon \nabla u \cdot R_\varepsilon v_\varepsilon + \langle (M - M_\varepsilon)\, u, R_\varepsilon v_\varepsilon\rangle_{H^{-1},H_0^1(\Omega)}, \quad (4.2.11)$$

$$\langle -\Delta r_\varepsilon, r_\varepsilon\rangle_{H^{-1},H_0^1(\Omega)} = \langle (M - M_\varepsilon)\, u, r_\varepsilon\rangle_{H^{-1},H_0^1(\Omega)} - \langle \nabla u\, Q_\varepsilon, r_\varepsilon\rangle_{H^{-1},H_0^1(\Omega)}$$

$$- 2\int_\Omega (W_\varepsilon - Id)\nabla u : \nabla r_\varepsilon - \int_\Omega (W_\varepsilon - Id)\Delta u \cdot r_\varepsilon + \int_\Omega \alpha_\varepsilon \nabla \cdot r_\varepsilon. \quad (4.2.12)$$

On the one hand, taking into account the weaker smoothness of u, and the fact that Q_ε and ∇W_ε are equal to zero in $\Omega - H_\varepsilon$, we bound (4.2.11):

$$\begin{aligned} |\langle \nabla[P_\varepsilon(\alpha_\varepsilon)], v_\varepsilon\rangle| \leqq\ & \|Id - W_\varepsilon\|_{L^2(\Omega)}\, \|\nabla u\|_{L^\infty(\Omega)}\, \|\nabla(R_\varepsilon v_\varepsilon)\|_{L^2(\Omega)} \\ & + \|\nabla r_\varepsilon\|_{L^2(\Omega)}\, \|\nabla(R_\varepsilon v_\varepsilon)\|_{L^2(\Omega)} \\ & + \|\nabla u\|_{L^\infty(\Omega)}\, \|\nabla W_\varepsilon\|_{L^2(H_\varepsilon)}\, \|R_\varepsilon v_\varepsilon\|_{L^2(H_\varepsilon)} \\ & + \|\nabla u\|_{L^\infty(\Omega)}\, \|Q_\varepsilon\|_{L^2(H_\varepsilon)}\, \|R_\varepsilon v_\varepsilon\|_{L^2(H_\varepsilon)} \\ & + \|u\|_{W^{1,\infty}(\Omega)}\, \|M - M_\varepsilon\|_{H^{-1}(\Omega)}\, \|R_\varepsilon v_\varepsilon\|_{H_0^1(\Omega)}. \end{aligned} \quad (4.2.13)$$

But, adapting Lemma 3.4.1, which furnishes an optimal Poincaré inequality, we easily prove that for each $\phi_\varepsilon \in H^1(H_\varepsilon)$, which is equal to zero on the boundaries of the holes T_i^ε, we have

$$\|\phi_\varepsilon\|_{L^2(H_\varepsilon)} \leqq C\varepsilon^{1/2}\, \|\nabla \phi_\varepsilon\|_{L^2(H_\varepsilon)} \quad (4.2.14)$$

where the constant C does not depend on ε. Then, recalling that q' is defined by $\frac{1}{q}+\frac{1}{q'}=1$, and applying (4.2.14) for $R_\varepsilon v_\varepsilon$, we convert (4.2.13) to

$$\|\alpha_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)/\mathbb{R}} \leqq C\|\nabla r_\varepsilon\|_{L^2(\Omega)} + C\|u\|_{W^{1,\infty}(\Omega)}[\|M_\varepsilon - M\|_{H^{-1}(\Omega)} + \|Id - W_\varepsilon\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla W_\varepsilon\|_{L^2(\Omega)} + \varepsilon^{1/2}\|Q_\varepsilon\|_{L^2(\Omega)}]. \tag{4.2.15}$$

On the other hand, recalling that $\nabla\cdot r_\varepsilon = -W_\varepsilon : \nabla u = (Id - W_\varepsilon) : \nabla u$ is equal to 0 in the holes T_i^ε, we can bound the last term of (4.2.12) by

$$\left|\int_\Omega \alpha_\varepsilon \nabla\cdot r_\varepsilon\right| = \left|\int_{\Omega_\varepsilon} \alpha_\varepsilon \nabla\cdot r_\varepsilon\right| \leqq C\|\nabla u\|_{L^\infty(\Omega)}\|Id - W_\varepsilon\|_{L^q(\Omega_\varepsilon)}\|\alpha_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)/\mathbb{R}}$$

$$\text{with } \frac{1}{q}+\frac{1}{q'}=1.$$

An integration by parts yields

$$\int_\Omega (W_\varepsilon - Id)\Delta u\, r_\varepsilon = -\int_\Omega (W_\varepsilon - Id)\nabla u \nabla r_\varepsilon - \int_\Omega r_\varepsilon \nabla W_\varepsilon \nabla u.$$

Then, recalling that $\nabla W^\varepsilon = 0$ in $\Omega - H_\varepsilon$, we bound (4.2.12) by

$$\begin{aligned}\|\nabla r_\varepsilon\|^2_{L^2(\Omega)} \leqq{}& \|u\|_{W^{1,\infty}(\Omega)}\|\nabla r_\varepsilon\|_{L^2(\Omega)}\|M_\varepsilon - M\|_{H^{-1}(\Omega)}\\ &+ \|\nabla u\|_{L^\infty(\Omega)}\|r_\varepsilon\|_{L^2(H_\varepsilon)}\|Q_\varepsilon\|_{L^2(H_\varepsilon)}\\ &+ C\|\nabla u\|_{L^\infty(\Omega)}\|\nabla r_\varepsilon\|_{L^2(\Omega)}\|Id - W_\varepsilon\|_{L^2(\Omega)}\\ &+ C\|\nabla u\|_{L^\infty(\Omega)}\|r_\varepsilon\|_{L^2(H_\varepsilon)}\|\nabla W_\varepsilon\|_{L^2(H_\varepsilon)}\\ &+ C\|u\|_{W^{1,\infty}(\Omega)}\|Id - W_\varepsilon\|_{L^q(\Omega_\varepsilon)}\|\alpha_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)/\mathbb{R}}.\end{aligned} \tag{4.2.16}$$

Applying that Poincaré inequality (4.2.14) for r_ε, we obtain

$$\|\nabla r_\varepsilon\|^2_{L^2(\Omega)} \leqq C\|u\|_{W^{1,\infty}(\Omega)}\|\nabla r_\varepsilon\|_{L^2(\Omega)}[\|M_\varepsilon - M\|_{H^{-1}(\Omega)} + \|Id - W_\varepsilon\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla W_\varepsilon\|_{L^2(\Omega)} + \varepsilon^{1/2}\|Q_\varepsilon\|_{L^2(\Omega)}] + C\|u\|_{W^{1,\infty}(\Omega)}\|Id - W_\varepsilon\|_{L^q(\Omega_\varepsilon)}\|\alpha_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)/\mathbb{R}}. \tag{4.2.17}$$

Adding (4.2.3) and (4.2.5) for $N = 2$, and adding the estimates (4.2.7) for $N \geqq 3$ (note that these estimates holds in Ω_ε and are different from (4.2.8), which hold in Ω), we obtain

$$\begin{aligned}&\|Id - W_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leqq C\varepsilon^{\frac{q+1}{q}} \quad \text{for } q > N \geqq 3,\\ &\|Id - W_\varepsilon\|_{L^q(\Omega)} \leqq C\varepsilon \quad \text{for } q \geqq 1 \text{ and } N = 2.\end{aligned} \tag{4.2.18}$$

Previous computations in this subsection give

$$\begin{aligned}&\|Id - W_\varepsilon\|_{L^2(\Omega)} \leqq C\varepsilon, \quad \|\nabla W_\varepsilon\|_{H^{-1}(\Omega)} \leqq C\varepsilon,\\ &\|\nabla W_\varepsilon\|_{L^2(\Omega)} \leqq C, \quad \|Q_\varepsilon\|_{L^2(\Omega)} \leqq C.\end{aligned} \tag{4.2.19}$$

In order to bound $\|M_\varepsilon - M\|_{H^{-1}(\Omega)}$, we apply Lemma 2.4.2 in the set $Q = H_\varepsilon$ and take $h_\varepsilon = \mu_k^\varepsilon - \mu_k$ to obtain

$$\|h_\varepsilon\|_{H^{-1}(H_\varepsilon)} \leqq \varepsilon \left(\frac{|H_\varepsilon|}{2^N}\right)^{1/2} \|\nabla v\|_{L^2(P)}. \tag{4.2.20}$$

Here v is the unique solution of the problem

$$\text{Find } v \in H_p^1(P) \text{ such that}$$
$$-\Delta v = h \quad \text{in } P = (-1; +1)^N$$

with h formally defined by $h\left(\frac{x}{\varepsilon}\right) = h_\varepsilon(x)$, *i.e.*, for $y \in P$,

$$h(y) = \frac{1}{\varepsilon}\left[\frac{2}{C_0}(-e_k + 4(e_k \cdot e_r)\, e_r)\,[1 + o(1)]\,\delta_0^1 - \frac{2\pi}{C_0} e_k \delta_{H_0}\right] \quad \text{for } N = 2,$$

$$h(y) = \frac{1}{\varepsilon}\left[\frac{2^{N-2}C_0^{N-2}}{S_N}[F_k + N(F_k \cdot e_r)\, e_r]\,\delta_0^{1/2} - \frac{C_0^{N-2}}{2^{N-1}} F_k\, \delta_{H_0} + o(1)\,\delta_0^{1/2}\right]$$
$$\text{for } N \geqq 3$$

and with δ_{H_0} defined by

$$\langle \delta_{H_\varepsilon}, \phi \rangle = \varepsilon^N \left\langle \frac{1}{\varepsilon} \delta_{H_0}, \phi(x) \right\rangle \quad \text{for each } \phi \in D(\mathbb{R}^N).$$

Then we easily check that $\|\nabla v\|_{L^2(P)} \leqq \frac{C}{\varepsilon}$. Because $|H_\varepsilon| \leqq C\varepsilon$, we merely deduce from (4.2.20) that $\|\mu_k^\varepsilon - \mu_k\|_{H^{-1}(H_\varepsilon)} \leqq C\varepsilon^{1/2}$. Recalling that $\mu_k^\varepsilon - \mu_k = 0$ in $\Omega - H_\varepsilon$, we obtain

$$\|M_\varepsilon - M\|_{H^{-1}(\Omega)} \leqq C\varepsilon^{1/2}. \tag{4.2.21}$$

Now, introducing the estimates (4.2.18), (4.2.19), and (4.2.21) in both inequalities (4.2.15) and (4.2.17) yields the desired result (4.1.10):

$$\|r_\varepsilon\|_{H_0^1(\Omega)} \leqq C\varepsilon^{1/2}\,\|u\|_{W^{1,\infty}(\Omega)},$$

$$\|\alpha_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)/\mathbb{R}} \leqq C\varepsilon^{1/2}\,\|u\|_{W^{1,\infty}(\Omega)} \quad \text{with } 1 < q' < \frac{N}{N-1}.$$

Q.E.D.

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