

Localization of high-frequency waves propagating in a locally periodic medium

G. Allaire

Centre de Mathématiques Appliquées, École Polytechnique,
91128 Palaiseau, France (gregoire.allaire@polytechnique.fr)

L. Friz

Departamento de Ciencias Básicas, Facultad de Ciencias,
Universidad del Bío-Bío, Avenida Andrés Bello s/n,
Casilla 447, Chillán, Chile (lfriz@roble.fdo-may.ubiobio.cl)

(MS received 9 January 2009; accepted 2 November 2009)

We study the homogenization and localization of high-frequency waves in a locally periodic media with period ε . We consider initial data that are localized Bloch-wave packets, i.e. that are the product of a fast oscillating Bloch wave at a given frequency ξ and of a smooth envelope function whose support is concentrated at a point x with length scale $\sqrt{\varepsilon}$. We assume that (ξ, x) is a stationary point in the phase space of the Hamiltonian $\lambda(\xi, x)$, i.e. of the corresponding Bloch eigenvalue. Upon rescaling at size $\sqrt{\varepsilon}$, we prove that the solution of the wave equation is approximately the sum of two terms with opposite phases which are the product of the oscillating Bloch wave and of two limit envelope functions which are the solution of two Schrödinger type equations with quadratic potential. Furthermore, if the full Hessian of the Hamiltonian $\lambda(\xi, x)$ is positive definite, then localization takes place in the sense that the spectrum of each homogenized Schrödinger equation is made of a countable sequence of finite multiplicity eigenvalues with exponentially decaying eigenfunctions.

1. Introduction

We consider the wave equation in a locally periodic medium with small period $\varepsilon > 0$ and study its homogenization, i.e. its limit when ε goes to zero. Our wave equation is

$$\left. \begin{aligned} \rho^\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) &= 0 && \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ u_\varepsilon(0) &= u_\varepsilon^0 && \text{in } \mathbb{R}^N, \\ \frac{\partial u_\varepsilon}{\partial t}(0) &= u_\varepsilon^1 && \text{in } \mathbb{R}^N, \end{aligned} \right\} \quad (1.1)$$

where

$$\rho^\varepsilon(x) := \rho\left(x, \frac{x}{\varepsilon}\right) \quad \text{and} \quad A^\varepsilon(x) := A\left(x, \frac{x}{\varepsilon}\right) \quad (1.2)$$

and the unknown $u_\varepsilon(t, x)$ is a function from $\mathbb{R}^+ \times \mathbb{R}^N$ into $\mathbb{C}$. We assume that the coefficients $A(x, y)$ and $\rho(x, y)$ are real and sufficiently smooth bounded functions defined on $\mathbb{R}^N \times \mathbb{T}^N$, where $\mathbb{T}^N$ is the flat unit torus, i.e. the unit cell $(0, 1)^N$

equipped with periodic boundary conditions (see §2 for more precise smoothness assumptions). The initial data $u_\varepsilon^0, u_\varepsilon^1$ are highly oscillating in resonance with the period ε . More precisely, they are given in terms of so-called Bloch waves $\psi_n(x, y, \xi)e^{2i\pi\xi^n \cdot y}$, where ψ_n is an eigenfunction, with corresponding eigenvalue $\lambda_n(x, \xi)$, of the Bloch spectral cell problem [9, 22]

$$-(\operatorname{div}_y + 2\pi i\xi)(A(x, y)(\nabla_y + 2\pi i\xi)\psi_n) = \rho(x, y)\lambda_n(x, \xi)\psi_n \quad \text{in } \mathbb{T}^N. \quad (1.3)$$

By standard arguments of spectral theory, (1.3) admits a countable sequence of real increasing eigenvalues $(\lambda_n)_{n \geq 1}$, repeating each value as many times as its multiplicity, with corresponding periodic eigenfunctions normalized in $L^2(\mathbb{T}^N)$ by

$$\int_{\mathbb{T}^N} \rho(x, y) |\psi_n(x, y, \xi)|^2 dy = 1. \quad (1.4)$$

The Bloch parameter ξ is usually interpreted as a reduced wave number and the square root of the eigenvalue is the time frequency defined by

$$\omega_n(x, \xi) = \sqrt{\lambda_n(x, \xi)}. \quad (1.5)$$

Such Bloch-wave initial data are called high frequency. In this context, the homogenized limit of (1.1) can be studied by means of geometric optic method, also-called WKB asymptotic expansion [9, 15] (this can be made rigorous by using the notion of semiclassical measures, or Wigner transforms [16, 17]). We do not intend to give a full account of this theory here (we may also refer to [5, §6] for a brief review). Rather, we content ourselves by loosely stating that the asymptotic behaviour of the solution u_ε of (1.1) is given by the superposition of two waves, the amplitudes of which are solutions of Liouville transport equations, and the phases of which are the solutions of two eikonal equations. Solving these equations is somehow equivalent to solving, in the phase space $(x, \xi) \in \mathbb{R}^N \times \mathbb{T}^N$, the following two Hamiltonian systems:

$$\dot{x} = \nabla_\xi(\pm\omega_n(x, \xi)), \quad \dot{\xi} = -\nabla_x(\pm\omega_n(x, \xi)), \quad (1.6)$$

where the Hamiltonian $\pm\omega_n(x, \xi)$ is precisely the time frequency associated through (1.5) to the n th Bloch eigenvalue of (1.3).

In the case of a purely periodic medium, i.e. when the coefficients ρ and A do not depend on x , the Hamiltonian systems (1.6) simplify considerably and the corresponding eikonal equations for the phases have explicit global solutions. In such a case, one can go beyond the geometric optic time scale and study long time dispersive effects for monochromatic initial data [5, 6] (see [13] for constant or smooth coefficients). Similar dispersive effects have also been described in the physics literature [19, 24].

We stick to the case of locally periodic medium, i.e. the coefficients ρ and A depend both on x and y . However, we focus on a special instance of initial data, so that (1.6) has a trivial solution and a more precise analysis is required in order to describe the asymptotic behaviour of (1.1). We consider initial data which are concentrating at a critical point $(x^n, \xi^n) \in \mathbb{R}^N \times \mathbb{T}^N$ in the phase space, i.e.

$$\nabla_\xi \lambda_n(x^n, \xi^n) = \nabla_x \lambda_n(x^n, \xi^n) = 0, \quad (1.7)$$

which, of course, implies the same for the Hamiltonian

$$\nabla_{\xi}\omega_n(x^n, \xi^n) = \nabla_x\omega_n(x^n, \xi^n) = 0.$$

More precisely, for such a given critical point (x^n, ξ^n) and for given functions $v^0 \in H^1(\mathbb{R}^N)$ and $v^1 \in H^2(\mathbb{R}^N)$, we choose initial data which are wave packets or coherent states, i.e.

$$u_{\varepsilon}^0(x) = \psi_n\left(x^n, \frac{x}{\varepsilon}, \xi^n\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} v^0\left(\frac{x - x^n}{\sqrt{\varepsilon}}\right), \quad (1.8)$$

$$u_{\varepsilon}^1(x) = \frac{1}{\varepsilon} \psi_n\left(x^n, \frac{x}{\varepsilon}, \xi^n\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} v^1\left(\frac{x - x^n}{\sqrt{\varepsilon}}\right), \quad (1.9)$$

where the scale of focusing near x^n is exactly $\sqrt{\varepsilon}$. For the critical point (x^n, ξ^n) , we use the notation $\omega_n = \sqrt{\lambda_n(x^n, \xi^n)}$.

Our first result (theorems 3.1 and 3.2) shows that the solution of equation (1.1) is asymptotically given by

$$\begin{aligned} u_{\varepsilon}(t, x) \approx & \psi_n\left(x^n, \frac{x}{\varepsilon}, \xi^n\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} \\ & \times \left(\exp\left\{i \frac{\omega_n t}{\varepsilon}\right\} v^+\left(t, \frac{x - x^n}{\sqrt{\varepsilon}}\right) + \exp\left\{-i \frac{\omega_n t}{\varepsilon}\right\} v^-\left(t, \frac{x - x^n}{\sqrt{\varepsilon}}\right) \right), \end{aligned} \quad (1.10)$$

where $v^{\pm}(t, z)$ are the unique solutions of the two homogenized equations

$$\left. \begin{aligned} \pm 2i \frac{\partial v^{\pm}}{\partial t} - \operatorname{div}(A^* \nabla v^{\pm}) + \operatorname{div}(v^{\pm} B^* z) + c^* v^{\pm} + v^{\pm} D^* z \cdot z &= 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ v^{\pm}(0, z) &= \frac{1}{2} \left(v^0(z) \pm \frac{1}{i\omega_n} v^1(z) \right) \quad \text{in } \mathbb{R}^N, \end{aligned} \right\} \quad (1.11)$$

where the tensorial coefficients are the full Hessian of the time frequency

$$\begin{aligned} A^* &= \frac{1}{8\pi^2} \nabla_{\xi} \nabla_{\xi} \omega_n(x^n, \xi^n), & B^* &= \frac{1}{2i\pi} \nabla_{\xi} \nabla_x \omega_n(x^n, \xi^n), \\ D^* &= \frac{1}{2} \nabla_x \nabla_x \omega_n(x^n, \xi^n) \end{aligned}$$

and the constant c^* is defined by (3.4). In principle, a Schrödinger equation such as (1.11) shows the dispersive nature of the envelope functions $v^{\pm}$ in the ansatz (1.10) (as explained in [5]). However, the presence of a quadratic potential and a ‘convective’ term in (1.11) changes dramatically its interpretation.

Our second result is the following. If the full Hessian of the Hamiltonian or time frequency $\nabla \nabla \omega_n(x^n, \xi^n)$ is positive definite (or negative definite), then the resolvent of (1.11) is compact, implying that (1.11) admits a countable family of eigenfunctions, with exponential decay at infinity and forming an orthonormal basis of $L^2(\mathbb{R}^N)$. In other words, any eigenmode of (1.11) is localized in space. One can interpret this localization phenomenon by saying that the wave is ‘trapped’ exponentially close to x^n . Physically, this effect is well-known and is effectively used for trapping light in perturbed photonic crystals [8] or fibres [23]. Our results are a

partial explanation of this phenomenon, since real photonic crystals feature line or point defects in periodic geometries while we consider instead smooth variations, with respect to x , of the coefficients of the wave equation.

A similar result was already explained in the context of quantum mechanics or, more precisely, for solid-state physics, where a Schrödinger equation with periodic coefficients describes the wave function of an electron in a periodic crystal [3] (see also [4] for the corresponding eigenvalue problem). There, localization is a well-known phenomenon which is called Anderson localization if the periodicity perturbation is random [7, 11]. Localization can also appear for classical waves (see [14] and the references therein). Usually, localization is obtained by introducing some randomness into a periodic medium. One original feature of our work is that localization is produced by a deterministic modulation of the periodic coefficients of the wave equation.

Our assumption about the existence of a critical point (x^n, ξ^n) for the Hamiltonian, the Hessian of which is positive definite (or negative definite), is reasonable. It happens, at least generically, at the bottom or top of each Bloch band. On the other hand, we do not require the existence of a strict gap but, merely, of a smooth critical point of the Hamiltonian.

Let us finish this discussion by emphasizing that we talk about ‘localization’ only in the case of a positive definite (or negative definite) full Hessian of the Hamiltonian $\nabla \omega_n(x^n, \xi^n)$, which implies pure point spectrum of (1.11) with exponentially decaying eigenfunctions. In all other cases, the homogenized equation (1.11) for the envelope function has some essential spectrum and, therefore, no special property of localization of its eigenfunctions. In other words, our result of localization is proposition 3.6 rather than theorems 3.1 and 3.2. In our view, ‘localization’ should not be confused with what we may call ‘concentration’, which merely means that we can build approximate solutions of the wave equation (1.1) that have a support concentrating around a single point in the physical space. This latter phenomenon of ‘concentration’ can be studied in a more general framework than that of theorems 3.1 and 3.2: in particular, it may occur with purely periodic coefficients and can be checked by simple WKB or Wigner-measure arguments (it corresponds to a zero-group velocity $\nabla_{\xi} \omega_n(x^n, \xi^n) = 0$ and a constant envelope function, as was shown in [17]). On the other hand, our type of ‘localization’ can take place only with a macroscopic modulation of the periodic coefficients in (1.1) and yields a more precise asymptotic behaviour of the envelope function than ‘concentration’.

The content of our paper is as follows. Section 2 gives some basic properties of Bloch waves and two-scale convergence, as well as our main assumptions. Our main results are precisely stated in § 3. Section 4 is devoted to the proof of the required *a priori* estimates. Sections 5 and 6 are concerned with the proofs of our convergence results.

2. Preliminaries

In this section we describe our notation, state our assumptions and give some preliminary results concerning the Bloch spectral problem. The coefficients $A(x, y)$ and $\rho(x, y)$ are real and uniformly bounded Carathéodory functions defined on $\mathbb{R}^N \times \mathbb{T}^N$, i.e. they belong to $L^\infty(\mathbb{T}^N; C_b(\mathbb{R}^N))$. The density is uniformly bounded

from below by a positive constant, $\rho(x, y) \geq \rho_0 > 0$, and the matrix A is symmetric and uniformly coercive, i.e. there exists $\nu > 0$ such that $A(x, y)\zeta \cdot \zeta \geq \nu|\zeta|^2$ for any $\zeta \in \mathbb{R}^N$. Our hypotheses are as follows.

HYPOTHESIS 2.1. *There exist $x^n \in \mathbb{R}^N$ and $\xi^n \in \mathbb{T}^N$ such that*

- (i) $\lambda_n(x^n, \xi^n)$ is a simple eigenvalue;
- (ii) (x^n, ξ^n) is a critical point of $\lambda_n(x, \xi)$, i.e. $\nabla_x \lambda_n(x^n, \xi^n) = \nabla_\xi \lambda_n(x^n, \xi^n) = 0$;
- (iii) the eigenfunction $\psi_n(x^n, \cdot, \xi^n)$ belongs to $W^{1,\infty}(\mathbb{T}^N)$.

HYPOTHESIS 2.2. *The coefficients $A(x, y)$ and $\rho(x, y)$ are of class C^2 with respect to the variable x in a neighbourhood of $x = x^n$, and they admit the following second-order Taylor expansion:*

$$A(x, y) = A(x_n, y) + (x - x_n) \cdot \nabla_x A(x_n, y) + \frac{1}{2}(x - x_n) \nabla_x \nabla_x A(x_n, y)(x - x_n) + o(|x - x_n|^2)$$

and similarly for ρ .

We denote by $\nabla \nabla \lambda_n$ the full Hessian matrix of the function $\lambda_n(x, \xi)$ evaluated at the point (x^n, ξ^n) , i.e.

$$\nabla \nabla \lambda_n = \begin{bmatrix} \nabla_x \nabla_x \lambda_n & \nabla_\xi \nabla_x \lambda_n \\ \nabla_\xi \nabla_x \lambda_n & \nabla_\xi \nabla_\xi \lambda_n \end{bmatrix} (x^n, \xi^n).$$

REMARK 2.3. Because of the simplicity hypothesis 2.1(i), it is perfectly legitimate to differentiate the eigenvalue λ_n as much as we need. Because of hypothesis 2.1(ii), the Hessians of the eigenvalue λ_n and of the time frequency ω_n are proportional at the point (x^n, ξ^n) , i.e. $2\omega_n \nabla \nabla \omega_n = \nabla \nabla \lambda_n$.

Hypothesis 2.1(iii) holds true, for example, if the coefficients $A(x, y)$ and $\rho(x, y)$ are piecewise smooth with respect to y .

For the sake of notational simplicity, we define

$$\left. \begin{aligned} A_0(y) &:= A(x^n, y), & \lambda_n &:= \lambda_n(x^n, \xi^n), \\ \psi_n(y) &:= \psi(x^n, y, \xi^n), & \rho_0(y) &:= \rho(x^n, y) \end{aligned} \right\} \quad (2.1)$$

and, similarly, for the derivatives we set

$$A_{1,h}(y) := \frac{\partial A}{\partial x_h}(x^n, y), \quad A_{2,lh}(y) := \frac{\partial^2 A}{\partial x_l \partial x_h}(x^n, y) \quad \text{for } l, h = 1, \dots, N.$$

Analogous notation holds for all derivatives of ρ , ψ_n and λ_n with respect to the x variables and the ξ variables evaluated at $x = x^n$ and $\xi = \xi^n$.

Recall that we have three space variables corresponding to three different scales, x , $y := (x - x^n)/\varepsilon$ and $z := (x - x^n)/\sqrt{\varepsilon}$. For any function $\theta(x, y)$ defined on $\mathbb{R}^N \times \mathbb{T}^N$, we define

$$\theta^\varepsilon(x) := \theta\left(x, \frac{x}{\varepsilon}\right) = \theta\left(x^n + \sqrt{\varepsilon}z, \frac{z}{\sqrt{\varepsilon}} + \frac{x^n}{\varepsilon}\right) := \tilde{\theta}^\varepsilon(z). \quad (2.2)$$

In what follows, the symbols div_y and ∇_y are used to denote the divergence and gradient operators which act with respect to the y variable, while div and ∇ will indicate the divergence and gradient operators which act with respect to the x or z variable, according to the context.

We introduce the operator $\mathbb{A}_n(x, \xi)$, defined for $\psi \in L^2(\mathbb{T}^N)$ by

$$\mathbb{A}_n(x, \xi)\psi := -(\operatorname{div}_y + 2i\pi\xi)(A(x, y)(\nabla_y + 2i\pi\xi)\psi) - \lambda_n(x, \xi)\rho(x, y)\psi. \quad (2.3)$$

Under hypotheses 2.1 and 2.2 we can differentiate the Bloch spectral equation (1.3) in a neighbourhood of the point (x^n, ξ^n) in the phase space [18]. Denoting the canonical basis of $\mathbb{R}^N$ by $(e_k)_{1 \leq k \leq N}$, the first derivatives satisfy

$$\mathbb{A}_n(x, \xi) \frac{\partial \psi_n}{\partial \xi_k} = 2\pi i e_k A(\nabla_y + 2i\pi\xi)\psi_n + (\operatorname{div}_y + 2i\pi\xi)(A 2\pi i e_k \psi_n) + \rho \frac{\partial \lambda_n}{\partial \xi_k} \psi_n \quad (2.4)$$

and

$$\mathbb{A}_n(x, \xi) \frac{\partial \psi_n}{\partial x_l} = (\operatorname{div}_y + 2i\pi\xi) \left(\frac{\partial A}{\partial x_l} (\nabla_y + 2i\pi\xi)\psi_n \right) + \frac{\partial \rho}{\partial x_l} \lambda_n \psi_n + \rho \frac{\partial \lambda_n}{\partial x_l} \psi_n. \quad (2.5)$$

In the same manner, we can obtain formulae for the second-order derivatives, namely

$$\begin{aligned} \mathbb{A}_n(x, \xi) \frac{\partial^2 \psi_n}{\partial x_h \partial \xi_k} &= (\operatorname{div}_y + 2i\pi\xi) \frac{\partial A}{\partial x_h} (\nabla_y + 2i\pi\xi) \frac{\partial \psi_n}{\partial \xi_k} \\ &\quad + 2\pi i e_k \frac{\partial A}{\partial x_h} (\nabla_y + 2i\pi\xi)\psi_n + (\operatorname{div}_y + 2i\pi\xi) \left(\frac{\partial A}{\partial x_h} 2\pi i e_k \psi_n \right) \\ &\quad + 2\pi i e_k A(\nabla_y + 2i\pi\xi) \frac{\partial \psi_n}{\partial x_h} + (\operatorname{div}_y + 2i\pi\xi) \left(A 2\pi i e_k \frac{\partial \psi_n}{\partial x_h} \right) \\ &\quad + \frac{\partial \rho}{\partial x_h} \lambda_n \frac{\partial \psi_n}{\partial \xi_k} + \rho \frac{\partial \lambda_n}{\partial x_h} \frac{\partial \psi_n}{\partial \xi_k} + \frac{\partial \rho}{\partial x_h} \frac{\partial \lambda_n}{\partial \xi_k} \psi_n \\ &\quad + \rho \frac{\partial^2 \lambda_n}{\partial x_h \partial \xi_k} \psi_n + \rho \frac{\partial \lambda_n}{\partial \xi_k} \frac{\partial \psi_n}{\partial x_h}, \\ \mathbb{A}_n(x, \xi) \frac{\partial^2 \psi_n}{\partial x_h \partial x_l} &= (\operatorname{div}_y + 2i\pi\xi) \frac{\partial^2 A}{\partial x_h \partial x_l} (\nabla_y + 2i\pi\xi)\psi_n + \frac{\partial^2 \rho}{\partial x_h \partial x_l} \lambda_n \psi_n \\ &\quad + \rho \frac{\partial^2 \lambda_n}{\partial x_h \partial x_l} \psi_n + (\operatorname{div}_y + 2i\pi\xi) \frac{\partial A}{\partial x_h} (\nabla_y + 2i\pi\xi) \frac{\partial \psi_n}{\partial x_l} \\ &\quad + (\operatorname{div}_y + 2i\pi\xi) \frac{\partial A}{\partial x_l} (\nabla_y + 2i\pi\xi) \frac{\partial \psi_n}{\partial x_h} + \frac{\partial \rho}{\partial x_h} \lambda_n \frac{\partial \psi_n}{\partial x_l} \\ &\quad + \frac{\partial \rho}{\partial x_l} \lambda_n \frac{\partial \psi_n}{\partial x_h} + \rho \frac{\partial \lambda_n}{\partial x_h} \frac{\partial \psi_n}{\partial x_l} + \rho \frac{\partial \lambda_n}{\partial x_l} \frac{\partial \psi_n}{\partial x_h} \\ &\quad + \frac{\partial \rho}{\partial x_l} \frac{\partial \lambda_n}{\partial x_h} \psi_n + \frac{\partial \rho}{\partial x_h} \frac{\partial \lambda_n}{\partial x_l} \psi_n, \end{aligned}$$

$$\begin{aligned}
 \mathbb{A}_n(x, \xi) \frac{\partial^2 \psi_n}{\partial \xi_l \partial \xi_k} &= 2\pi i e_l A (\nabla_y + 2\pi i \xi) \frac{\partial \psi_n}{\partial \xi_k} + (\operatorname{div}_y + 2\pi i \xi) A 2\pi i e_l \frac{\partial \psi_n}{\partial \xi_k} \\
 &\quad + 2\pi i e_k A (\nabla_y + 2\pi i \xi) \frac{\partial \psi_n}{\partial \xi_l} + (\operatorname{div}_y + 2\pi i \xi) A 2\pi i e_k \frac{\partial \psi_n}{\partial \xi_l} \\
 &\quad + \rho \frac{\partial \lambda_n}{\partial \xi_l} \frac{\partial \psi_n}{\partial \xi_k} + \rho \frac{\partial \lambda_n}{\partial \xi_k} \frac{\partial \psi_n}{\partial \xi_l} - 4\pi^2 e_k A e_l \psi_n \\
 &\quad - 4\pi^2 e_l A e_k \psi_n + \rho \frac{\partial^2 \lambda_n}{\partial \xi_l \partial \xi_k} \psi_n.
 \end{aligned}$$

By integrating these equations for the second-order derivatives against ψ_n , recalling the normalization (1.4) of the eigenfunctions and taking $x = x^n$, we obtain the following formulae that will be useful later.

LEMMA 2.4. *Under hypotheses 2.1 and 2.2, the following equalities hold:*

$$\begin{aligned}
 &\int_{\mathbb{T}^N} \frac{1}{2\pi i} \left[A_{1,h} (\nabla_y + 2\pi i \xi^n) \frac{\partial \psi_n}{\partial \xi_k} \cdot (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n - \rho_{1,h} \lambda_n \frac{\partial \psi_n}{\partial \xi_k} \bar{\psi}_n \right] dy \\
 &\quad + \int_{\mathbb{T}^N} \left[A_{1,h} e_k \psi_n (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n + A_0 e_k \frac{\partial \psi_n}{\partial x_h} \cdot (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n \right] dy \\
 &\quad - \int_{\mathbb{T}^N} \left[e_k \bar{\psi}_n A_{1,h} (\nabla_y + 2\pi i \xi^n) \psi_n + e_k \bar{\psi}_n A_0 (\nabla_y + 2\pi i \xi^n) \frac{\partial \psi_n}{\partial x_h} \right] dy \\
 &\quad - \frac{1}{2\pi i} \frac{\partial^2 \lambda_n}{\partial x_h \partial \xi_k} = 0,
 \end{aligned} \tag{2.6}$$

$$\begin{aligned}
 &\int_{\mathbb{T}^N} \left[A_{2,lh} (\nabla_y + 2\pi i \xi^n) \psi_n \cdot (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n - \left(\rho_{2,lh} \lambda_n + \rho \frac{\partial^2 \lambda_n}{\partial x_h \partial x_l} \right) |\psi_n|^2 \right] dy \\
 &\quad + \int_{\mathbb{T}^N} \left[A_{1,h} (\nabla_y + 2\pi i \xi^n) \frac{\partial \psi_n}{\partial x_l} \cdot (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n - \rho_{1,h} \lambda_n \frac{\partial \psi_n}{\partial x_l} \bar{\psi}_n \right] dy \\
 &\quad + \int_{\mathbb{T}^N} \left[A_{1,l} (\nabla_y + 2\pi i \xi^n) \frac{\partial \psi_n}{\partial x_h} \cdot (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n - \rho_{1,l} \lambda_n \frac{\partial \psi_n}{\partial x_h} \bar{\psi}_n \right] dy = 0,
 \end{aligned} \tag{2.7}$$

$$\begin{aligned}
 &\int_{\mathbb{T}^N} \left[2\pi i e_k A_0 (\nabla_y + 2\pi i \xi^n) \frac{\partial \psi_n}{\partial \xi_l} \bar{\psi}_n - \left(A_0 2\pi i e_k \frac{\partial \psi_n}{\partial \xi_l} \right) (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n \right] dy \\
 &\quad + \int_{\mathbb{T}^N} \left[2\pi i e_l A_0 (\nabla_y + 2\pi i \xi^n) \frac{\partial \psi_n}{\partial \xi_k} \bar{\psi}_n - \left(A_0 2\pi i e_l \frac{\partial \psi_n}{\partial \xi_k} \right) (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n \right] dy \\
 &\quad - \int_{\mathbb{T}^N} [4\pi^2 e_k A_0 e_l |\psi_n|^2 + 4\pi^2 e_l A_0 e_k |\psi_n|^2] dy + \frac{\partial^2 \lambda_n}{\partial \xi_l \partial \xi_k} = 0.
 \end{aligned} \tag{2.8}$$

We also recall the variational formulations of

$$\psi_n^\varepsilon(x) = \psi_n(x^n, x/\varepsilon, \xi^n)$$

and of its derivatives.

LEMMA 2.5. *Let $\varphi(z)$ be a smooth, compactly supported function defined from $\mathbb{R}^N$ into $\mathbb{C}$. Under hypotheses 2.1 and 2.2, the following equalities hold:*

$$\int_{\mathbb{R}^N} [A_0^\varepsilon(\nabla_y + 2\pi i \xi^n) \psi_n^\varepsilon \cdot (\sqrt{\varepsilon} \nabla - 2\pi i \xi^n) \bar{\varphi}(z) - \rho_0^\varepsilon \lambda_n^\varepsilon \psi_n^\varepsilon \bar{\varphi}(z)] dz = 0, \quad (2.9)$$

$$\begin{aligned} & \int_{\mathbb{R}^N} \left[A_0^\varepsilon(\nabla_y + 2\pi i \xi^n) \frac{\partial \psi_n^\varepsilon}{\partial \xi_k} \cdot (\sqrt{\varepsilon} \nabla - 2\pi i \xi^n) \bar{\varphi}(z) - \rho_0^\varepsilon \lambda_n^\varepsilon \frac{\partial \psi_n^\varepsilon}{\partial \xi_k} \bar{\varphi}(z) \right] dz \\ & + \int_{\mathbb{R}^N} [-2\pi i e_k A_0^\varepsilon(\nabla_y + 2\pi i \xi^n) \psi_n^\varepsilon \bar{\varphi}(z) + A_0^\varepsilon 2\pi i e_k \psi_n^\varepsilon (\sqrt{\varepsilon} \nabla - 2\pi i \xi^n) \bar{\varphi}(z)] dz = 0, \end{aligned} \quad (2.10)$$

$$\begin{aligned} & \int_{\mathbb{R}^N} \left[A_0^\varepsilon(\nabla_y + 2\pi i \xi^n) \frac{\partial \psi_n^\varepsilon}{\partial x_l} \cdot (\sqrt{\varepsilon} \nabla - 2\pi i \xi^n) \bar{\varphi}(z) - \rho_0^\varepsilon \lambda_n^\varepsilon \frac{\partial \psi_n^\varepsilon}{\partial x_l} \bar{\varphi}(z) \right] dz \\ & + \int_{\mathbb{R}^N} [A_{1,l}^\varepsilon(\nabla_y + 2\pi i \xi^n) \psi_n^\varepsilon \cdot (\sqrt{\varepsilon} \nabla - 2\pi i \xi^n) \bar{\varphi}(z) - \rho_{1,l}^\varepsilon \lambda_n^\varepsilon \psi_n^\varepsilon \bar{\varphi}(z)] dz = 0. \end{aligned} \quad (2.11)$$

We recall the notion of two-scale convergence [1,20] with a small parameter $\delta > 0$, which will be equal to $\sqrt{\varepsilon}$ in what follows.

PROPOSITION 2.6. *Let f_δ be a sequence uniformly bounded in $L^2(\mathbb{R}^N)$.*

- (i) *There exists a subsequence, still denoted by f_δ , and a limit $f_0(x, y) \in L^2(\mathbb{R}^N \times \mathbb{T}^N)$, such that f_δ two-scale converges weakly to f_0 in the sense that*

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^N} f_\delta(x) \phi\left(x, \frac{x}{\delta}\right) dx = \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} f_0(x, y) \phi(x, y) dx dy$$

for all functions $\phi(x, y) \in L^2(\mathbb{R}^N; C(\mathbb{T}^N))$.

- (ii) *Assume further that f_δ two-scale converges weakly to f_0 and that*

$$\lim_{\delta \rightarrow 0} \|f_\delta\|_{L^2(\mathbb{R}^N)} = \|f_0\|_{L^2(\mathbb{R}^N \times \mathbb{T}^N)}.$$

Then f_δ is said to two-scale converge strongly to its limit f_0 in the sense that, if f_0 is smooth enough, e.g. $f_0(x, y) \in L^2(\mathbb{R}^N; C(\mathbb{T}^N))$, we have

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^N} \left| f_\delta(x) - f_0\left(x, \frac{x}{\delta}\right) \right|^2 dx = 0.$$

- (iii) *Assume that $\delta \nabla f_\delta$ is also uniformly bounded in $L^2(\mathbb{R}^N)^N$. Then there exists a subsequence, still denoted by f_δ , and a limit $f_0(x, y) \in L^2(\mathbb{R}^N; H^1(\mathbb{T}^N))$, such that f_δ two-scale converges weakly to $f_0(x, y)$ and $\delta \nabla f_\delta$ two-scale converges weakly to $\nabla_y f_0(x, y)$.*

3. Main results

In order to apply the two-scale convergence of proposition 2.6, we first need to remove the oscillating phase in u_ε and rescale it to the concentration scale $\sqrt{\varepsilon}$. This is the purpose of the changes of unknowns (3.1) and (3.5) which are necessary to pass to the limit and obtain the homogenized equations.

THEOREM 3.1. Assume that hypotheses 2.1 and 2.2 hold true and that the initial data u_ε^0 and u_ε^1 are of the form (1.8) and (1.9), respectively, with $v^0 \in H^1(\mathbb{R}^N)$ and $v^1 \in H^2(\mathbb{R}^N)$. Define

$$v_\varepsilon^+ \left(t, \frac{x - x^n}{\sqrt{\varepsilon}} \right) = u_\varepsilon(t, x) \exp \left\{ -i \frac{\omega_n t}{\varepsilon} \right\} \exp \left\{ -2i\pi \frac{\xi^n \cdot x}{\varepsilon} \right\}, \quad (3.1)$$

where u_ε is the solution of equation (1.1). Then $v_\varepsilon^+(t, z)$ two-scale converges weakly to $\psi_n(y)v^+(t, z)$ and v^+ is the unique solution of the homogenized Schrödinger equation

$$\left. \begin{aligned} 2i \frac{\partial v^+}{\partial t} - \operatorname{div}(A^* \nabla v^+) + \operatorname{div}(v^+ B^* z) + c^* v^+ + v^+ D^* z \cdot z &= 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ v^+(0, z) &= \frac{1}{2} \left(v^0(z) + \frac{1}{i\omega_n} v^1(z) \right) \quad \text{in } \mathbb{R}^N, \end{aligned} \right\} \quad (3.2)$$

where

$$\left. \begin{aligned} A^* &= \frac{1}{8\pi^2} \nabla_\xi \nabla_\xi \omega_n(x^n, \xi^n), & B^* &= \frac{1}{2i\pi} \nabla_\xi \nabla_x \omega_n(x^n, \xi^n), \\ D^* &= \frac{1}{2} \nabla_x \nabla_x \omega_n(x^n, \xi^n) \end{aligned} \right\} \quad (3.3)$$

and c^* is given by

$$c^* = \int_{\mathbb{T}^N} \left[A(\nabla_y + 2i\pi \xi^n) \psi_n \cdot \frac{\partial \bar{\psi}_n}{\partial x_k} e_k - A(\nabla_y - 2i\pi \xi^n) \frac{\partial \bar{\psi}_n}{\partial x_k} \cdot \psi_n e_k - A_{1,k}(\nabla_y - 2i\pi \xi^n) \bar{\psi}_n \cdot \psi_n e_k \right] dy. \quad (3.4)$$

A similar result holds true for

$$v_\varepsilon^- \left(t, \frac{x - x^n}{\sqrt{\varepsilon}} \right) = u_\varepsilon(t, x) \exp \left\{ i \frac{\omega_n t}{\varepsilon} \right\} \exp \left\{ -2i\pi \frac{\xi^n \cdot x}{\varepsilon} \right\}, \quad (3.5)$$

namely, $v_\varepsilon^-(t, z)$ two-scale converges weakly to $\psi_n(y)v^-(t, z)$, where v^- is the unique solution of the homogenized Schrödinger equation

$$\left. \begin{aligned} -2i \frac{\partial v^-}{\partial t} - \operatorname{div}(A^* \nabla v^-) + \operatorname{div}(v^- B^* z) + c^* v^- + v^- D^* z \cdot z &= 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ v^-(0, z) &= \frac{1}{2} \left(v^0(z) - \frac{1}{i\omega_n} v^1(z) \right) \quad \text{in } \mathbb{R}^N. \end{aligned} \right\} \quad (3.6)$$

Note that, if v^0 and v^1 are real-valued functions, then we deduce that $v^- = \overline{v^+}$. Theorem 3.1 gives two different limit behaviours for u_ε , according to the two different phases in (3.1) and (3.5). However, each of these limits carry only half of the initial data. The next result explains that the sum of these two waves is a valid approximation of the solution u_ε of (1.1). In other words we now state a strong two-scale convergence result instead of a weak one as in theorem 3.1.

THEOREM 3.2. Assume that hypotheses 2.1 and 2.2 hold true, and assume that $v^0 \in H^2(\mathbb{R}^N)$ and $v^1 \in H^2(\mathbb{R}^N)$. If $\xi^n = 0$, assume furthermore that

$$\left. \begin{aligned} v^1 &\in L^1(\mathbb{R}^N) \quad \text{if } N \geq 3, \\ \int_{\mathbb{R}^N} v^1(z) \, dz &= 0 \quad \text{and} \quad \mathcal{F}v^1 \in C^{0,\alpha}(B_0) \quad \text{with } \alpha > 1 - \frac{N}{2} \quad \text{if } N \leq 2, \end{aligned} \right\} \quad (3.7)$$

where $\mathcal{F}v^1(\xi)$ denotes the Fourier transform of $v^1(x)$ and B_0 is a small open ball around the origin. Define the ansatz

$$\begin{aligned} u_\varepsilon^{\text{approx}}(t, x) &= \psi_n\left(x^n, \frac{x}{\varepsilon}, \xi^n\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} \\ &\quad \times \left(\exp\left\{i \frac{\omega_n t}{\varepsilon}\right\} v^+\left(t, \frac{x - x^n}{\sqrt{\varepsilon}}\right) + \exp\left\{-i \frac{\omega_n t}{\varepsilon}\right\} v^-\left(t, \frac{x - x^n}{\sqrt{\varepsilon}}\right) \right), \end{aligned} \quad (3.8)$$

where $v^\pm(t, z)$ are the solutions of the two homogenized equations (3.2) and (3.6). It then satisfies

$$\lim_{\varepsilon \rightarrow 0} \frac{\|u_\varepsilon(t, x) - u_\varepsilon^{\text{approx}}(t, x)\|_{L^2((0,T) \times \mathbb{R}^N)}}{\|u_\varepsilon^{\text{approx}}(t, x)\|_{L^2((0,T) \times \mathbb{R}^N)}} = 0$$

for any final time $T > 0$.

REMARK 3.3. Theorem 3.2 gives a relative error going to zero. Indeed, it is easily shown, upon rescaling at scale $\sqrt{\varepsilon}$, that $\|u_\varepsilon^{\text{approx}}(t, x)\|_{L^2((0,T) \times \mathbb{R}^N)}$ is of order $\varepsilon^{N/4}$. Actually, this estimate for $u_\varepsilon^{\text{approx}}$ is valid in $L^2(\mathbb{R}^N)$ for almost every time t . However, the error estimate requires a time integration and is not valid for almost every time t .

REMARK 3.4. Assumption (3.7) is technical and is used merely in the *a priori* estimate of lemma 6.2. We do not know whether or not it is absolutely necessary. However, the assumption that v^1 has zero average is reminiscent of a similar assumption used to prove that the solution of the wave equation in a periodic domain is uniformly bounded in $L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^N))$ (see, for example, [2, § 2.3.2]).

Before we prove theorems 3.1 and 3.2, let us analyse the homogenized Schrödinger equation (3.2). We define the following unbounded operator acting in $L^2(\mathbb{R}^N)$:

$$\mathbb{A}^* \phi := -\operatorname{div}(A^* \nabla \phi) + \operatorname{div}(\phi B^* z) + c^* \phi + \phi D^* z \cdot z, \quad (3.9)$$

which already appears in the study of localization for the Schrödinger equation [3] (beware that the star symbol in (3.9) means ‘homogenized’ and not adjoint). We show that (3.2) or, equivalently, (3.6) are well-posed.

PROPOSITION 3.5 (Allaire and Palombaro [3, proposition 3.4]). *The operator $\mathbb{A}^*$ defined in (3.9) is essentially self-adjoint. As a consequence, there exists a unique solution $v^+(t, z)$ of (3.2) in $C(\mathbb{R}^+; L^2(\mathbb{R}^N))$. Furthermore, it satisfies the energy conservation*

$$\|v^+(t, \cdot)\|_{L^2(\mathbb{R}^N)} = \|v^+(0, \cdot)\|_{L^2(\mathbb{R}^N)} \quad \text{for all } t \in \mathbb{R}^+.$$

Proof. We thank the anonymous referee for pointing out a flaw in the original proof of this result in [3], where we ignored the fact that the domain of the unbounded operator $\mathbb{A}^*$ may be different from that of its adjoint. We thus wrongly concluded that $\mathbb{A}^*$ is self-adjoint. We shall now merely prove that it is essentially self-adjoint, i.e. that its closure is self-adjoint. This last property is sufficient for our purposes since it implies that $\mathbb{A}^*$ has a unique self-adjoint extension. We briefly indicate how to modify the proof in [3]. Firstly, a simple integration by parts shows that $\mathbb{A}^*$ is symmetric on $C_0^\infty(\mathbb{R}^N)$ because $((1/2i)\operatorname{tr} B^* + \operatorname{Im} c^*) = 0$. This last identity is obtained by a combination of a formula for c^* , deduced from (2.4) multiplied by $\partial\bar{\psi}_n/\partial x_k$, and another formula for $\nabla_\xi \nabla_x \lambda_n$ (and, thus, for B^*), deduced from (2.6), where we plug in (2.4) multiplied by $\partial\bar{\psi}_n/\partial x_h$. Secondly, we use [22, theorem 10.37, volume II] to prove that $\mathbb{A}^*$ is essentially self-adjoint. Introducing the self-adjoint operator $N = -\Delta + |x|^2 + 1$ with domain $H^2(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx)$, we can easily check the assumptions of this theorem, namely, that there exists a constant $C > 0$ such that, for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\|\mathbb{A}^* \phi\|_{L^2(\mathbb{R}^N)} \leq C \|N\phi\|_{L^2(\mathbb{R}^N)} \quad \text{and} \quad |\langle \mathbb{A}^* \phi, N\phi \rangle - \langle N\phi, \mathbb{A}^* \phi \rangle| \leq C \|N^{1/2} \phi\|_{L^2(\mathbb{R}^N)}^2.$$

Thirdly, by semigroup theory [10, 21], we can deduce from the uniqueness of the self-adjoint extension of $\mathbb{A}^*$ that there exists a unique solution of (3.2) in $C(\mathbb{R}^+; L^2(\mathbb{R}^N))$ which may not belong to $C(\mathbb{R}^+; H^1(\mathbb{R}^N))$ if the matrix A^* is not positive definite or positive negative. The energy conservation for (3.2) is just a consequence of the symmetry of $\mathbb{A}^*$. $\square$

We now recall a compactness result of [3], which is at the root of the localization phenomenon.

PROPOSITION 3.6 (Allaire and Palombaro [3, proposition 3.5]). *Assume that the matrix $\nabla \nabla \omega_n$ is positive definite (or, equivalently, positive negative). Then the resolvent of $\mathbb{A}^*$ is compact in $L^2(\mathbb{R}^N)$ and there exists an orthonormal basis $(\varphi_n)_{n \geq 1}$ of eigenfunctions of $\mathbb{A}^*$ which decay exponentially, i.e. for each n , there exists a constant $\gamma_n > 0$ such that*

$$e^{\gamma_n |z|^2} \varphi_n(z), \quad e^{\gamma_n |z|^2} \nabla \varphi_n(z) \in L^2(\mathbb{R}^N). \quad (3.10)$$

REMARK 3.7. As mentioned in the introduction, by ‘localization’ we mean the case when the homogenized equations (3.2) and (3.6) for the two envelope functions have pure point spectrum with exponentially decaying eigenfunctions. Therefore, rather than theorems 3.1 and 3.2, proposition 3.6 is our localization result. In particular, what we call ‘localization’ should not be confused with ‘concentration’, which we define as the possibility of having sequences of solutions of the wave equation (1.1) concentrating around a single point in the physical space. This latter phenomenon is provided by theorems 3.1 and 3.2, but it can be obtained in the simpler setting of purely periodic coefficients by means of well-known WKB or Wigner-measure arguments. In this latter setting, ‘concentration’ means a zero-group velocity $\nabla_\xi \omega_n(x^n, \xi^n) = 0$ and a constant envelope function. This was already derived in [17].

The proof of theorem 3.1 relies on the change of unknowns (3.1). It is then possible to pass to the two-scale limit in the equation satisfied by $v_\varepsilon^+(t, z)$. We introduce the

following notation:

$$\tilde{\rho}^\varepsilon(z) = \rho\left(x^n + \sqrt{\varepsilon}z, \frac{z}{\sqrt{\varepsilon}} + \frac{x^n}{\varepsilon}\right) \quad (3.11)$$

and similarly for $\tilde{A}^\varepsilon$ and $\tilde{\psi}_n^\varepsilon$. A simple computation yields

$$\left. \begin{aligned} & \tilde{\rho}^\varepsilon \left(2i\omega_n \frac{\partial v_\varepsilon^+}{\partial t} + \varepsilon \frac{\partial^2 v_\varepsilon^+}{\partial t^2} \right) \\ & - \left(\operatorname{div}_z + \frac{2i\pi\xi^n}{\sqrt{\varepsilon}} \right) \cdot \left(\tilde{A}^\varepsilon \left(\nabla_z + \frac{2i\pi\xi^n}{\sqrt{\varepsilon}} \right) v_\varepsilon^+ \right) \\ & = \tilde{\rho}^\varepsilon \frac{\lambda_n}{\varepsilon} v_\varepsilon^+ \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ & v_\varepsilon^+(0) = \tilde{\psi}_n^\varepsilon v^0 \quad \text{in } \mathbb{R}^N, \\ & \frac{\partial v_\varepsilon^+}{\partial t}(0) = \frac{1}{\varepsilon} \tilde{\psi}_n^\varepsilon (-i\omega_n v^0 + v^1) \quad \text{in } \mathbb{R}^N. \end{aligned} \right\} \quad (3.12)$$

The solution $v_\varepsilon^+(t, z)$ of (3.12) satisfies a suitable *a priori* estimate for using the notion of two-scale convergence.

LEMMA 3.8. *For any final time $T > 0$, there exists $C(T) > 0$ independent of ε such that the solution of (3.12) satisfies*

$$\begin{aligned} & \|v_\varepsilon^+\|_{L^\infty((0,T);L^2(\mathbb{R}^N))} + \varepsilon \left\| \frac{\partial v_\varepsilon^+}{\partial t} \right\|_{L^\infty((0,T);L^2(\mathbb{R}^N))} \\ & + \sqrt{\varepsilon} \|\nabla_z v_\varepsilon^+\|_{L^\infty((0,T);L^2(\mathbb{R}^N)^N)} \\ & \leq C(T) (\|v^0\|_{H^1(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}). \end{aligned} \quad (3.13)$$

Notation In what follows, we shall assume without loss of generality that $x^n = 0$. This is always possible by a simple translation and it simplifies the writing of many formulae.

4. *A priori* estimates (proof of lemma 3.8)

Although the statement of lemma 3.8 involves v_ε^+ , we shall prove an *a priori* estimate for u_ε , the solution of (1.1). Then, remarking that, by virtue of (3.1),

$$\begin{aligned} & \varepsilon^{N/4} \|v_\varepsilon^+(t, \cdot)\|_{L^2(\mathbb{R}^N)} = \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^N)}, \\ & \varepsilon^{N/4} \left\| \varepsilon \frac{\partial v_\varepsilon^+}{\partial t}(t, \cdot) \right\|_{L^2(\mathbb{R}^N)} = \left\| \varepsilon \frac{\partial u_\varepsilon}{\partial t}(t, \cdot) - i\omega u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R}^N)} \end{aligned}$$

and

$$\varepsilon^{N/4} \|\sqrt{\varepsilon} \nabla_z v_\varepsilon^+(t, \cdot)\|_{L^2(\mathbb{R}^N)^N} = \|\varepsilon \nabla_x u_\varepsilon(t, \cdot) + 2i\pi \xi u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^N)^N},$$

we can easily deduce (3.13) from the corresponding estimates on u_ε .

In a first step, we obtain the usual energy conservation by multiplying equation (1.1) by $\partial \bar{u}_\varepsilon / \partial t$ and integrating by parts:

$$\frac{d}{dt} E^\varepsilon(t) = 0 \quad \text{with } E^\varepsilon(t) = \frac{1}{2} \int_{\mathbb{R}^N} \left[\rho^\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 + A^\varepsilon \nabla u_\varepsilon \cdot \nabla \bar{u}_\varepsilon \right] dx.$$

Since the initial data u_ε^0 and u_ε^1 are defined by (1.8) and (1.9) respectively, and because ψ_n belongs to $L^\infty(\mathbb{T}^N)$, we deduce

$$\begin{aligned} \int_{\mathbb{R}^N} \rho^\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t}(0) \right|^2 dx &= \frac{\varepsilon^{N/2}}{\varepsilon^2} \int_{\mathbb{R}^N} \rho \left(\sqrt{\varepsilon} z, \frac{z}{\sqrt{\varepsilon}} \right) \left| \psi_n \left(\frac{z}{\sqrt{\varepsilon}} \right) v^1(z) \right|^2 dz \\ &\leq C \varepsilon^{N/2-2} \|v^1\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

A similar estimate holds for $\nabla u_\varepsilon(0)$ because hypothesis 2.1(iii) tells us that $\nabla_y \psi_n$ belongs to $L^\infty(\mathbb{T}^N)^N$:

$$\int_{\mathbb{R}^N} A^\varepsilon \nabla u_\varepsilon(0) \cdot \nabla \overline{u_\varepsilon(0)} dx \leq C \varepsilon^{N/2-2} (\|v^0\|_{L^2(\mathbb{R}^N)}^2 + \varepsilon \|\nabla v^0\|_{L^2(\mathbb{R}^N)}^2),$$

which implies

$$E^\varepsilon(0) \leq C \varepsilon^{N/2-2} (\|v^0\|_{H^1(\mathbb{R}^N)}^2 + \|v^1\|_{L^2(\mathbb{R}^N)}^2).$$

This is only at this point that we use hypothesis 2.1(iii) on the smoothness of ψ_n . Note that we always have $\psi_n \in L^\infty(\mathbb{T}^N)$ by standard elliptic regularity, and that we could have replaced 2.1(iii) by an additional regularity of v^0 (using integration by parts as in the proof of lemma 4.1, below). In any case, we obtain

$$\varepsilon \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^N))} + \varepsilon \|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^N)^N)} \leq C \varepsilon^{N/4} (\|v^0\|_{H^1(\mathbb{R}^N)} + \|v^1\|_{L^2(\mathbb{R}^N)}).$$

In a second step, we obtain an estimate for u_ε in $L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^N))$ following an argument of [5] based on a classical idea of time regularization. For a given $\alpha \neq 0$, we introduce a time primitive of u_ε :

$$\Psi_\varepsilon(x, t) = e^{-\alpha t} \left[\int_0^t e^{\alpha s} u_\varepsilon(x, s) ds + \chi_\varepsilon(x) \right], \quad (4.1)$$

where χ_ε is defined as the unique solution in $H^1(\mathbb{R}^N)$ of the time-independent equation

$$-\operatorname{div} \left(A \left(x, \frac{x}{\varepsilon} \right) \nabla \chi_\varepsilon \right) + \alpha^2 \rho \left(x, \frac{x}{\varepsilon} \right) \chi_\varepsilon = \alpha \rho \left(x, \frac{x}{\varepsilon} \right) u_\varepsilon^0 - \rho \left(x, \frac{x}{\varepsilon} \right) u_\varepsilon^1 \quad \text{in } \mathbb{R}^N. \quad (4.2)$$

A simple computation shows that Ψ_ε satisfies

$$\left. \begin{aligned} \rho \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 \Psi_\varepsilon}{\partial t^2} - \operatorname{div} \left(A \left(x, \frac{x}{\varepsilon} \right) \nabla \Psi_\varepsilon \right) &= 0 && \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ \Psi_\varepsilon(0) &= \chi_\varepsilon && \text{in } \mathbb{R}^N, \\ \frac{\partial \Psi_\varepsilon}{\partial t}(0) &= u_\varepsilon^0 - \alpha \chi_\varepsilon && \text{in } \mathbb{R}^N. \end{aligned} \right\} \quad (4.3)$$

The interest of (4.3) is that its initial data is one order smaller in ε than those of (1.1) as stated in the following lemma.

LEMMA 4.1. *The solution of (4.2) satisfies*

$$\|\chi_\varepsilon\|_{H^1(\mathbb{R}^N)} \leq C \varepsilon^{N/4} (\|v^0\|_{L^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}).$$

Postponing the proof of lemma 4.1 for a moment, we are now in a position to prove that

$$\|u_\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^N))} \leq C(T)\varepsilon^{N/4}(\|v^0\|_{L^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}), \quad (4.4)$$

which concludes the proof of lemma 3.8. Indeed, since $u_\varepsilon = \partial\Psi_\varepsilon/\partial t + \alpha\Psi_\varepsilon$, the standard energy conservation for (4.3), together with lemma 4.1, implies that

$$\left\| \frac{\partial\Psi_\varepsilon}{\partial t} \right\|_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^N))} \leq C\varepsilon^{N/4}(\|v^0\|_{L^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}). \quad (4.5)$$

Since $\Psi_\varepsilon(t) = \chi_\varepsilon + \int_0^t (\partial\Psi_\varepsilon/\partial t)(s) \, ds$, we deduce (4.4).

Proof of lemma 4.1. Without loss of generality, we take $\alpha = 1$. By multiplying (4.2) by $\bar{\chi}_\varepsilon$, we obtain

$$\int_{\mathbb{R}^N} \left(A\left(x, \frac{x}{\varepsilon}\right) \nabla \chi_\varepsilon \cdot \nabla \bar{\chi}_\varepsilon + \rho\left(x, \frac{x}{\varepsilon}\right) |\chi_\varepsilon|^2 \right) dx = \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) (u_\varepsilon^0 - u_\varepsilon^1) \bar{\chi}_\varepsilon dx.$$

First, we can easily check that

$$\int_{\mathbb{R}^N} |\rho^\varepsilon u_\varepsilon^0 \bar{\chi}_\varepsilon| \, dx \leq C\varepsilon^{N/4} \|v^0\|_{L^2(\mathbb{R}^N)} \|\chi_\varepsilon\|_{L^2(\mathbb{R}^N)}.$$

Since

$$u_\varepsilon^1 = \varepsilon^{-1} \psi_n\left(\frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi_n \cdot x}{\varepsilon}\right\} v^1\left(\frac{x}{\sqrt{\varepsilon}}\right),$$

we define

$$\Delta_\varepsilon := \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \psi_n\left(\frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi_n \cdot x}{\varepsilon}\right\} v^1\left(\frac{x}{\sqrt{\varepsilon}}\right) \bar{\chi}_\varepsilon(x) \, dx.$$

Let us now prove that

$$|\Delta_\varepsilon| \leq C\varepsilon^{N/4+1} \|v^1\|_{H^2(\mathbb{R}^N)} \|\chi_\varepsilon\|_{H^1(\mathbb{R}^N)}. \quad (4.6)$$

By [5, lemma 4.4], there exists a solution $\zeta \in C^2(\mathbb{R}^N; C(\mathbb{T}^N))^N$ of

$$-\operatorname{div}_y(\zeta(x, y) e^{2i\pi \xi^n \cdot y}) = \rho(x, y) \psi_n(y) e^{2i\pi \xi^n \cdot y}.$$

Then

$$\begin{aligned} & \rho\left(x, \frac{x}{\varepsilon}\right) \psi_n\left(\frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} \\ &= -\varepsilon \operatorname{div} \left[\zeta\left(x, \frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} \right] + \varepsilon \operatorname{div}_x \zeta\left(x, \frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\}. \end{aligned}$$

Using ζ allows us to rewrite

$$\begin{aligned} \Delta_\varepsilon &= \varepsilon \int_{\mathbb{R}^N} \left[\zeta\left(x, \frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} \right. \\ &\quad \times \left(v^1\left(\frac{x}{\sqrt{\varepsilon}}\right) \nabla \bar{\chi}_\varepsilon(x) + \frac{1}{\sqrt{\varepsilon}} \bar{\chi}_\varepsilon(x) \nabla v^1\left(\frac{x}{\sqrt{\varepsilon}}\right) \right) \Big] dx \\ &\quad + \varepsilon \int_{\mathbb{R}^N} \operatorname{div}_x \zeta\left(x, \frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} v^1\left(\frac{x}{\sqrt{\varepsilon}}\right) \bar{\chi}_\varepsilon(x) \, dx. \end{aligned} \quad (4.7)$$

The first and last terms on the right-hand side of (4.7) are of order ε as expected, but the second term is not. We therefore perform another integration by parts using the solution $\theta \in C^2(\mathbb{R}^N; C(\mathbb{T}^N)^{N^2})$ of

$$-\operatorname{div}_y(\theta(x, y)e^{2i\pi\xi^n \cdot y}) = \zeta(x, y)e^{2i\pi\xi^n \cdot y},$$

which is given by another application of [5, lemma 4.4]. Thus, we obtain

$$\begin{aligned} & \sqrt{\varepsilon} \int_{\mathbb{R}^N} \zeta\left(x, \frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} \cdot \bar{\chi}^\varepsilon(x) \nabla v^1\left(\frac{x}{\sqrt{\varepsilon}}\right) dx \\ &= \varepsilon^{3/2} \int_{\mathbb{R}^N} \theta\left(x, \frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} \cdot \nabla\left(\bar{\chi}^\varepsilon \nabla v^1\left(\frac{x}{\sqrt{\varepsilon}}\right)\right) dx \\ &+ \varepsilon^{3/2} \int_{\mathbb{R}^N} \operatorname{div}_x \theta\left(x, \frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} \cdot \left(\bar{\chi}^\varepsilon \nabla v^1\left(\frac{x}{\sqrt{\varepsilon}}\right)\right) dx, \end{aligned}$$

where all terms can be bounded by the Cauchy–Schwarz inequality, using the fact that ζ and θ are bounded functions. This leads to the desired result (4.6) and completes the proof of the lemma. $\square$

5. Weak convergence (proof of theorem 3.1)

For the sake of notational simplicity, we now drop the tilde notation, introduced in (3.11), which indicates a function of z instead of x . In a first step, we multiply equation (3.12) by a test function $\varepsilon \bar{\phi}^\varepsilon = \varepsilon \bar{\phi}(t, z, z/\sqrt{\varepsilon})$, where $\phi(t, z, y)$ is a smooth, compactly supported function defined on $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{T}^N$. Integrating by parts and using hypothesis 2.2, we obtain

$$\begin{aligned} & -2i\omega_n \varepsilon \int_0^{+\infty} \int_{\mathbb{R}^N} \rho^\varepsilon v_\varepsilon^+ \frac{\partial \bar{\phi}^\varepsilon}{\partial t} dz dt + \varepsilon^2 \int_0^{+\infty} \int_{\mathbb{R}^N} \rho^\varepsilon v_\varepsilon^+ \frac{\partial^2 \bar{\phi}^\varepsilon}{\partial t^2} dz dt \\ & -2i\omega_n \varepsilon \int_{\mathbb{R}^N} \rho^\varepsilon v_\varepsilon^+(0) \bar{\phi}^\varepsilon(0) dz - \varepsilon^2 \int_{\mathbb{R}^N} \rho^\varepsilon v_\varepsilon^+(0) \frac{\partial \bar{\phi}^\varepsilon}{\partial t}(0) dz \\ & + \varepsilon^2 \int_{\mathbb{R}^N} \rho^\varepsilon \frac{\partial v_\varepsilon^+}{\partial t}(0) \bar{\phi}^\varepsilon(0) dz \\ & + \int_0^{+\infty} \int_{\mathbb{R}^N} [A_0^\varepsilon + A_{1,h}^\varepsilon \sqrt{\varepsilon} z_h + \frac{1}{2} A_{2,lh}^\varepsilon \varepsilon z_l z_h + o(\varepsilon)] \\ & \quad \times (\sqrt{\varepsilon} \nabla + 2i\pi \xi^n) v_\varepsilon^+ \cdot (\sqrt{\varepsilon} \nabla - 2i\pi \xi^n) \bar{\phi}^\varepsilon dz dt \\ & - \int_0^{+\infty} \int_{\mathbb{R}^N} [\rho_0^\varepsilon + \rho_{1,h}^\varepsilon \sqrt{\varepsilon} z_h + \frac{1}{2} \rho_{2,lh}^\varepsilon \varepsilon z_l z_h + o(\varepsilon)] \lambda_n v_\varepsilon^+ \bar{\phi}^\varepsilon dz dt = 0. \end{aligned}$$

Note that the above $o(\varepsilon)$ holds true in the $L^\infty(\mathbb{R}^N)$ norm since the test function ϕ has compact support in z . From proposition 2.6 and because of the *a priori* estimate (3.13), there exist a subsequence and a limit $v^*(t, z, y) \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^N; H^1(\mathbb{T}^N)))$ such that $v_\varepsilon^+(t, z)$ and $\sqrt{\varepsilon} \nabla v_\varepsilon^+(t, z)$ two-scale converge to v^* and $\nabla_y v^*$, respectively. Passing to the two-scale limit in the above equality, we deduce a variational formulation for

$$-(\operatorname{div}_y + 2i\pi \xi^n)(A_0(y)(\nabla_y + 2i\pi \xi^n)v^*) = \rho_0 \lambda_n v^*.$$

By the simplicity of λ_n (hypothesis 2.1), there exists a scalar function $v^+(t, z) \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^N))$ such that

$$v^*(t, z, y) = v^+(t, z)\psi_n(y).$$

In a second step, we multiply equation (3.12) by the complex conjugate of

$$\Phi_\varepsilon(t, z) = \psi_n^\varepsilon(z)\phi(t, z) + \sqrt{\varepsilon} \sum_{k=1}^N \left(\frac{1}{2\pi i} \frac{\partial \psi_n^\varepsilon}{\partial \xi_k} \frac{\partial \phi}{\partial z_k} + z_k \frac{\partial \psi_n^\varepsilon}{\partial x_k} \phi \right),$$

where $\phi(t, z)$ is a smooth, compactly supported function defined on $\mathbb{R}^+ \times \mathbb{R}^N$. We decompose the resulting variational formulation in several pieces and pass to the limit in each of them separately.

The time derivative terms in (3.12) yield, first, by two successive integrations by parts,

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}^N} \varepsilon \rho^\varepsilon \frac{\partial^2 v_\varepsilon^+}{\partial t^2} \bar{\Phi}^\varepsilon \, dt \, dz \\ &= - \int_0^{+\infty} \int_{\mathbb{R}^N} \varepsilon \rho^\varepsilon \frac{\partial v_\varepsilon^+}{\partial t} \left[\bar{\psi}_n^\varepsilon \frac{\partial \bar{\phi}}{\partial t} + \sqrt{\varepsilon} \sum_{k=1}^N \left(-\frac{1}{2\pi i} \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial^2 \bar{\phi}}{\partial t \partial z_k} + z_k \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \frac{\partial \bar{\phi}}{\partial t} \right) \right] \\ &\quad - \int_{\mathbb{R}^N} \varepsilon \rho^\varepsilon \frac{\partial v_\varepsilon^+}{\partial t}(0) \left[\bar{\psi}_n^\varepsilon \bar{\phi}(0) + \sqrt{\varepsilon} \sum_{k=1}^N \left(-\frac{1}{2\pi i} \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial \bar{\phi}}{\partial z_k}(0) + z_k \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \bar{\phi}(0) \right) \right] \\ &= \int_0^{+\infty} \int_{\mathbb{R}^N} \varepsilon \rho^\varepsilon v_\varepsilon^+ \left[\bar{\psi}_n^\varepsilon \frac{\partial^2 \bar{\phi}}{\partial t^2} + \sqrt{\varepsilon} \sum_{k=1}^N \left(-\frac{1}{2\pi i} \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial^3 \bar{\phi}}{\partial t^2 \partial z_k} + z_k \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \frac{\partial^2 \bar{\phi}}{\partial t^2} \right) \right] \\ &\quad + \int_{\mathbb{R}^N} \varepsilon \rho^\varepsilon v_\varepsilon^+(0) \left[\bar{\psi}_n^\varepsilon \frac{\partial \bar{\phi}}{\partial t}(0) + \sqrt{\varepsilon} \sum_{k=1}^N \left(-\frac{1}{2\pi i} \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial^2 \bar{\phi}}{\partial t \partial z_k}(0) + z_k \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \frac{\partial \bar{\phi}}{\partial t}(0) \right) \right] \\ &\quad - \int_{\mathbb{R}^N} \varepsilon \rho^\varepsilon \frac{\partial v_\varepsilon^+}{\partial t}(0) \left[\bar{\psi}_n^\varepsilon \bar{\phi}(0) + \sqrt{\varepsilon} \sum_{k=1}^N \left(-\frac{1}{2\pi i} \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial \bar{\phi}}{\partial z_k}(0) + z_k \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \bar{\phi}(0) \right) \right] \\ &= - \int_{\mathbb{R}^N} \rho^\varepsilon |\psi_n^\varepsilon|^2 (-i\omega_n v^0 + v^1) \bar{\phi}(0) + o(1) \end{aligned} \tag{5.1}$$

and second, by a single integration by parts,

$$\begin{aligned} & 2i\omega_n \int_0^{+\infty} \int_{\mathbb{R}^N} \rho^\varepsilon \frac{\partial v_\varepsilon^+}{\partial t} \bar{\Phi}^\varepsilon \, dt \, dz \\ &= -2i\omega_n \int_0^{+\infty} \int_{\mathbb{R}^N} \rho^\varepsilon v_\varepsilon^+ \left[\bar{\psi}_n^\varepsilon \frac{\partial \bar{\phi}}{\partial t} + \sqrt{\varepsilon} \sum_{k=1}^N \left(-\frac{1}{2\pi i} \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial^2 \bar{\phi}}{\partial t \partial z_k} + z_k \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \frac{\partial \bar{\phi}}{\partial t} \right) \right] \\ &\quad - 2i\omega_n \int_{\mathbb{R}^N} \rho^\varepsilon v_\varepsilon^+(0) \left[\bar{\psi}_n^\varepsilon(0) \bar{\phi} + \sqrt{\varepsilon} \sum_{k=1}^N \left(-\frac{1}{2\pi i} \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial \bar{\phi}}{\partial z_k}(0) + z_k \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \bar{\phi}(0) \right) \right] \\ &= -2i\omega_n \int_0^{+\infty} \int_{\mathbb{R}^N} \rho^\varepsilon v_\varepsilon^+ \bar{\psi}_n^\varepsilon \frac{\partial \bar{\phi}}{\partial t} - 2i\omega_n \int_{\mathbb{R}^N} \rho^\varepsilon |\psi_n^\varepsilon|^2 v^0 \bar{\phi}(0) + o(1). \end{aligned} \tag{5.2}$$

Passing to the two-scale limit in (5.1) added to (5.2) and recalling the normalization (1.4), we find that

$$-2i\omega_n \int_0^{+\infty} \int_{\mathbb{R}^N} v^+ \frac{\partial \bar{\phi}}{\partial t} dt dz - \int_{\mathbb{R}^N} (i\omega_n v^0 + v^1) \bar{\phi}(0) dz. \quad (5.3)$$

We further decompose the test function Φ_ε as follows:

$$\Phi_\varepsilon = \Phi_\varepsilon^1 + \Phi_\varepsilon^2 \cdot z \quad \text{with } \Phi_\varepsilon^2 = \sqrt{\varepsilon} \sum_{k=1}^N \frac{\partial \psi_n^\varepsilon}{\partial x_k} \phi(t, z) e_k.$$

In the remaining terms of the variational formulation of (3.12), we replace A^ε by its Taylor expansion (according to hypothesis 2.2) and we look first to those terms which are of zero order with respect to z . They are

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}^N} \left[A_0^\varepsilon \left(\nabla + \frac{2\pi i \xi^n}{\sqrt{\varepsilon}} \right) v_\varepsilon^+ \cdot \left(\left(\nabla - \frac{2\pi i \xi^n}{\sqrt{\varepsilon}} \right) \bar{\Phi}_\varepsilon^1 + \bar{\Phi}_\varepsilon^2 \right) - \frac{1}{\varepsilon} \rho_0^\varepsilon \lambda_n \Phi_\varepsilon^1 \right] dz dt \\ &= \int_0^{+\infty} \int_{\mathbb{R}^N} \left[\frac{1}{\varepsilon} A_0^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \cdot (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n^\varepsilon \bar{\phi} - \frac{1}{\varepsilon} \rho_0^\varepsilon \lambda_n \bar{\psi}_n^\varepsilon v_\varepsilon^+ \bar{\phi} \right] dz dt \\ &\quad - \frac{1}{2i\pi} \int_0^{+\infty} \int_{\mathbb{R}^N} \left[\frac{1}{\sqrt{\varepsilon}} A_0^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \cdot (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial \bar{\phi}}{\partial z_k} \right. \\ &\quad \left. - \frac{1}{\sqrt{\varepsilon}} \rho_0^\varepsilon \lambda_n v_\varepsilon^+ \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial \bar{\phi}}{\partial z_k} \right] dz dt \\ &\quad + \int_0^{+\infty} \int_{\mathbb{R}^N} \frac{1}{\sqrt{\varepsilon}} A_0^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \cdot \bar{\psi}_n^\varepsilon \nabla \bar{\phi} dz dt \\ &\quad + \int_0^{+\infty} \int_{\mathbb{R}^N} -\frac{1}{2\pi i} A_0^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \cdot \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \nabla \frac{\partial \bar{\phi}}{\partial z_k} dz dt \\ &\quad + \int_0^{+\infty} \int_{\mathbb{R}^N} A_0^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \cdot \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \bar{\phi} e_k dz dt. \end{aligned} \quad (5.4)$$

Using equation (2.9) with $\varphi = v_\varepsilon^+ \bar{\phi}$, and equation (2.10) with $\varphi = v_\varepsilon^+ (\partial \bar{\phi} / \partial z_k)$, we rewrite the first two integrals on the right-hand side of (5.4) as follows:

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}^N} -\frac{1}{\sqrt{\varepsilon}} A_0^\varepsilon (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n^\varepsilon \cdot v_\varepsilon^+ \nabla \bar{\phi} dz dt \\ &+ \int_0^{+\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2\pi i} A_0^\varepsilon (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \cdot v_\varepsilon^+ \nabla \frac{\partial \bar{\phi}}{\partial z_k} \right. \\ &\quad \left. + \frac{1}{\sqrt{\varepsilon}} A_0^\varepsilon e_k \cdot v_\varepsilon^+ \frac{\partial \bar{\phi}}{\partial z_k} (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n^\varepsilon \right. \\ &\quad \left. - \frac{1}{\sqrt{\varepsilon}} A_0^\varepsilon \bar{\psi}_n^\varepsilon e_k \cdot (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) \left(v_\varepsilon^+ \frac{\partial \bar{\phi}}{\partial z_k} \right) \right] dz dt. \end{aligned}$$

Combining the above terms with the other terms in (5.4) and passing to the two-scale limit in (5.4) yields

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \left[\frac{1}{2\pi i} A_0 \psi_n (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n}{\partial \xi_k} - \frac{1}{2\pi i} A_0 \frac{\partial \bar{\psi}_n}{\partial \xi_k} (\nabla_y + 2\pi i \xi^n) \psi_n - A_0 |\psi_n|^2 e_k \right] \\ & \quad \times v^+ \nabla \frac{\partial \bar{\phi}}{\partial z_k} dy dz dt + \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} A_0 (\nabla_y + 2\pi i \xi^n) \psi_n \cdot \frac{\partial \bar{\psi}_n}{\partial x_k} v^+ \bar{\phi} e_k dy dz dt. \end{aligned} \quad (5.5)$$

By equation (2.8), the first integral of (5.5) is equal to

$$\int_0^{+\infty} \int_{\mathbb{R}^N} A^* \nabla v^+ \nabla \bar{\phi} dz dt, \quad (5.6)$$

while the second integral of (5.5) contributes to the first term in formula (3.4) for c^* .

Next we consider the terms that are linear in z in the variational formulation of (3.12):

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}^N} \left[A_0^\varepsilon \left(\nabla + \frac{2\pi i \xi^n}{\sqrt{\varepsilon}} \right) v_\varepsilon^+ \cdot (z \nabla \bar{\psi}_\varepsilon^2) - \frac{1}{\varepsilon} \rho_0^\varepsilon \lambda_n v_\varepsilon^+ \bar{\psi}_\varepsilon^2 z \right] dz dt \\ & + \int_0^{+\infty} \int_{\mathbb{R}^N} \left[A_{1,k}^\varepsilon \sqrt{\varepsilon} z_k \left(\nabla + \frac{2\pi i \xi^n}{\sqrt{\varepsilon}} \right) v_\varepsilon^+ \cdot \left(\left(\nabla - \frac{2\pi i \xi^n}{\sqrt{\varepsilon}} \right) \bar{\psi}_\varepsilon^1 + \bar{\psi}_\varepsilon^2 \right) \right] dz dt \\ & = \int_0^{+\infty} \int_{\mathbb{R}^N} \left[\frac{1}{\sqrt{\varepsilon}} A_0^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \right. \\ & \quad \times (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \bar{\phi} z_k - \frac{1}{\sqrt{\varepsilon}} \lambda_n \rho_0^\varepsilon v_\varepsilon^+ \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \bar{\phi} z_k \left. \right] dz dt \\ & + \int_0^{+\infty} \int_{\mathbb{R}^N} \left[\frac{1}{\sqrt{\varepsilon}} A_{1,k}^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \right. \\ & \quad \times (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n^\varepsilon \bar{\phi} z_k - \frac{1}{\sqrt{\varepsilon}} \lambda_n \rho_{1,k}^\varepsilon v_\varepsilon^+ \bar{\psi}_n^\varepsilon \bar{\phi} z_k \left. \right] dz dt \\ & + \int_0^{+\infty} \int_{\mathbb{R}^N} \left[A_0^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \right. \\ & \quad \times \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \nabla \bar{\phi} z_k + A_{1,k}^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \cdot \psi_n^\varepsilon \nabla \bar{\phi} z_k \left. \right] dz dt \\ & - \frac{1}{2\pi i} \int_0^{+\infty} \int_{\mathbb{R}^N} \left[A_{1,h}^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \cdot (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial \bar{\phi}}{\partial z_k} z_h \right. \\ & \quad \left. - \lambda_n \rho_{1,h}^\varepsilon v_\varepsilon^+ \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \frac{\partial \bar{\phi}}{\partial z_k} z_h \right] dz dt \\ & + \int_0^{+\infty} \int_{\mathbb{R}^N} \left[\sqrt{\varepsilon} A_{1,h}^\varepsilon (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \right. \\ & \quad \times \left(-\frac{1}{2\pi i} \frac{\partial \bar{\psi}_n^\varepsilon}{\partial \xi_k} \nabla \frac{\partial \bar{\phi}}{\partial z_k} + \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \bar{\phi} e_k \right) z_h \left. \right] dz dt. \end{aligned} \quad (5.7)$$

By equation (2.11) with $\varphi = v_\varepsilon^+ \bar{\phi} z_k$, and since

$$\nabla(\phi z_k) = \phi e_k + z_k \nabla \phi,$$

the sum of the first two integrals on the right-hand side of (5.7) gives

$$\begin{aligned} & - \int_0^{+\infty} \int_{\mathbb{R}^N} \left(A_0^\varepsilon (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \cdot v_\varepsilon^+ \nabla(\bar{\phi} z_k) \right. \\ & \quad \left. + A_{1,k}^\varepsilon (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n^\varepsilon \cdot v_\varepsilon^+ \nabla(\bar{\phi} z_k) \right) dz dt. \end{aligned} \quad (5.8)$$

Therefore, passing to the two-scale limit in (5.7) we have

$$\begin{aligned} & - \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \left[A_0 (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n}{\partial x_k} \cdot v^+ \psi_n \bar{\phi} e_k \right. \\ & \quad \left. + A_{1,k} (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n \cdot v^+ \psi_n \bar{\phi} e_k \right] dy dz dt \\ & - \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \left[A_0 (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n}{\partial x_k} \cdot v^+ \psi_n z_k \nabla \bar{\phi} \right. \\ & \quad \left. + A_{1,k} (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n \cdot v^+ \psi_n z_k \nabla \bar{\phi} \right] dy dz dt \\ & + \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \left[A_0 (\nabla_y + 2\pi i \xi^n) \psi_n \cdot v^+ \frac{\partial \bar{\psi}_n}{\partial x_k} z_k \nabla \bar{\phi} \right. \\ & \quad \left. + A_{1,k} (\nabla_y + 2\pi i \xi^n) \psi_n \cdot v^+ \bar{\psi}_n z_k \nabla \bar{\phi} \right] dy dz dt \\ & - \frac{1}{2\pi i} \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \left[A_{1,h} (\nabla_y + 2\pi i \xi^n) \psi_n \cdot (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n}{\partial \xi_k} v^+ z_h \frac{\partial \bar{\phi}}{\partial z_k} \right. \\ & \quad \left. - \lambda_n \rho_{1,h} v^+ \psi_n \frac{\partial \bar{\psi}_n}{\partial \xi_k} \frac{\partial \bar{\phi}}{\partial z_k} z_h \right] dy dz dt. \end{aligned} \quad (5.9)$$

By equation (2.6) it follows that the last integral in (5.9) is equal to

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \left[A_{1,h} \psi_n e_k \cdot (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n \right. \\ & \quad \left. + A_0 \psi_n e_k \cdot (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n}{\partial x_h} \right] v^+ z_h \frac{\partial \bar{\phi}}{\partial z_k} dy dz dt \\ & - \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \left[A_{1,h} \bar{\psi}_n e_k \cdot (\nabla_y + 2\pi i \xi^n) \psi_n \right. \\ & \quad \left. + A_0 \frac{\partial \bar{\psi}_n}{\partial x_h} e_k \cdot (\nabla_y + 2\pi i \xi^n) \psi_n \right] v^+ z_h \frac{\partial \bar{\phi}}{\partial z_k} dy dz dt \\ & - \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \frac{1}{2\pi i} \frac{\partial^2 \lambda_n}{\partial x_h \partial \xi_k} |\psi_n|^2 v^+ z_h \frac{\partial \bar{\phi}}{\partial z_k} dy dz dt. \end{aligned} \quad (5.10)$$

Note that the first and the second line of (5.10) cancel out with the second and third line of (5.9), respectively, and (5.9) therefore reduces to

$$\begin{aligned} & - \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \left[A_0(\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n}{\partial x_k} \cdot v^+ \psi_n \bar{\phi} e_k \right. \\ & \quad \left. + A_{1,k}(\nabla_y - 2\pi i \xi^n) \bar{\psi}_n \cdot v^+ \psi_n \bar{\phi} e_k \right] dy dz dt \\ & \quad - \int_0^{+\infty} \int_{\mathbb{R}^N} \frac{1}{2i\pi} \frac{\partial^2 \lambda_n}{\partial x_h \partial \xi_k} v^+ \frac{\partial \bar{\phi}}{\partial z_k} z_h dz dt. \end{aligned} \quad (5.11)$$

Finally, we consider all quadratic in z terms in the variational formulation of (3.12):

$$\begin{aligned} & \frac{1}{2} \int_0^{+\infty} \int_{\mathbb{R}^N} \left[A_{2,lh}^\varepsilon \varepsilon z_l z_h \left(\nabla + \frac{2i\pi \xi^n}{\sqrt{\varepsilon}} \right) v_\varepsilon^+ \right. \\ & \quad \left. \times \left(\left(\nabla - \frac{2i\pi \xi^n}{\sqrt{\varepsilon}} \right) \bar{\psi}_\varepsilon^1 + \bar{\psi}_\varepsilon^2 \right) - \lambda_n \rho_{2,lh}^\varepsilon z_l z_h v_\varepsilon^+ \bar{\psi}_\varepsilon^1 \right] dz dt \\ & + \int_0^{+\infty} \int_{\mathbb{R}^N} \left[A_{1,k}^\varepsilon \varepsilon z_k \left(\nabla - \frac{2i\pi \xi^n}{\sqrt{\varepsilon}} \right) v_\varepsilon^+ \cdot (z \nabla \bar{\psi}_\varepsilon^2) - \lambda_n \rho_{1,k}^\varepsilon z_k v_\varepsilon^+ z \cdot \bar{\psi}_\varepsilon^2 \right] dz dt \\ & = \frac{1}{2} \int_0^{+\infty} \int_{\mathbb{R}^N} A_{2,lh}^\varepsilon \varepsilon z_l z_h (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \cdot (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n^\varepsilon \bar{\phi} dz dt \\ & \quad + \frac{1}{2} \int_0^{+\infty} \int_{\mathbb{R}^N} A_{1,h}^\varepsilon z_h (\sqrt{\varepsilon} \nabla + 2\pi i \xi^n) v_\varepsilon^+ \cdot z_k (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \bar{\phi} dz dt \\ & \quad - \int_0^{+\infty} \int_{\mathbb{R}^N} \frac{1}{2} \lambda_n \rho_{2,lh}^\varepsilon z_l z_h v_\varepsilon^+ \bar{\psi}_n^\varepsilon \bar{\phi} dz dt \\ & \quad - \int_0^{+\infty} \int_{\mathbb{R}^N} \lambda_n \rho_{1,h}^\varepsilon z_h v_\varepsilon^+ z_k \frac{\partial \bar{\psi}_n^\varepsilon}{\partial x_k} \bar{\phi} dz dt + o(1), \end{aligned} \quad (5.12)$$

which give, on passing to the two-scale limit,

$$\begin{aligned} & \frac{1}{2} \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} [A_{2,lh}(\nabla_y + 2\pi i \xi^n) \psi_n \cdot (\nabla_y - 2\pi i \xi^n) \bar{\psi}_n - \lambda_n \rho_{2,lh} \psi_n \bar{\psi}_n] \\ & \quad \times v^+ \bar{\phi} z_l z_h dy dz dt \\ & + \int_0^{+\infty} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \left[A_{1,h}(\nabla_y + 2\pi i \xi^n) \psi_n \cdot (\nabla_y - 2\pi i \xi^n) \frac{\partial \bar{\psi}_n}{\partial x_k} - \lambda_n \rho_{1,h} \psi_n \frac{\partial \bar{\psi}_n}{\partial x_k} \right] \\ & \quad \times v^+ \bar{\phi} z_l z_h dy dz dt. \end{aligned} \quad (5.13)$$

Now, using equation (2.7), we find that (5.13) reduces to

$$\int_0^{+\infty} \int_{\mathbb{R}^N} \frac{1}{2} \frac{\partial^2 \lambda_n}{\partial x_l \partial x_h} v^+ \bar{\phi} z_l z_h dz dt. \quad (5.14)$$

Summing together (5.3), (5.5), (5.6), (5.11) and (5.14) yields the weak formulation of (3.2). We know by proposition 3.5 that (3.2) admits a unique solution. Therefore, the entire sequence v_ε^+ , and not merely a subsequence, converges. Of course, a symmetric proof works for the other sequence v_ε^- corresponding to the opposite phase.

6. Strong convergence (proof of theorem 3.2)

Recall that $u_\varepsilon^{\text{approx}}$ has an L^2 norm of order $\varepsilon^{N/4}$, so we expect the same for u_ε . Therefore, we define the difference between them in the scale of the z variable. In other words we define $r_\varepsilon(t, z)$ as

$$\begin{aligned} r_\varepsilon\left(t, \frac{x}{\sqrt{\varepsilon}}\right) &= u_\varepsilon(t, x) - \psi_n\left(\frac{x}{\varepsilon}\right) \exp\left\{2i\pi \frac{\xi^n \cdot x}{\varepsilon}\right\} \\ &\quad \times \left(e^{i\omega_n t/\varepsilon} v^+\left(t, \frac{x}{\sqrt{\varepsilon}}\right) + e^{-i\omega_n t/\varepsilon} v^-\left(t, \frac{x}{\sqrt{\varepsilon}}\right)\right). \end{aligned} \quad (6.1)$$

Theorem 3.2 proves that $r_\varepsilon(t, z)$ converges strongly to zero in $L^2((0, T) \times \mathbb{R}^N)$.

We begin with a technical lemma, the proof of which is postponed for a moment.

LEMMA 6.1. *The solution of (1.1) satisfies*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{N/2}} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) |u_\varepsilon(t, x)|^2 dx dt = \|v^+\|_{L^2((0, T) \times \mathbb{R}^N)}^2 + \|v^-\|_{L^2((0, T) \times \mathbb{R}^N)}^2. \quad (6.2)$$

Equipped with lemma 6.1 we are now able to prove theorem 3.2. We develop the following norm:

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \rho\left(\sqrt{\varepsilon}z, \frac{z}{\sqrt{\varepsilon}}\right) |r_\varepsilon(t, z)|^2 dz dt \\ &= \frac{1}{\varepsilon^{N/2}} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \left|r_\varepsilon\left(t, \frac{x}{\sqrt{\varepsilon}}\right)\right|^2 dx dt \\ &= \frac{1}{\varepsilon^{N/2}} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) |u_\varepsilon(t, x)|^2 dx dt \\ &\quad + \frac{1}{\varepsilon^{N/2}} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \left|\psi_n\left(\frac{x}{\varepsilon}\right) v^+\left(t, \frac{x}{\sqrt{\varepsilon}}\right)\right|^2 dx dt \\ &\quad + \frac{1}{\varepsilon^{N/2}} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \left|\psi_n\left(\frac{x}{\varepsilon}\right) v^-\left(t, \frac{x}{\sqrt{\varepsilon}}\right)\right|^2 dx dt \\ &\quad - \frac{2}{\varepsilon^{N/2}} \mathcal{R} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) v_\varepsilon^+(t, x) \bar{\psi}_n\left(\frac{x}{\varepsilon}\right) \overline{v^+}\left(t, \frac{x}{\sqrt{\varepsilon}}\right) dx dt \\ &\quad - \frac{2}{\varepsilon^{N/2}} \mathcal{R} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) v_\varepsilon^-(t, x) \bar{\psi}_n\left(\frac{x}{\varepsilon}\right) \overline{v^-}\left(t, \frac{x}{\sqrt{\varepsilon}}\right) dx dt \\ &\quad + \frac{2}{\varepsilon^{N/2}} \mathcal{R} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \left|\psi_n\left(\frac{x}{\varepsilon}\right)\right|^2 \\ &\quad \quad \times \overline{v^+}\left(t, \frac{x}{\sqrt{\varepsilon}}\right) v^-\left(t, \frac{x}{\sqrt{\varepsilon}}\right) e^{-2i\omega_n t/\varepsilon} dx dt, \end{aligned} \quad (6.3)$$

where v_ε^+ , and v_ε^- are defined by (3.1) and (3.5), while $\mathcal{R}$ denotes the real part of a complex number. By virtue of lemma 6.1 we can pass to the limit in the first integral on the right-hand side of (6.3). By using the changes of variables $z = x/\sqrt{\varepsilon}$ and recalling that

$$\left| \rho\left(\sqrt{\varepsilon}z, \frac{z}{\sqrt{\varepsilon}}\right) - \rho\left(0, \frac{z}{\sqrt{\varepsilon}}\right) \right| \leq C \min\{\sqrt{\varepsilon}z, 1\}, \quad (6.4)$$

it is easy to pass to the limit in the second and third integrals on the right-hand side of (6.3) to obtain

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{N/2}} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \left| \psi_n\left(\frac{x}{\varepsilon}\right) v^\pm\left(t, \frac{x}{\sqrt{\varepsilon}}\right) \right|^2 dx dt = \|v^\pm\|_{L^2((0,T) \times \mathbb{R}^N)}^2.$$

We then pass to the two-scale limit in the fourth and fifth integrals on the right-hand side of (6.3):

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^{N/2}} \mathcal{R} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) v_\varepsilon^\pm(t, x) \bar{\psi}_n\left(\frac{x}{\varepsilon}\right) \overline{v^\pm\left(t, \frac{x}{\sqrt{\varepsilon}}\right)} dx dt \\ = 2\mathcal{R} \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} \rho_0(y) |\psi_n(y)|^2 |v^\pm(t, z)|^2 dy dz dt \\ = 2\|v^\pm\|_{L^2((0,T) \times \mathbb{R}^N)}^2. \end{aligned}$$

To show that the limit of the last integral on the right-hand side of (6.3) is actually zero, we take advantage of the time-oscillating phase. We define

$$K_\varepsilon(t) = \frac{1}{\varepsilon^{N/2}} \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \left| \psi_n\left(\frac{x}{\varepsilon}\right) \right|^2 v^+\left(t, \frac{x}{\sqrt{\varepsilon}}\right) v^-\left(t, \frac{x}{\sqrt{\varepsilon}}\right) dx,$$

which is a continuous function of time (because $v^\pm$ belong to $C(\mathbb{R}^+; L^2(\mathbb{R}^N))$) and which converges uniformly to a continuous limit $K(t)$ as ε goes to zero. Therefore, since $\omega_n \neq 0$,

$$\lim_{\varepsilon \rightarrow 0} \int_0^T K_\varepsilon(t) e^{-2i\omega_n t/\varepsilon} dt = 0.$$

We thus obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^N} \rho\left(\sqrt{\varepsilon}z, \frac{z}{\sqrt{\varepsilon}}\right) |r_\varepsilon(t, z)|^2 dz dt = 2\|v^+\|^2 + 2\|v^-\|^2 - 2\|v^+\|^2 - 2\|v^-\|^2 = 0,$$

which completes the proof of theorem 3.2.

It remains to prove lemma 6.1: a long task partitioned in several other lemmas. The main idea is to use the energy conservations for the ε -oscillating wave equation and the homogenized Schrödinger equations to show that the initial energy of the wave equation converges to the sum of the initial energies of the two homogenized Schrödinger equations. However, there are additional technical difficulties. Firstly, the left-hand side of (6.2) does not involve the energy of the wave equation (1.1). Therefore, we rely on the energy of a time primitive of (1.1) (as introduced in § 4) and on an energy equipartition (see lemma 6.5). Secondly, refined *a priori* estimates

are necessary and a second time regularization has to be introduced (see lemmas 6.4 and 6.2).

Let us start by recalling the definition (4.1) of the time primitive of the solution u_ε of (1.1):

$$\Psi_\varepsilon(x, t) = e^{-\alpha t} \left[\int_0^t e^{\alpha s} u_\varepsilon(x, s) ds + \chi_\varepsilon(x) \right]$$

with χ_ε solution of (4.2). We know that Ψ_ε is the solution of another wave equation, (4.3). We improve the *a priori* estimate of lemma 4.1 on the initial datum $\chi_\varepsilon(x)$. Surprisingly, the result is different according to the value of ξ^n .

LEMMA 6.2. *Define the sequence*

$$w_\varepsilon(z) = \frac{1}{\varepsilon} \chi_\varepsilon(\sqrt{\varepsilon} z) \exp \left\{ -2i\pi \frac{\xi^n \cdot z}{\sqrt{\varepsilon}} \right\}. \quad (6.5)$$

If $\xi^n \neq 0$, it satisfies

$$\sqrt{\varepsilon} \|\nabla_z w_\varepsilon\|_{L^2(\mathbb{R}^N)} + \|w_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C(\xi^n) (\|v^0\|_{L^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}), \quad (6.6)$$

where the constant $C(\xi^n)$ does not depend on ε and it two-scale converges strongly to $-(\psi_n(y)/\lambda_n v^1(z))$, while $\sqrt{\varepsilon} \nabla w_\varepsilon(z)$ two-scale converges strongly to

$$-\frac{\nabla_y \psi_n(y)}{\lambda_n v^1(z)}.$$

If $\xi^n = 0$, it satisfies

$$\sqrt{\varepsilon} \|\nabla_z w_\varepsilon\|_{L^2(\mathbb{R}^N)} + \sqrt{\varepsilon} \|w_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C(\|v^0\|_{L^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}), \quad (6.7)$$

where the constant $C(\xi^n)$ does not depend on ε and $\sqrt{\varepsilon} \nabla w_\varepsilon(z)$ two-scale converges strongly to $-(\nabla_y \psi_n(y)/\lambda_n) v^1(z)$.

We postpone the proof of lemma 6.2 for a moment and we introduce the energy functional for Ψ_ε :

$$E(\Psi_\varepsilon)(t) = \frac{1}{2} \int_{\mathbb{R}^N} \left(\rho \left(x, \frac{x}{\varepsilon} \right) \left| \frac{\partial \Psi_\varepsilon}{\partial t} \right|^2 + A \left(x, \frac{x}{\varepsilon} \right) \nabla \Psi_\varepsilon \cdot \nabla \bar{\Psi}_\varepsilon \right) dx.$$

LEMMA 6.3. *The energy of the initial data satisfies*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{N/2}} E(\Psi_\varepsilon)(0) = \frac{1}{2} \left(\|v^0\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{\omega_n^2} \|v^1\|_{L^2(\mathbb{R}^N)}^2 \right).$$

Proof. In view of the wave equation (4.3), we have

$$E(\Psi_\varepsilon)(0) = \frac{1}{2} \int_{\mathbb{R}^N} \left(\rho \left(x, \frac{x}{\varepsilon} \right) |u_\varepsilon^0(x) - \alpha \chi_\varepsilon(x)|^2 + A \left(x, \frac{x}{\varepsilon} \right) \nabla \chi_\varepsilon(x) \cdot \nabla \bar{\chi}_\varepsilon(x) \right) dx.$$

By the change of variables $z = x/\sqrt{\varepsilon}$, we rewrite

$$E(\Psi_\varepsilon)(0) = \frac{\varepsilon^{N/2}}{2} \int_{\mathbb{R}^N} \left(\tilde{\rho}^\varepsilon |\tilde{u}_\varepsilon^0(z) - \alpha \tilde{\chi}_\varepsilon(z)|^2 + \frac{1}{\varepsilon} \tilde{A}^\varepsilon \nabla_z \tilde{\chi}_\varepsilon(z) \cdot \nabla_z \bar{\tilde{\chi}}_\varepsilon(z) \right) dz$$

with the notation (3.11) for $\tilde{\rho}^\varepsilon(x) = \rho(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon})$ (and similarly for other functions) which satisfies (6.4). From lemma 6.2 we deduce that, as ε goes to 0,

$$\int_{\mathbb{R}^N} \tilde{\rho}^\varepsilon |\tilde{u}_\varepsilon^0 - \alpha \tilde{\chi}_\varepsilon|^2 dz = \int_{\mathbb{R}^N} \tilde{\rho}^\varepsilon |\tilde{\psi}_n^\varepsilon v^0 - \alpha \varepsilon w_\varepsilon|^2 dz \rightarrow \|v^0\|_{L^2(\mathbb{R}^N)}^2$$

because, for any value of ξ^n , $\sqrt{\varepsilon}w_\varepsilon$ is bounded in $L^2(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} \left(\frac{1}{\varepsilon} \tilde{A}^\varepsilon \nabla_z \tilde{\chi}_\varepsilon \cdot \nabla_z \bar{\tilde{\chi}}_\varepsilon \right) dz = \int_{\mathbb{R}^N} \tilde{A}^\varepsilon (\sqrt{\varepsilon} \nabla_z + 2i\pi \xi^n) w_\varepsilon \cdot (\sqrt{\varepsilon} \nabla_z - 2i\pi \xi^n) \bar{w}_\varepsilon dz$$

converges to

$$\begin{aligned} \frac{1}{\lambda_n^2} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} A_0(y) (\nabla_y + 2i\pi \xi^n) \psi_n(y) \\ \times (\nabla_y - 2i\pi \xi^n) \bar{\psi}_n(y) |v^1(z)|^2 dy dz = \frac{1}{\lambda_n} \int_{\mathbb{R}^N} |v^1(z)|^2 dz \end{aligned}$$

because of the strong two-scale convergence of lemma 6.2 (note that, when $\xi^n = 0$, we do not need the convergence of w_ε). Summing these two limits completes the proof of lemma 6.3. $\square$

Although Ψ_ε has a finite non-vanishing energy, as shown in lemma 6.3, its amplitude is asymptotically vanishing (but of course not its first derivatives).

LEMMA 6.4. *For any final time $T > 0$ there exists a constant $C(T) > 0$, independent of ε , such that*

$$\|\Psi_\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^N))} \leq C(T) \varepsilon^{N/4+1/2} (\|v^0\|_{H^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}). \quad (6.8)$$

Proof. As in § 4, where we introduced a time primitive Ψ_ε of u_ε , we now iterate this regularization procedure and introduce a new time primitive Φ_ε of Ψ_ε , defined as

$$\Phi_\varepsilon(x, t) = e^{-t} \left[\int_0^t e^s \Psi_\varepsilon(x, s) ds + \pi_\varepsilon \right],$$

where π_ε is the unique solution in $H^1(\mathbb{R}^N)$ of the equation

$$-\operatorname{div} \left(A \left(x, \frac{x}{\varepsilon} \right) \nabla \pi_\varepsilon \right) + \rho \left(x, \frac{x}{\varepsilon} \right) \pi_\varepsilon = 2\rho \left(x, \frac{x}{\varepsilon} \right) \chi_\varepsilon - \rho \left(x, \frac{x}{\varepsilon} \right) u_\varepsilon^0. \quad (6.9)$$

Then Φ_ε satisfies

$$\left. \begin{aligned} \rho \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 \Phi_\varepsilon}{\partial t^2} - \operatorname{div} \left(A \left(x, \frac{x}{\varepsilon} \right) \nabla \Phi_\varepsilon \right) &= 0 && \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ \Phi_\varepsilon(0) &= \pi_\varepsilon && \text{in } \mathbb{R}^N, \\ \frac{\partial \Phi_\varepsilon}{\partial t}(0) &= \chi_\varepsilon - \pi_\varepsilon && \text{in } \mathbb{R}^N. \end{aligned} \right\} \quad (6.10)$$

From lemma 6.2, we know that

$$\|\chi_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C \varepsilon^{N/4+1/2} (\|v^0\|_{L^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}). \quad (6.11)$$

Note that the estimate (6.11) is optimal when $\xi^n = 0$ but is pessimistic when $\xi^n \neq 0$, because the factor $\varepsilon^{N/4+1/2}$ could be replaced by $\varepsilon^{N/4+1}$. However, we do not need this refinement in what follows. Reproducing the arguments of the proof of lemma 4.1, we can prove that

$$\left| \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) u_\varepsilon^0 \bar{\pi}_\varepsilon(x) dx \right| \leq C \varepsilon^{N/4+1} \|v^0\|_{H^2(\mathbb{R}^N)} \|\pi_\varepsilon\|_{H^1(\mathbb{R}^N)}.$$

Therefore, from the energy equality of (6.9) we deduce that

$$\|\pi_\varepsilon\|_{H^1(\mathbb{R}^N)} \leq C \varepsilon^{N/4+1/2} (\|v^0\|_{H^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}). \quad (6.12)$$

The standard energy conservation for (6.10), together with (6.11) and (6.12), imply that

$$\left\| \frac{\partial \Phi_\varepsilon}{\partial t} \right\| \leq C \varepsilon^{N/4+1/2} (\|v^0\|_{H^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}) \quad (6.13)$$

which, combined with the relations

$$\Phi_\varepsilon(t) = \pi_\varepsilon + \int_0^t \frac{\partial \Phi_\varepsilon}{\partial t}(s) ds \quad \text{and} \quad \Psi_\varepsilon = \frac{\partial \Phi_\varepsilon}{\partial t} + \Phi_\varepsilon,$$

yields the desired result (6.8). $\square$

We now prove a result on the equipartition of the energy which shows that the kinetic energy is essentially equal to the deformation energy.

LEMMA 6.5. *The following equality holds:*

$$\int_0^T E(\Psi_\varepsilon(t)) dt = \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \left| \frac{\partial \Psi_\varepsilon}{\partial t} \right|^2 dx dt + o(\varepsilon^{N/2}).$$

Proof. We multiply equation (4.3) by $\bar{\Psi}_\varepsilon(t, x)\phi(t)$, where ϕ is a smooth function defined on $\mathbb{R}^+$. Integrating by parts, we obtain

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^N} \phi(t) \rho\left(x, \frac{x}{\varepsilon}\right) \left| \frac{\partial \Psi_\varepsilon}{\partial t} \right|^2 dx dt + \int_0^T \int_{\mathbb{R}^N} \phi(t) A\left(x, \frac{x}{\varepsilon}\right) \nabla \Psi_\varepsilon \cdot \nabla \bar{\Psi}_\varepsilon dx dt \\ & - \int_0^T \frac{\partial \phi}{\partial t}(t) \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \frac{\partial \Psi_\varepsilon}{\partial t} \bar{\Psi}_\varepsilon dx dt + \left[\int_{\mathbb{R}^N} \phi(t) \rho\left(x, \frac{x}{\varepsilon}\right) \frac{\partial \Psi_\varepsilon}{\partial t} \bar{\Psi}_\varepsilon dx \right]_0^T = 0. \end{aligned} \quad (6.14)$$

From (6.8) and (4.5) we have that, for almost every $t \in [0, T]$,

$$\left| \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \frac{\partial \Psi_\varepsilon}{\partial t} \bar{\Psi}_\varepsilon dx \right| \leq C(T) \varepsilon^{N/2+1/2}.$$

Thus, (6.14) can be rewritten as

$$\int_0^T \phi(t) \left[\int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \left| \frac{\partial \Psi_\varepsilon}{\partial t} \right|^2 dx - \int_{\mathbb{R}^N} A\left(x, \frac{x}{\varepsilon}\right) \nabla \Psi_\varepsilon \cdot \nabla \bar{\Psi}_\varepsilon dx \right] dt = o(\varepsilon^{N/2}). \quad (6.15)$$

Taking $\phi \equiv 1$ yields the desired result. $\square$

Proof of lemma 6.1. The limit of the initial energy for the solution Ψ_ε of the wave equation (4.3) is given by lemma 6.3 and it coincides with the sum of the initial energies of the homogenized Schrödinger equations (3.2) and (3.6). Indeed, we have

$$\begin{aligned} \|v^+(0, \cdot)\|_{L^2(\mathbb{R}^N)}^2 + \|v^-(0, \cdot)\|_{L^2(\mathbb{R}^N)}^2 &= \frac{1}{2} \left(\|v^0\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{\omega_n^2} \|v^1\|_{L^2(\mathbb{R}^N)}^2 \right) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{N/2}} E(\Psi_\varepsilon)(0). \end{aligned}$$

From this equality, using the energy conservations for the wave equation (4.3) and the homogenized Schrödinger equations (as proved in proposition 3.5), we deduce that

$$\|v^+(t, \cdot)\|_{L^2(\mathbb{R}^N)}^2 + \|v^-(t, \cdot)\|_{L^2(\mathbb{R}^N)}^2 = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{N/2}} E(\Psi_\varepsilon)(t).$$

Integrating in time and recalling the energy equipartition of lemma 6.5 yields

$$\|v^+\|_{L^2((0,T) \times \mathbb{R}^N)}^2 + \|v^-\|_{L^2((0,T) \times \mathbb{R}^N)}^2 = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{N/2}} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) \left| \frac{\partial \Psi_\varepsilon}{\partial t} \right|^2 dx dt.$$

Finally, recalling $u_\varepsilon = (\partial \Psi_\varepsilon / \partial t) + \alpha \Psi_\varepsilon$ and using lemma 6.4, which says that Ψ_ε converges to 0 in a suitable weighted norm, we obtain

$$\|v^+\|_{L^2((0,T) \times \mathbb{R}^N)}^2 + \|v^-\|_{L^2((0,T) \times \mathbb{R}^N)}^2 = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{N/2}} \int_0^T \int_{\mathbb{R}^N} \rho\left(x, \frac{x}{\varepsilon}\right) |u_\varepsilon(t, x)|^2 dx dt,$$

which is the desired result. $\square$

Proof of lemma 6.2. We begin with the easy case, when $\xi^n \neq 0$. From the definition (6.5) of w_ε , we have

$$\chi_\varepsilon(\sqrt{\varepsilon}z) = \varepsilon w_\varepsilon(z) \exp \left\{ -2i\pi \frac{\xi^n \cdot z}{\sqrt{\varepsilon}} \right\},$$

which, upon differentiating, yields

$$(\nabla_x \chi_\varepsilon)(\sqrt{\varepsilon}z) = (\sqrt{\varepsilon} \nabla_z w_\varepsilon + 2i\pi \xi^n w_\varepsilon) \exp \left\{ -2i\pi \frac{\xi^n \cdot z}{\sqrt{\varepsilon}} \right\}.$$

From lemma 4.1, we know that

$$\|(\nabla_x \chi_\varepsilon)(\sqrt{\varepsilon}z)\|_{L^2(\mathbb{R}^N)} \leq C(\|v^0\|_{L^2(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}).$$

For $\xi^n \neq 0$, there exists a positive constant $C(\xi^n) > 0$ (see [12]) such that

$$\|\sqrt{\varepsilon} \nabla_z w_\varepsilon + 2i\pi \xi^n w_\varepsilon\|_{L^2(\mathbb{R}^N)} \geq C(\xi^n) (\|\sqrt{\varepsilon} \nabla_z w_\varepsilon\|_{L^2(\mathbb{R}^N)} + \|w_\varepsilon\|_{L^2(\mathbb{R}^N)}), \quad (6.16)$$

which implies the desired estimate (6.6). We now study the two-scale convergence of the sequence w_ε , which is a solution of

$$-(\sqrt{\varepsilon} \nabla + 2i\pi \xi^n) \cdot (A^\varepsilon(\sqrt{\varepsilon} \nabla + 2i\pi \xi^n) w_\varepsilon) + \varepsilon^2 \rho^\varepsilon w_\varepsilon = \rho^\varepsilon \psi_n^\varepsilon(\varepsilon v^0 - v^1). \quad (6.17)$$

We multiply equation (6.17) by $\bar{\phi}^\varepsilon = \bar{\phi}(z, z/\sqrt{\varepsilon})$, where $\phi(z, y)$ is a smooth, compactly supported function defined on $\mathbb{R}^N \times \mathbb{T}^N$. Integrating by parts and using hypothesis 2.2 yields

$$\begin{aligned} & \int_{\mathbb{R}^N} [A_0^\varepsilon + o(1)] (\sqrt{\varepsilon} \nabla + 2i\pi \xi^n) w_\varepsilon \cdot (\sqrt{\varepsilon} \nabla - 2i\pi \xi^n) \bar{\phi}^\varepsilon \, dz \\ & + \varepsilon^2 \int_{\mathbb{R}^N} [\rho_0^\varepsilon + o(1)] w_\varepsilon \bar{\phi}^\varepsilon \, dz = \int_{\mathbb{R}^N} [\rho_0^\varepsilon + o(1)] \psi_n^\varepsilon (\varepsilon v^0 - v^1) \bar{\phi}^\varepsilon \, dz. \end{aligned} \quad (6.18)$$

Because of the *a priori* estimate (6.6), there exist a subsequence and a limit,

$$w_0(z, y) \in L^2(\mathbb{R}^N; H^1(\mathbb{T}^N)),$$

such that w_ε and $\sqrt{\varepsilon} \nabla w_\varepsilon$ two-scale converge weakly to w_0 and $\nabla_y w_0$, respectively. Passing to the two-scale limit, we obtain

$$-(\operatorname{div}_y + 2i\pi \xi^n) A_0(y) (\nabla_y + 2i\pi \xi^n) w_0 = -\rho_0(y) v^1(z) \psi_n(y).$$

By the simplicity of λ_n , we deduce that w_0 is given by

$$w_0(z, y) = -\frac{\psi_n(y)}{\lambda_n} v^1(z),$$

and the uniqueness of this limit implies that the entire sequence converges. The strong two-scale convergence is easily obtained by replacing the test function ϕ^ε by w_ε in (6.18) and passing to the limit on the right-hand side, which implies the convergence of the energies in the left-hand side and thus the strong two-scale convergence according to proposition 2.6(ii).

Now suppose that $\xi^n = 0$. The above proof no longer works because inequality (6.16) is not valid for $\xi^n = 0$. It turns out that w_ε is no longer bounded in $L^2(\mathbb{R}^N)$ but is precisely of order $\varepsilon^{-1/2}$. We cannot use simple *a priori* estimates to prove this result and we instead rely on a comparison with a four-term asymptotic expansion. We construct a two-scale asymptotic expansion for a version of (6.17) with purely periodic coefficients (the macroscopic variable $\sqrt{\varepsilon}z$ being replaced by 0):

$$-\varepsilon \operatorname{div}_z \left(A_0 \left(\frac{z}{\sqrt{\varepsilon}} \right) \nabla_z w_\varepsilon^0 \right) + \varepsilon^2 \rho_0 \left(\frac{z}{\sqrt{\varepsilon}} \right) w_\varepsilon^0 = \rho_0 \left(\frac{z}{\sqrt{\varepsilon}} \right) \psi_n \left(\frac{z}{\sqrt{\varepsilon}} \right) (\varepsilon v^0 - v^1), \quad (6.19)$$

where A_0 and ρ_0 are defined by (2.1). Recall that, by virtue of hypothesis 2.2, the difference between $A(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon})$ and $A_0(z/\sqrt{\varepsilon})$ is small, and the same is true for ρ (see (6.4)). The ansatz for w_ε^0 is taken to be

$$\tilde{w}_\varepsilon(z) = \frac{w^*(z)}{\sqrt{\varepsilon}} + w^0 \left(z, \frac{z}{\sqrt{\varepsilon}} \right) + \sqrt{\varepsilon} w^1 \left(z, \frac{z}{\sqrt{\varepsilon}} \right) + \varepsilon w^2 \left(z, \frac{z}{\sqrt{\varepsilon}} \right).$$

Plugging this into (6.19), we obtain the following cascade of equations:

$$-\operatorname{div}_y (A_0(y) \nabla_y w^0(z, y) + A_0(y) \nabla_z w^*(z)) = -\rho_0(y) \psi_n(y) v^1(z),$$

$$\begin{aligned}
-\operatorname{div}_y(A_0(y)\nabla_y w^1(z, y)) &= \operatorname{div}_z(A_0(y)\nabla_y w^0(z, y)) + \operatorname{div}_y(A_0(y)\nabla_z w^0) \\
&\quad + \operatorname{div}_z(A_0(y)\nabla_z w^*(z)), \\
-\operatorname{div}_y(A_0(y)\nabla_y w^2(z, y)) &= \operatorname{div}_z(A_0(y)\nabla_y w^1(z, y)) + \operatorname{div}_y(A_0(y)\nabla_z w^1(z, y)) \\
&\quad + \operatorname{div}_z(A_0(y)\nabla_z w^0(z, y)) + \rho_0(y)\psi_n(y)v^0(z).
\end{aligned}$$

The first equation allows us to compute $w^0(z, y)$ up to an unknown function $\hat{w}_0(z)$:

$$w^0(z, y) = \hat{w}_0(z) - \frac{\psi_n(y)v^1(z)}{\lambda_n} + \sum_{i=1}^N \frac{\partial w^*}{\partial z_i}(z)\chi_i(y),$$

where χ_i , $i = 1, \dots, N$, are the usual periodic solutions of the cell problems

$$-\operatorname{div}_y(A(y)(\nabla_y \chi_i + e_i)) = 0 \quad \text{in } \mathbb{T}^N.$$

In order to solve the second equation, we must first check the following compatibility condition:

$$\operatorname{div}_z(A^H \nabla_z w^*(z)) - \frac{1}{\lambda_n} \operatorname{div}_z \left(v^1(z) \int_{\mathbb{T}^N} A(y) \nabla_y \psi_n(y) dy \right) = 0, \quad (6.20)$$

where A^H is the classical homogenized matrix defined by

$$A^H e_i = \int_{\mathbb{T}^N} A(y) (\nabla_y \chi_i + e_i) dy.$$

Equation (6.20) has a unique (up to an additive constant) solution $w^*(z)$ in the space

$$D^{1,2}(\mathbb{R}^N) = \{\phi \in H_{\text{loc}}^1(\mathbb{R}^N) \text{ such that } \nabla \phi \in L^2(\mathbb{R}^N)^N\}.$$

It is not obvious, however, that w^* belongs to $L^2(\mathbb{R}^N)$. Since equation (6.20) has constant coefficients, by Fourier analysis we can deduce that

$$|\mathcal{F}w^*(\xi)| \leq C \frac{|\mathcal{F}v^1(\xi)|}{|\xi|}, \quad (6.21)$$

where $\mathcal{F}\phi(\xi)$ denotes the Fourier transform of $\phi(x)$. Assumption (3.7) (see remark 3.4 for comments) is precisely designed, so we can deduce from (6.21) that w^* does indeed belong to $L^2(\mathbb{R}^N)$. Knowing w^* , we can compute the solution of the second equation (up to the addition of an unknown function $\hat{w}_1(z)$) as

$$w^1(z, y) = \sum_{i=1}^N \frac{\partial \hat{w}_0}{\partial z_i}(z)\chi_i(y) + \tilde{w}_1(z, y),$$

where $\tilde{w}_1(z, y)$ depends on z only linearly through $\nabla_z v^1$ and $\nabla_z \nabla_z w^*$ (which itself depends only on $\nabla_z v^1$) but not on $\hat{w}_0$.

In order to solve the third equation (for w^2), we must first check the following compatibility condition:

$$\operatorname{div}_z(A^H \nabla_z \hat{w}_0) + \operatorname{div}_z \int_{\mathbb{T}^N} A(y) \left[\nabla_y \tilde{w}_1 - \frac{\psi_n}{\lambda_n} \nabla_z v^1 + \sum_{i=1}^N \chi_i \nabla_z \frac{\partial w^*}{\partial z_i} \right] dy = 0,$$

which again admits a unique (up to an additive constant) solution $\hat{w}_0 \in D^{1,2}(\mathbb{R}^N)$ if $v^1 \in H^1(\mathbb{R}^N)$. Since the coefficients are constant in the above equation, by Fourier analysis we can easily check that the $L^2(\mathbb{R}^N)$ norm of $\hat{w}_0$ is bounded by the $L^2(\mathbb{R}^N)$ norm of v^1 .

Since we have proved that both w^* and $\hat{w}_0$ belong to $H^1(\mathbb{R}^N)$, it is now obvious that the two-scale asymptotic expansion $\tilde{w}_\varepsilon$ satisfies the desired *a priori* estimate (6.7), whatever the choices of the underdetermined additive functions $\hat{w}_1(z)$ and $\hat{w}_2(z)$. To obtain the same for w_ε , we bound their difference $\delta_\varepsilon(z) = w_\varepsilon(z) - \tilde{w}_\varepsilon(z)$, which satisfies

$$-\varepsilon \operatorname{div}_z(A^\varepsilon \nabla_z \delta_\varepsilon) + \varepsilon^2 \rho^\varepsilon \delta_\varepsilon = \varepsilon^{3/2} f^\varepsilon,$$

where

$$\begin{aligned} f^\varepsilon(z) = & -\rho\left(z, \frac{z}{\sqrt{\varepsilon}}\right) \left(w^*(z) + \sqrt{\varepsilon} w^0\left(z, \frac{z}{\sqrt{\varepsilon}}\right) + \varepsilon w^1\left(z, \frac{z}{\sqrt{\varepsilon}}\right) + \varepsilon^{3/2} w^2\left(z, \frac{z}{\sqrt{\varepsilon}}\right) \right) \\ & - \varepsilon^{-1/2} \operatorname{div}_z \left(\left[A_0\left(\frac{z}{\sqrt{\varepsilon}}\right) - A\left(\sqrt{\varepsilon} z, \frac{z}{\sqrt{\varepsilon}}\right) \right] \nabla_z \tilde{w}_\varepsilon \right) \\ & \times [-\operatorname{div}_z(A_0(y) \nabla_z w^1(z, y)) - \sqrt{\varepsilon} \operatorname{div}_z(A_0(y) \nabla_z w^2(z, y))] \left(y = \frac{z}{\sqrt{\varepsilon}} \right) \\ & \times [-\operatorname{div}_z(A_0(y) \nabla_y w^2(z, y)) - \operatorname{div}_y(A_0(y) \nabla_z w^2(z, y))] \left(y = \frac{z}{\sqrt{\varepsilon}} \right). \end{aligned}$$

We check that

$$\int_{\mathbb{R}^N} f^\varepsilon \delta_\varepsilon \, dz \leq C(\|\delta_\varepsilon\|_{L^2(\mathbb{R}^N)} + \varepsilon^{-1/2} \|\nabla \delta_\varepsilon\|_{L^2(\mathbb{R}^N)})$$

because of hypothesis 2.2 and (6.4). Therefore, we obtain the following inequality:

$$\varepsilon \|\nabla_z \delta_\varepsilon\|_{L^2(\mathbb{R}^N)^N}^2 + \varepsilon^2 \|\delta_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 \leq C\sqrt{\varepsilon}(\varepsilon \|\delta_\varepsilon\|_{L^2(\mathbb{R}^N)} + \sqrt{\varepsilon} \|\nabla \delta_\varepsilon\|_{L^2(\mathbb{R}^N)}),$$

which implies

$$\sqrt{\varepsilon} \|\delta_\varepsilon\|_{L^2(\mathbb{R}^N)} + \|\nabla \delta_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C$$

and, in turn, this implies the desired estimate (6.7). We pass to the weak two-scale limit in equation (6.17) with $\xi^n = 0$, as before. The strong two-scale convergence of $\sqrt{\varepsilon} \nabla w_\varepsilon$ is obtained by remarking that $\sqrt{\varepsilon} \|\nabla \delta_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C\sqrt{\varepsilon}$. This completes the proof of the lemma. $\square$

Acknowledgements

This work was partly supported by the ECOS project of cooperation between Chile and France (project no. C04E07) and by the Chair of “Mathematical modelling and numerical simulation”, F-EADS, École Polytechnique INRIA. G.A. was partly supported by the DEFI project at INRIA Saclay, Île de France. L.F. was partly supported by grant “Dirección de Investigación” (no. 073609 1/R), Universidad del Bío-Bío and by FONDECYT Grant no. 1090510. The authors thank for its hospitality the Centro de Modelamiento Matemático (Santiago de Chile), where part of this work was completed.

References

- 1 G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Analysis* **23** (1992), 1482–1518.
- 2 G. Allaire. *Numerical analysis and optimization. An introduction to mathematical modelling and numerical simulation* (Oxford University Press, 2007).
- 3 G. Allaire and M. Palombaro. Localization for the Schrödinger equation in a locally periodic medium. *SIAM J. Math. Analysis* **38** (2006), 127–142.
- 4 G. Allaire and A. Piatnitsky. Uniform spectral asymptotics for singularly perturbed locally periodic operators. *Commun. PDEs* **27** (2002), 705–725.
- 5 G. Allaire, M. Palombaro and J. Rauch. Diffractive behavior of the wave equation in periodic media: weak convergence analysis. *Annali Mat. Pura Appl.* **188** (2009), 561–589.
- 6 G. Allaire, M. Palombaro and J. Rauch. Diffractive geometric optics for Bloch wave packets. (Submitted.)
- 7 P. W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.* **109** (1958), 1492–1505.
- 8 H. Benisty and C. Weisbuch. Photonic crystals. In *Progress in optics*, vol. 49 (Elsevier, 2006).
- 9 A. Bensoussan, J. L. Lions and G. Papanicolaou. *Asymptotic analysis in periodic structures* (Amsterdam: North-Holland, 1978).
- 10 H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* (Amsterdam: North-Holland, 1973).
- 11 R. Carmona and J. Lacroix. *Spectral theory of random Schrödinger operators* (Birkhäuser, 1990).
- 12 C. Conca, J. Planchard and M. Vanninathan. *Fluids and periodic structures*, Research in Applied Mathematics vol. 38 (Paris: J. Wiley and Masson, 1995).
- 13 P. Donnat, J.-L. Joly, G. Metivier and J. Rauch. Diffractive nonlinear geometric optics. *Séminaire sur les Equations aux Dérivées Partielles, 1995–1996, École Polytechnique, Palaiseau, no. 17* (1996).
- 14 A. Figotin and A. Klein. Localization of classical waves. I. Acoustic waves. *Commun. Math. Phys.* **180** (1996), 439–482.
- 15 G. Francfort and F. Murat. Oscillations and energy densities in the wave equation. *Commun. PDEs* **17** (1992), 1785–1865.
- 16 P. Gérard. Mesures semi-classiques et ondes de Bloch. *Séminaire sur les equations aux Dérivées Partielles, 1990–1991, École Polytechnique, Palaiseau, no. 16* (1991).
- 17 G. Gérard, P. Markowich, N. Mauser and F. Poupaud. Homogenization limits and Wigner transforms. *Commun. Pure Appl. Math.* **50** (1997), 323–379.
- 18 T. Kato. *Perturbation theory for linear operators* (Springer, 1980).
- 19 C. Martin de Sterke and J. Sipe. Envelope-function approach for the electrodynamics of nonlinear periodic structures. *Phys. Rev. A* **38** (1998), 5149–5165.
- 20 G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Analysis* **20** (1989), 608–623.
- 21 A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences vol. 44 (Springer, 1983).
- 22 B. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis and self-adjointness* (New York: Academic Press, 1975).
- 23 P. St. J. Russell. Photonic crystal fibers. *J. Lightwave Technol.* **24** (2006), 4729–4749.
- 24 J. Sipe and H. Winful. Nonlinear Schrödinger solitons in a periodic structure. *Optics Lett.* **13** (1988), 132–133.

(Issued 8 October 2010)