# KINETIC SCHEMES FOR GAS DYNAMICS OF REAL GASES OR TWO-PHASE MIXTURES

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**Abstract:** Kinetic, or Boltzmann, schemes are numerical schemes based on a kinetic formulation of a hyperbolic system of conservation laws. For Euler equations describing the motion of a compressible inviscid fluid, there has been a renewed interest in this class of numerical algorithms after the recent work of Perthame. One drawback of his theory is that it applies only to so-called perfect gas which have a very simple thermodynamic state law. The goal of this paper is to present some numerical generalizations for real gas and equilibrated two-phase flows.

## 1 Introduction

This paper is devoted to the numerical study of so-called kinetic schemes for gas dynamics simulations of real gases and two-phase flows. Kinetic schemes (also called Boltzmann schemes) are a class of flux splitting schemes for the numerical solution of non-linear hyperbolic systems of conservation laws (see e.g. [9]). The main idea in a kinetic scheme is first to replace the original hyperbolic system by an equivalent transport equation for a distribution function depending on an additional variable (equivalent to a microscopic velocity), then to compute a solution of this transport equation, and finally to take moments of the distribution function to obtain approximate solutions of the original system. In the context of fluid mechanics this idea is very similar to the well-known formal asymptotics that leads to Euler equations when the mean free-path goes to zero in Boltzmann equation (see e.g. [1]). Recent works have shown that there exists an exact equivalence between some hyperbolic systems (mainly scalar conservation laws or the Euler isentropic system) and corresponding kinetic formulations which allows to prove new existence and stability results [12], [13], [14]. However, here we are only concerned with the application of such ideas to numerical computations. Although kinetic schemes have been used since the seventies, there has been recently a renewed interest in these schemes due to the fundamental work of Perthame [15], [16]. We shall follow his approach in the sequel.

Let us describe the kinetic approach to the numerical solution of hyperbolic systems of conservation laws. To simplify the presentation, we shall restrict ourselves to the one-dimensional case, but all the details go through in higher dimensions. Let us denote the time variable by $t \in \mathbb{R}^+$ and the space variable by $x \in \mathbb{R}$. Let $F$ be a smooth function from $\mathbb{R}^p$ into $\mathbb{R}^p$, with $p \geq 1$. The unknown is a vector $u(t,x) \in \mathbb{R}^p$, solution of

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \\ u(t=0,x) = u_0(x). \end{array} \right. \tag{1}$$

As is well-known, system (1) is supplemented with an entropy condition in order to select a possibly unique relevant physical solution. In other words, for some convex function $S(u)$ (called the entropy) and some entropy flux $G(u)$ we ask the solution $u(t,x)$ to satisfy a further inequality $\frac{\partial S(u)}{\partial t} + \frac{\partial G(u)}{\partial x} \leq 0$.

The main idea is to introduce a new variable $v$ (that can be thought of as a microscopic velocity) and a distribution function $f(t,x,v)$ (that can be interpreted as a particles density at time $t$, position $x$, and velocity $v$) such that its moments with respect to $v$ are precisely equal to the solution $u(t,x)$ and to the flux $F(u(t,x))$

$$\int f(t,x,v)dv \;=\; u(t,x), \; \int vf(t,x,v)dv \;=\; F\left(u(t,x)\right).$$

Remark that if $u$ is a vector $f$ may also need to be a vector, except if some other microscopic variables are introduced and each component of $u$ is recovered by different moments of $f$ (for example, an internal energy, see e.g. [15]). A kinetic formulation of the hyperbolic system (1) (if it exists) is a Boltzmann-type equation for the distribution function $f$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = Q(f), \tag{2}$$

where $Q(f)$ is a collision-type operator which forces $f$ to relax to an equilibrium function $\mathcal{M}(x,v)$ which is the unique solution of the equation $Q(\mathcal{M}) = 0$. For the true Boltzmann equation, $\mathcal{M}(x,v)$ is called a Maxwellian and is a gaussian function in the $v$ variable. The basis of kinetic schemes is **to solve in $f$ rather than in $u$** and to recover the desired unknown $u$ by taking moments of $f$.

As already said, a careful choice of $Q$ allows in some cases to prove a strict equivalence between (1) and (2) (the entropy condition of (1) being recovered from the $H$-theorem for the kinetic equation (2)). However, for numerical purposes a precise form of the collision operator $Q$ is not needed, and we simply require to know the equilibrium function $\mathcal{M}(x,v)$. Then, we can solve (2) by splitting the transport phase and the relaxation or collision phase. At each time step, we first solve the free transport equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0,$$

then we apply the collision operator

$$\frac{\partial f}{\partial t} = Q(f).$$

Therefore, denoting by $\Delta t$ the time step, a Boltzmann or kinetic numerical scheme can be decomposed in two steps

1. Convection (or free transport of $f$) with initial data $f^0(x,v)$

$$\begin{cases} \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0 & \text{for } 0 < t < \Delta t, \\ f(t=0,x,v) = f^0(x,v) & \text{for } t = 0, \end{cases}$$

   which admits the explicit solution $f(t,x,v) = f^0(x - tv, v)$.

2. Collision (or relaxation to equilibrium) : project the previous solution $f$ onto the equilibrium function $\mathcal{M}(x,v)$ in order to obtain a new initial data $f^1(x,v)$ for the next convection step

$$f^1(x,v) = \mathcal{M}(x,v) \text{ with } \begin{cases} \int \mathcal{M}(x,v)dv = \int f(\Delta t,x,v)dv \\ \int v\mathcal{M}(x,v)dv = \int vf(\Delta t,x,v)dv \end{cases}$$

To define the collision step we implicitely assume that the equilibrium function $\mathcal{M}(x,v)$ is uniquely determined by its zero and first order moments in $v$.

Actually it turns out that, upon spatial discretization, a kinetic scheme simply reduces to a more classical flux splitting scheme. In other words, the $v$ variable can be eliminated and a kinetic scheme can be implemented solely in terms of $x$ and $t$. More precisely, discretizing the computational domains in cells of size $\Delta x$ and denoting by $u_i(t)$ the average of $u(t,x)$ in cell $i$, the above kinetic scheme yields

$$u_i(\Delta t) = u_i(0) - \frac{\Delta t}{\Delta x}\left[F\left(u_i(0), u_{i+1}(0)\right) - F\left(u_{i-1}(0), u_i(0)\right)\right],$$

where $F(u_L, u_R)$ is the numerical flux function depending on the choice of the equilibrium function $\mathcal{M}(x,v)$. Different choices of the equilibrium function lead to different numerical schemes. In particular, if $\mathcal{M}(x,v)$ has a compact support in $v$, the resulting numerical scheme has a finite speed of propagation.

Kinetic schemes enjoy a few useful properties by their very definition. Since $f$ satisfies a linear transport equation, it remains always positive if the equilibrium function $\mathcal{M}$ is positive. This yields a maximum principle for $u$ (for example, in gas dynamics the density and pressure remain positive for all time). The so-called $H$-theorem for the kinetic equation (2) has its counterpart as a discrete entropy principle for the numerical scheme if the equilibrium function $\mathcal{M}$ is chosen as the minimum of some entropy minimization problem (see [15]). Kinetic schemes are naturally multi-dimensional by simply averaging a truely multi-dimensional kinetic free transport equation (one-dimensional splitting is not necessary). They can easily be extended to second order in space by a MUSCL approach.

In the context of the numerical simulation of Euler equations, describing the motion of an inviscid compressible gas, these ideas have already been explored by many individuals, including [3], [4], [5], [6], [10], [11], [15], [16], [17], [18], [19], and [21]. For scalar conservation laws, kinetic schemes have also been used in [2].

One drawback of kinetic schemes for gas dynamics is that they are only defined for so-called perfect gas which have a very simple thermodynamic state law. The goal of this paper is to try to generalize the kinetic approach to more general state laws of real gas and even to apply it for a homogeneous model of two-phase flow. The research described here is part of the PhD thesis of the second author [24].

# 2 Isentropic Euler equations

In this section we briefly recall two well-known kinetic schemes for the isentropic system of Euler equations

$$\left\{ \begin{array}{ll} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} & = 0 \\ \frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} & = 0 \end{array} \right. \tag{3}$$

supplemented with the perfect gas state law $p = \rho^\gamma$ with $\gamma > 1$, and the entropy condition $\frac{\partial S}{\partial t} + \frac{\partial S u}{\partial x} \leq 0$ with $S = \rho u^2 + \frac{2p}{\gamma(\gamma-1)}$. Each kinetic scheme is defined by an equilibrium function $\mathcal{M}(x,v)$ which will be the initial data of the linear transport equation $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$ solved between two consecutive time steps.

## 2.1 Kaniel scheme

The idea of Kaniel [10] for computing numerical solutions of (3) is to use the following type of equilibrium function

$$\mathcal{M}(x,v) = \chi(|v-u|) 1_{|v-u|<\alpha(\rho)},$$

where the shape $\chi$ of the equilibrium function and the cut-off function $\alpha$ depends only on the pressure law, i.e. on $\gamma$. In order to recover the macroscopic unknowns of (3), we ask that $\mathcal{M}$ satisfies

$$\int_{|v|<\alpha(\rho)} \chi(|v|)\, dv = \rho. \tag{4}$$

Then, automatically since $\chi(|v|)$ is even, one has

$$\int \mathcal{M}(x,v)\, dv = \rho(x) \text{ and } \int v\mathcal{M}(x,v)\, dv = \rho(x)u(x).$$

Condition (4) yields also the consistency with the first equation of (3) (conservation of mass) since, up to first order in time, it is obtained by averaging the linear transport equation with respect to

$v$. In order to obtain the consistency with the second equation (conservation of momentum), we add another constraint

$$\int_{|v|<\alpha(\rho)} |v|^2 \chi(|v|) dv = p. \tag{5}$$

Conditions (4) and (5) uniquely determine both functions $\chi$ and $\alpha$. A simple computation yields

$$\alpha(\rho) = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{\gamma \rho^{\gamma-1}}, \text{ and } \chi(\alpha) = \alpha \left( \frac{\partial^2 p}{\partial \rho^2}(\alpha) \right)^{-1} = \frac{\alpha^{\frac{3-\gamma}{\gamma-1}}}{(\gamma-1)\gamma^{\frac{1}{\gamma-1}}}.$$

This scheme is well-defined for any value of $\gamma > 1$, even when $\gamma > 3$. The width of the support of the Kaniel equilibrium function is exactly equal to the sound speed (at least in 1-D). However, we are unable to prove any discrete entropy property for this scheme (except when $\gamma = 3$).

## 2.2 Perthame scheme

The idea of Perthame [15] is to obtain an equilibirum function

$$\mathcal{M}(x, v) = \chi(|v - u|),$$

where $\chi(v)$ is a minimizer of a kinetic entropy defined by

$$H(f) = \int \left( \kappa f(v)^{\frac{\gamma+1}{3-\gamma}} + v^2 f(v) \right) dv$$

among all possible distribution functions $f$ satisfying the constraints

$$f \geq 0, \quad \int f(v) dv = \rho, \quad \int v^2 f(v) dv = p.$$

The constant $\kappa$ is chosen precisely equal to $\frac{2(3-\gamma)}{(\gamma-1)^2}(8I_\gamma/(\gamma-1))^{(\gamma-1)/(3-\gamma)}$ with $I_\gamma = \int_0^1 (1 - t^2)^{\frac{3-\gamma}{2(\gamma-1)}} dt$. For $1 < \gamma \leq 3$ this is a convex minimization problem which admits a unique minimizer given by

$$\chi(v) = \frac{1}{2I_\gamma} \sqrt{\frac{\gamma-1}{2}} \left( \rho^{\gamma-1} - \frac{\gamma-1}{2}(v-u)^2 \right)_+^{\frac{3-\gamma}{2(\gamma-1)}},$$

where $(\cdot)_+$ denotes the positive part function. Clearly, because of the constraints the Perthame equilibrium function is consistent with the system (3). Furthermore, due to the special choice of the constant $\kappa$, an easy computation shows that $H(\chi) = S$. Since the kinetic entropy $H(f)$ is simply transported as $f$ during the free transport step and then is decreased during the collision step because $H(\chi) \leq H(f)$, Perthame scheme satisfies a corresponding discrete entropy principle.

Let us remark that when $\gamma = 3$ Perthame and Kaniel schemes coincide and the equilibrium function is a simple step function. When $\gamma > 3$, Perthame scheme is not any longer entropic (i.e. the minimum of the kinetic entropy is $-\infty$), but the formula for $\chi$ still yields a well-defined numerical scheme (that actually behaves poorly).

# 3 Euler equations

In this section we introduce a family of Perthame kinetic schemes for the full system of Euler equations

$$\left\{ \begin{array}{ll} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} & = 0 \\ \frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} & = 0 \\ \frac{\partial E}{\partial t} + \frac{\partial (E+p)u}{\partial x} & = 0 \end{array} \right. \tag{6}$$

with the total energy $E = 1/2\rho u^2 + \rho e$ and the perfect gas state law $p = (\gamma - 1)\rho e$ with $\gamma > 1$, and the entropy condition $\frac{\partial S}{\partial t} + \frac{\partial Su}{\partial x} \leq 0$ for $S = \rho log\left(\frac{e}{\rho^{(\gamma-1)}}\right)$.

To define a kinetic scheme for system (6) another microscopic variable is required (a particle internal energy) or another distribution function. The second choice is more convenient and we work with two equilibrium functions $\mathcal{M}(x,v)$ and $\mathcal{N}(x,v)$ which will be the initial data of two linear transport equations of the type $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$ at each time step. For $1 < \gamma \leq 3$, the couple $(\mathcal{M}, \mathcal{N})$ is defined by

$$\mathcal{M}(x,v) = \chi(|v - u(x)|) \text{ and } \mathcal{N}(x,v) = \tilde{\chi}(|v - u(x)|),$$

where $(\chi, \tilde{\chi})$ is the unique minimizer of the convex kinetic entropy

$$H(f,g) = \int f(v)^p g(v)^q dv$$

among all possible non-negative distribution functions $(f,g)$ satisfying the constraints

$$\int f(v)dv = \rho, \quad \int \left(\frac{v^2}{2}f(v) + g(v)\right) dv = e,$$

with $p > 1$ and $q = (p-1)\frac{\gamma-3}{\gamma+1}$. This choice of $q$ ensures the consistency with system (6), i.e. that $\int v^2 f(v)dv = p$, and that $H(\chi, \tilde{\chi})$ is precisely an entropy of system (6). Defining $\lambda = \frac{q-1}{1-p-q}$ the minimizer is given by

$$\chi(x,v) = C_1\frac{\rho}{e}\left(1 - C_2\frac{v^2}{\rho^2}\right)_+^{\lambda}, \quad \tilde{\chi}(x,v) = C_3(e - C_4\rho^2)\chi(x,v)^{\frac{p}{1-q}},$$

where $C_1, C_2, C_3, C_4$ are positive constants (see [24] for their precise values). The case $p = +\infty$ is described in [11]. This choice of kinetic entropy $H(f,g)$ has the very interesting feature of giving rise to compactly supported equilibrium functions. This is a highly desirable property for numerical purposes since the resulting scheme will be less diffusive. In particular, a compact support in $v$ implies a finite speed of propagation for the numerical scheme.

The kinetic scheme associated to the equilibrium functions $(\chi, \tilde{\chi})$ is well-defined and entropic for $1 < \gamma \leq 3$. For $\gamma = 3$ it is very simple since $\tilde{\chi} = 0$ and $\chi$ is a step function. It is not defined for $\gamma > 3$.

# 4 Extension to real gas and two-phase mixtures

Most kinetic schemes are designed only for perfect gas, i.e. for a state law of the type $p = (\gamma-1)\rho e$ with a constant $\gamma > 1$. Indeed, for the kinetic schemes described above, the constant $\gamma$ is built in the equilibrium functions which in turn yields the numerical flux formula. Our goal here is to extend, at least from a numerical point of view, these kinetic shemes for real gas which may have very complicated state laws.

Part of our motivation is also to extend Boltzmann schemes for some two-phase flow models. Here we shall be concerned only with the so-called homogeneous equilibrated two-phase flow model which describes the motion of a mixture of liquid and vapor. For extensions to another two-phase flow model, the so-called bi-fluid model, we refer to [7]. The homogeneous equilibrated two-phase flow model is the system of Euler equations for the mixture with a diphasic state law which is very stiff and has discontinuous derivatives when phase changes occur. The main physical assumptions allowing to derive this model are that the two phases must be thermodynamically equilibrated and have the same speed, the same pressure, and the same temperature. A typical state law for

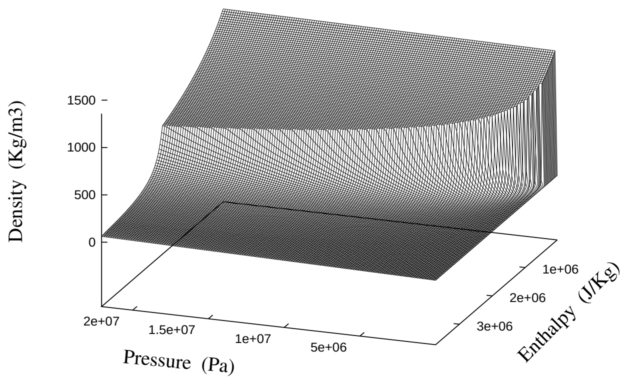


Figure 1: *State law* $\rho(p, h)$

a two-phase mixture of water and steam is that of the benchmark [8] (see figure 1). In the liquid region the mixture is almost incompressible, in the diphasic region it is almost isothermal, and in the vapor region it behaves like a real gas. Obviously it is a very severe example of real gas !

For such two-phase flow problems, many numerical schemes are available, but the state of the art is probably provided by Toumi's adaptation of Roe scheme [23]. The main trick is that the linearization depends on the thermodynamic states considered, and therefore contains more physics than a simple upwind scheme. Despite its advantages the modified Roe scheme is not the ultimate scheme, and better ones are always seeked. In particular, kinetic schemes have the following potential advantages : positivity of important variables, like density or pressure, is preserved (in the context of two-phase flows this could be helpful for the concentration to stay between 0 and 1), discrete entropy properties are easily established, and finally kinetic schemes have a natural multidimensional character.

Our strategy for extending the classical kinetic schemes to real gas and two-phase mixtures is to use a so-called *effective $\gamma$-coefficient technique*. In each cell and at each time step, for given thermodynamic conditions we shall compute a constant $\gamma$ such that the true state law is equivalent to a perfect gas state law ($p = (\gamma - 1)\rho e$). Then, locally in space and time the numerical fluxes are computed with one of the above kinetic scheme for this value of $\gamma$. In other words, the numerical fluxes depend on $\gamma$ and are therefore a nonlinear function of the thermodynamic variables. We consider two possible equivalences of the true state law and that of a perfect gas. First, we simply compute $\gamma$ by the obvious formula

$$\gamma = \frac{p}{\rho e} + 1, \tag{7}$$

and we use a kinetic scheme for the full system of Euler equations. An alternative is to assume that the flow is isentropic and to approximate the true state law by that of an isentropic perfect gas ($p = A\rho^\gamma$). In this case we compute two constants $A$ and $\gamma$ such that the pressure and the sound speed ($c = \sqrt{\frac{\partial p}{\partial \rho}(\rho, S)}$) coincide

$$\gamma = \frac{\rho c^2}{p}, \ A = \frac{p}{\rho^{\frac{\rho c^2}{p}}}.$$

Then, we use an isentropic kinetic scheme, but for the full system of Euler equations (i.e. two equilibrium functions are introduced like in section 3 but their shape is that of section 2). It turns out that this last procedure is more robust than the former one. The assumption of isentropicity is reasonable since in practice we consider only low Mach number flows without shocks (but with contact discontinuities).

The difficulty with such a $\gamma$ effective technique is that, due to the stiffness of the state law, the $\gamma$ coefficient may run out of bounds. In particular, the incompressible limit (when the fluid is a liquid like water) and the isothermal limit (when the fluid is an equilibrated two-phase mixture) have to be investigated to understand potential numerical degeneracy. This is the focus of next section.

# 5 Various asymptotic limits

This section is devoted to the study of asymptotic limits of the kinetic fluxes when $\gamma$ is close to some of its imposed bounds. For example, recall that $\gamma$ must always be larger than 1 and smaller than 3 (in 1-D) to ensure that a kinetic scheme is entropic.

## 5.1 Incompressible limit for Euler equations

The incompressible limit is equivalent to that of low Mach number, i.e. the sound speed is much larger than the average fluid velocity. For a perfect gas, this implies that $\gamma$ is close to 1. A careful inspection of the formula of section 3 shows that the equilibrium distribution function $\chi$ and $\tilde{\chi}$ converge to Dirac masses, and that upon a suitable rescaling they locally behave like a gaussian near the origin. This shows that the numerical fluxes are almost degenerate for $\gamma$ close to 1, and that a potential cure is to use a gaussian distribution function (which, as is well-known, gives rise to a very diffusive scheme).

In the case of the isentropic system of Euler equations, the incompressible limit (which is still equivalent to a very large sound speed corresponds to very large values of $\gamma$ (much bigger than 3). Clearly, the Perthame scheme is not very good in this regime : it is not isentropic and the equilibrium function blows up at both ends of its support. The Kaniel scheme seems to be not much better, its equilibrium function blows up too, but numerically it behaves in a better way for large $\gamma$ (for details, see [24]).

## 5.2 Isothermal limit for isentropic Euler equations

For a two-phase mixture as described in the previous section, there is another asymptotic regime in the diphasic region of the state law. Indeed, assuming thermodynamic equilibrium, if the two phases cohabit, their thermodynamical properties must lie on the saturation curve. In other words, for each individual phase the knowledge of the pressure is enough to determine all other quantities (temperature, enthalpy, etc.). The other variable is then the concentration of one phase. In practice it is almost an isothermal process. For an isentropic perfect gas, this implies that $\gamma$ is close to 1 in the diphasic regime. When $\gamma$ tends to 1, the equilibrium function of Perthame scheme converges to a gaussian

$$\lim_{\gamma \to 1} \chi(v) = \frac{1}{2\sqrt{2\pi}} \rho e^{-\frac{1}{2}v^2}.$$

Of course, this implies that, in the limit, the scheme is very diffusive (we refer to [24] for attempts to use a truncated gaussian in order to obtain a better kinetic scheme in this limit).

# 6 Shock tube in the incompressible limit

We have performed a shock tube test problem for the state law of [8] (see figure 1). The full system of Euler equations is used, and the fluid remains liquid (almost incompressible) through the all test. The precise intitial conditions may be found in [24], and we simply remark that it features a strong shock for a liquid (more than two bars). We compare the results of Perthame scheme with first a fixed value of $\gamma = 3$ (step function), and second a varying $\gamma$ fitted through (7) to a perfect gas state law. In the latter case, the effective $\gamma$ runs between 1.008 and 1.006. Remark

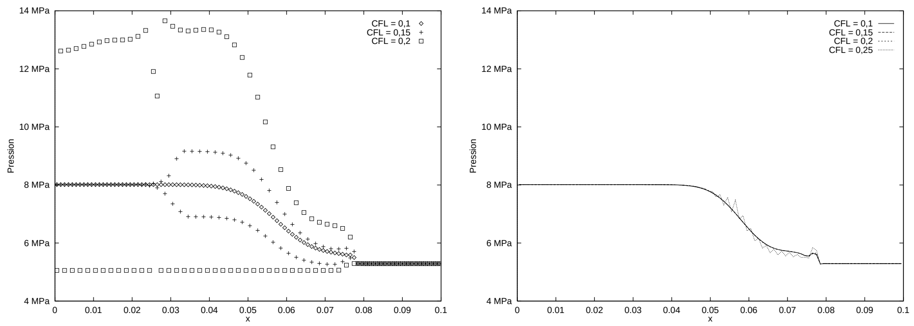


Figure 2: *Pressure, step function (left), entropic scheme (right)*

that the time step must be smaller than the CFL condition. Of course, the best results stem from the varying effective $\gamma$ strategy. Good results are also obtained with Kaniel scheme for which the numerical speed of propagation coincides with the sound speed.

# 7 Two-phase flow test

We also performed a classical benchmark in two-phase flow : the first CSNI Numerical Benchmark Problem [8]. It is a vertical tube filled initially with water that is heated on part of its length. The water begins to boil, preventing more water to enter the tube, but after the vapor has exited the tube, new water gets in and begin to boil. This cycle repeats in time and is damped only by dissipation and friction against the tube walls. The purpose of the test is to compute oscillations that are not damped too quickly by the numerical dissipation. We compare our solution to that of Toumi using a modified Roe scheme [23]. Remark that we need to use a finer grid with the kinetic sceme (Perthame scheme with effective $\gamma$). Nevertheless the results are worse : the oscillations are damped too much and their frequency is not correct.

As a conclusion we can say that kinetic schemes behave poorly for equilibrated two-phase flow (but Lax-Friedrichs scheme is even worse). Remark that the state law is very stiff and features discontinuities of thermodynamic derivatives at phase changes. However, the main problem with kinetic schemes is that they are known to be very diffusive through contact discontinuities which are the difficulties in the present subsonic situation. Notice also that our adaptation of kinetic schemes to general state laws is completely blind to any physical available information. This is in sharp contrast with the two-phase flow version of Roe scheme devised by Toumi [23] which uses the saturation curves to compute the linearized jacobian in the diphasic regime. Let us mention that kinetic schemes may be more adapted to two-fluid models (not equilibrated) [7].

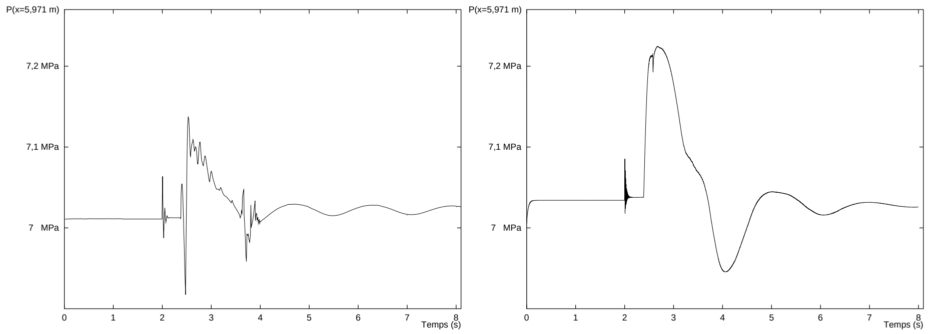


Figure 3: *Outlet pressure as a function of time. Modified Roe scheme of Toumi, 60 cells (left), kinetic scheme (source terms implicited), 300 cells (right)*

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