Ecole Polytechnique, Promotion X2009
Analyse Numérique et Optimisation (MAP431)
Contrôle Hors Classement du 12 avril 2011

Sujet proposé par François Alouges

IMPORTANT: Prière d'indiquer votre numéro de groupe de PC sur votre copie.

Problème (14 points). Let Ω be a regular bounded connected domain of $\mathbb{R}^N$, and Ω_1 a regular connected subdomain strictly included in Ω (which means $\bar{\Omega}_1 \subset \Omega$). In all the problem, $H_0^1(\Omega)$ is equipped with the scalar product

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx \,,$$

and we denote by $||u||_{H_0^1} = \sqrt{(u, u)}$ the associated norm. Let $f \in L^2(\Omega)$.

We consider for all $\varepsilon > 0$ the problem

$$\begin{aligned} &\text{Find} \quad u_\varepsilon \in H_0^1(\Omega), \text{ such that } \forall v \in H_0^1(\Omega), \\ &\int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dx + \frac{1}{\varepsilon} \int_{\Omega_1} \nabla u_\varepsilon \cdot \nabla v \, dx = \int_\Omega f v \, dx \,. \end{aligned} \tag{1}$$

1. Show that the problem (1) possesses a unique solution.
Assuming that the restriction to $\Omega \setminus \bar{\Omega}_1$ (resp. to Ω_1) of the solution u_ε of (1) belongs to $H^2(\Omega \setminus \bar{\Omega}_1)$ (resp. $H^2(\Omega_1)$) which boundary value problem does it verify on Ω?
Show also that u_ε is the unique solution to the minimization problem

$$\min_{u \in H_0^1(\Omega)} J_\varepsilon(u) \,,$$

where $J_\varepsilon(u) = \dfrac{1}{2} \displaystyle\int_\Omega |\nabla u|^2 \, dx + \dfrac{1}{2\varepsilon} \displaystyle\int_{\Omega_1} |\nabla u|^2 \, dx - \displaystyle\int_\Omega f u \, dx \,.$

2. Show that there exist two constants $C_1 > 0$ et $C_2 > 0$ independent of ε such that

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx \leq C_1 \text{ et } \int_{\Omega_1} |\nabla u_\varepsilon|^2 \, dx \leq C_2 \varepsilon \,.$$

Let V be the space defined by

$$V = \{u \in H_0^1(\Omega) \text{ such that } u|_{\Omega_1} \text{ is constant}\}.$$

We also consider the problem

$$\begin{aligned} &\text{Find} \quad u_0 \in V, \text{ such that } \forall v \in V, \\ &\int_\Omega \nabla u_0 \cdot \nabla v \, dx = \int_\Omega f v \, dx. \end{aligned} \tag{2}$$

3. Show that V is a closed subspace of $H_0^1(\Omega)$ and deduce that the problem (2) has a unique solution.

4. Show that

$$\forall v \in V, \quad \int_\Omega \nabla(u_\varepsilon - u_0) \cdot \nabla v \, dx = 0, \tag{3}$$

and deduce that

$$||u_\varepsilon||_{H_0^1} \geq ||u_0||_{H_0^1}.$$

5. Using (3), show that

$$||u_\varepsilon - u_0||_{H_0^1}^2 = \inf_{v \in V} ||u_\varepsilon - v||_{H_0^1}^2,$$

which means that u_0 is the projection on V of u_ε for the norm $H_0^1(\Omega)$.

6. We denote by $V^\perp$ the space orthogonal to V in $H_0^1(\Omega)$ for the scalar product $(\cdot,\cdot)$. Show that if $v \in V^\perp$ is such that $|v|_V^2 = \int_{\Omega_1} |\nabla v|^2 \, dx = 0$ then $v = 0$. Deduce that $|\cdot|_V$ is a norm on $V^\perp$.

7. Admitting that $|\cdot|_V$ defined herebefore is a norm on $V^\perp$ which is equivalent to the norm $H_0^1(\Omega)$, show that there exists $C > 0$ independent of ε such that

$$||u_\varepsilon - u_0||_{H_0^1} \leq C\sqrt{\epsilon},$$

and conclude that u_ε converges in $H^1(\Omega)$ to u_0 when ε tends to 0.

Exercice (6 points). We consider the advection equation posed on $\mathbb{R} \times \mathbb{R}^+$

$$\left[\begin{array}{l} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0 \text{ for } (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = u_0(x), \end{array} \right. \tag{4}$$

in which the sign of the velocity c is not prescribed. We discretize (4) with the scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\left(\theta \frac{u_{j+1}^n - u_j^n}{\Delta x} + (1-\theta)\frac{u_j^n - u_{j-1}^n}{\Delta x}\right) = 0\,,$$

in which Δt and Δx are temporal and spatial steps, $\theta \in [0,1]$ is a parameter which is independent of Δt and Δx, and u_j^n is an approximation of $u(n\Delta t, j\Delta x)$ for $k \in \mathbb{Z}$ and $n \in \mathbb{N}^*$. We assume in the rest of the exercise that $\lambda = c\dfrac{\Delta t}{\Delta x}$ is constant.

1. Study the L^2 stability of this scheme.

2. Show a convergence result of this scheme to the advection equation.