ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Written exam, March 4th, 2015 (2 hours)
(Copies à rendre en français ou en anglais)

1 Parametric optimization : 14 points

We consider a vibrating elastic membrane with a variable thickness $h(x)$, occupying at rest a plane domain Ω (a smooth bounded open set of $\mathbb{R}^2$) and clamped on its boundary. Denoting by λ the square of the vibration frequency and by $u(x)$ its modal displacement, the couple $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$, $u \neq 0$, is a solution of the eigenvalue problem

$$\begin{cases} -\mathrm{div}\,(h\nabla u) = \lambda\,\rho\,h\,u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $\rho > 0$ is the given constant material density. To emphasize its dependence with respect to h the solution of (1) is also denoted by $(\lambda(h), u(h))$. We consider only the first or smallest eigenvalue $\lambda(h)$ and we normalize the eigenfunction $u(h)$ by

$$\int_\Omega \rho h u^2 dx = 1. \tag{2}$$

The thickness belongs to the following space of admissible designs

$$\mathcal{U}_{ad} = \{h \in L^\infty(\Omega)\ , \quad h_{max} \geq h(x) \geq h_{min} > 0 \text{ in } \Omega\}\,.$$

The goal is to minimize the first eigenvalue

$$\inf_{h\in\mathcal{U}_{ad}} \lambda(h)\,. \tag{3}$$

We admit that, as a function from $\mathcal{U}_{ad}$ into $\mathbb{R}\times H_0^1(\Omega)$, the first eigenvalue and eigenfunction $(\lambda(h), u(h))$ is differentiable with respect to h.

1. Let $k \in L^\infty(\Omega)$ be a given function. We denote by $v = \langle u'(h), k\rangle$ the directional derivative of $u(h)$, solution of (1), in the direction k, and by $\Lambda = \langle \lambda'(h), k\rangle$ that of $\lambda(h)$. Give the boundary value problem satisfied by v as well as the normalization condition derived from (2).
2. By multiplying the equation for v by $u(h)$, find an expression for Λ in terms of $u(h)$ only.
3. We admit that the first eigenfunction $u(h)$ is positive and admits a unique point of maximum inside Ω, while its gradient does not vanish at any point on the boundary $\partial\Omega$. Prove that, if it exists, a minimizer of (3) must be of minimal thickness near the boundary and of maximal thickness near the maximum of $u(h)$.

4. We now replace (3) by the following objective function

$$\inf_{h\in\mathcal{U}_{ad}} \left\{ J(h) = \int_\Omega j\left(\frac{u(h)}{\|u(h)\|}\right) dx \right\}, \tag{4}$$

   where j is a given smooth bounded even function and $\|\phi\|$ denotes the $L^2(\Omega)$-norm of a function ϕ. Check that (4) is independent of the normalization choice for $u(h)$. Write the Lagrangian $\mathcal{L}(h, \hat\lambda, \hat u, \hat p)$, associated to (4), defined on $\mathcal{U}_{ad} \times \mathbb{R} \times H^1_0(\Omega) \times H^1_0(\Omega)$.

5. For a function $\hat u \neq 0$ compute the directional derivative in $L^2(\Omega)$ of

$$F(\hat u) = \int_\Omega j\left(\frac{\hat u}{\|\hat u\|}\right) dx$$

   and show that the directional derivative vanishes in the direction of $\hat u$.

6. Deduce the variational formulation of the adjoint boundary value problem, the solution of which is denoted by p.

7. Write the boundary value problem satisfied by the adjoint p. Show that the right hand side in the adjoint equation is orthogonal to u, the solution of (1). Show that, if p is a solution, then $(p + Cu)$ is another solution for any constant C. From the partial derivative of $\mathcal{L}$ with respect to $\hat\lambda$, find the normalization condition for p that determines the constant C.

8. Compute (at least formally) the derivative $J'(h)$ of (4).

2 Geometric optimization : 6 points

We consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$. For a given source term $f \in L^2(\mathbb{R}^N)$ and a given boundary condition $g \in H^1(\mathbb{R}^N)$, we define the solution $u \in H^1(\Omega)$ of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{5}$$

We minimize the objective function

$$\min_{\Omega\subset\mathbb{R}^N} \left\{ J(\Omega) = \int_\Omega j(u)\, dx \right\}, \tag{6}$$

where j is a smooth function satisfying

$$|j(v)| \le C(|v|^2 + 1) \quad \text{and} \quad |j'(v)| \le C(|v| + 1).$$

We use Hadamard's method of shape variations.

1. Write the Lagrangian corresponding to (6), taking care of the non-homogeneous Dirichlet boundary condition on $\partial\Omega$.
2. Deduce the adjoint problem, the solution of which is denoted by p.
3. Compute (formally) the shape derivative of (6).