Functional analysis and applications
by Grégoire Allaire
Basic course of the Master Program "Mathematical Modelling"
Ecole Polytechnique and Pierre et Marie Curie University
Academic year 2015-2016

1 Introduction

1.1 Motivation: pde's and optimization

1.2 Banach spaces

English: section 3.1 in [3].

1.3 Hilbert spaces

French: sections V.1 of [1]. English: sections 5.1 of [2], section 4.1 in [3].

1.4 Examples: Lebesgue L^p spaces

French: sections IV.1 and IV.2 of [1]. English: sections 4.1 and 4.2 of [2], section 3.4 in [3].

Measurable functions, simple functions, Young's inequality, Hölder inequality, completeness of L^p, norm convergence versus almost everywhere convergence.

2 Linear operators on Banach spaces

2.1 Definitions

Linear and continuous applications, duality, bidual. Dual space of L^p, Hahn-Banach theorem.

English: section 3.5 in [3].

2.2 The "great" theorems of Banach

Banach-Steinhaus theorem, open mapping theorem, closed graph theorem.

French: sections II.1, II.2 and II.3 of [1]. English: sections 2.1, 2.2 and 2.3 of [2], sections 5.3, 5.6 and 5.7 in [3].

2.3 Introduction to unbounded operators

Domain, adjoint

French: section II.6 of [1]. English: section 2.6 of [2].

3 Weak convergence

3.1 Definitions

Weak and weak-* sequential convergence.

French: section III.2 of [1]. English: section 3.2 of [2], section 5.12 in [3].

3.2 Sequential compactness

Proof of sequential weak-* compactness (by a diagonal sequence extraction). Statement of sequential weak compactness. Weak lower semi-continuity of convex functionals.

French: sections III.4, III.5 and III.6 of [1]. English: sections 3.4, 3.5 and 3.6 of [2].

3.3 A few words about topology

Link with the notion of topological space and definitions of weak topologies.

3.4 Application to Lebesgue L^p spaces

Examples (oscillation, concentration, escape at infinity, evanescence).

French: section IV.3 of [1]. English: section 4.3 of [2].

4 Compact operators and spectral analysis

4.1 Compact operators and Fredholm alternative

Definition of compact operators in a Hilbert space. Riesz lemma. Fredholm alternative in a Hilbert space. Application to integral equations.

French: sections V1.1 and VI.2 of [1]. English: sections 6.1 and 6.2 of [2].

4.2 Self-adjoint operators

Definition of self-adjoint operators in a Hilbert space. Eigenvalues and eigenfunctions. Spectral decomposition of self-adjoint compact operators.

French: section VI.4 of [1]. English: section 6.4 of [2].

5 Distributions, Fourier analysis and Sobolev spaces

5.1 Distributions

Definition, convergence of test functions in $C_c^\infty(\Omega)$, convergence of distributions

French: chapter 3 in [4]. English: section 6.3 in [3].

5.2 Fourier analysis

Definition in the Schwartz class and in $L^2(\mathbb{R}^N)$

French: chapter 9 of [5]. English: chapter 9 in [9].

5.3 Sobolev spaces

Definition of $W^{m,p}(\Omega)$ for an integer $m \in \mathbb{N}$. Definition of $H^s(\mathbb{R}^N)$ for a real $s \geq 0$. Definition of a C^1 smooth open set. Extension operator, partition of unity and the local charts argument. Non-local norm on $W^{s,p}(\Omega)$ for $0 < s < 1$ and its equivalence with the Fourier definition for $p = 2$. Negative Sobolev spaces as duals of $W_0^{m,p}(\Omega)$.

French: section IX.1 of [1]. English: section 9.1 of [2].

5.4 Embeddings and traces

Trace theorem from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega)$ (with proof by partition of unity and Fourier analysis). Sobolev embedding theorems (with proof in $\mathbb{R}^N$).

French: section IX.3 of [1]. English: section 9.3 of [2].

5.5 Compactness

Rellich and Rellich-Kondrachov theorems (with a proof by Fourier series in the torus).

French: section IX.3 of [1]. English: section 9.3 of [2].

5.6 Time dependent Sobolev spaces

Definition of $L^p((0,T);W^{m,q}(\Omega))$ and $C((0,T);W^{m,q}(\Omega))$. Aubin-Lions compactness lemma.

French: volume 1, chapter 3 of [6]. English: volume 1, chapter 3 of [7].

References

[1] H. Brézis *Analyse fonctionnelle*, Masson, Paris (1983).

[2] H. Brézis *Functional analysis, Sobolev spaces and partial differential equations*, Universitext. Springer, New York (2011).

[3] Ph. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications*, SIAM, Philadelphia (2013).

[4] F. Golse, *Distributions, analyse de Fourier, équations aux dérivées partielles*, Ecole Polytechnique (2011).

[5] F. Golse, Y. Laszlo, F. Pacard, C. Viterbo, *Analyse réelle et complexe*, Ecole Polytechnique (2013).

[6] J.-L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications,* Travaux et Recherches Mathmatiques, No. 17 Dunod, Paris 1968.

[7] J.-L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications,* Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972.

[8] W. McLean, *Strongly elliptic systems and boundary integral equations,* Cambridge University Press, Cambridge, 2000.

[9] W. Rudin, *Real and complex analysis*, Third edition. McGraw-Hill Book Co., New York (1987).

[10] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces,* Lecture Notes of the Unione Matematica Italiana, 3. Springer, Berlin; UMI, Bologna, 2007.