The Lavrentiev phenomena

by

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– CMAP Ecole Polytechnique –

Abstract

- 1. The Lavrentiev phenomenon
- 2. The Manià example
- 3. A class of Lagrangians without Lavrentiev phenomenon
- 4. A multi-dimensional variational problem
- 5. The case of higher-order Lagrangians
- 6. "$+\infty$-values" phenomenon

The Lavrentiev phenomenon

Let $L : [a,b] \times \mathbb{R}^N \times \mathbb{R}^N \to [-\infty, +\infty]$ be the Lagrangian function associated to an action functional

$$\mathcal{I}(x) = \int_a^b L(t, x(t), x'(t))dt$$

and consider the following sets of admissible trajectories:

$$\mathbf{AC}_*[a,b] = \{x \in \mathbf{AC}([a,b]; \mathbb{R}^N) : x(a) = A, x(b) = B\},$$
$$\mathbf{Lip}_*[a,b] = \{x \in \mathbf{Lip}([a,b]; \mathbb{R}^N) : x(a) = A, x(b) = B\}.$$

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The action $\mathcal{I}$ exhibits the Lavrentiev phenomenon (LP) whenever

$$\inf_{\mathbf{AC}_*[a,b]} \mathcal{I} < \inf_{\mathbf{Lip}_*[a,b]} \mathcal{I}.$$

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- We cannot calculate a minimizer by using a standard finite-element method.
- The set of trajectories is a fundamental part of the physical model.

The Manià example

- The action

$$\mathcal{I}(x) = \int_{-1}^{1} x'^{6}(t)[x^3(t) - t]^2 dt,$$

with boundary conditions $x(-1) = -1$, $x(1) = 1$, exhibits (LP), i.e.

$$\inf_{\mathbf{AC}_*[0,1]} \mathcal{I} < \inf_{\mathbf{Lip}_*[0,1]} \mathcal{I}.$$

($\bar{x}(t) = \sqrt[3]{t}$ is a minimizer for $\mathcal{I}$ in $\mathbf{AC}_*[-1,1]$.)

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- (LP) persists under perturbations of the Lagrangians:

$$\int_{-1}^{1} \{{x'}^6(t)[x^3(t) - t]^2 + \epsilon |x'(t)|^{5/4}\} dt, \quad x(-1) = -1,\ x(1) = 1,$$

exhibits (LP) for any "small" ϵ.

- (LP) persists under perturbations of the boundary conditions:

$$\int_{-1}^{1} {x'}^6(t)[x^3(t) - t]^2 dt, \quad x(t_{-1}) = x_{-1},\ x(t_1) = x_1,$$

where $(t_{-1}, x_{-1}) \in B((-1, -1), \epsilon)$, $(t_1, x_1) \in B((1, 1), \epsilon)$, exhibits (LP) for any "small" ϵ.

A class of Lagrangians without (LP)

Theorem [A. Cellina, A. F., E.M. Marchini]. Let $x : [a, b] \to \mathbb{R}^N$ be a trajectory in $\mathbf{AC}[a, b]$. Assume that:

1. $L_1(x, \xi), \cdots, L_m(x, \xi)$: Im$[x] \times \mathbb{R}^N \to \mathbb{R}$ are continuous and convex in ξ;
2. $\psi_1, \cdots, \psi_m$: $[a, b] \times$ Im$[x] \to [c, +\infty)$ are continuous, with $c > 0$;
3. $\mathcal{I}(x) = \int_a^b \sum_{i=1}^m L_i(x(t), x'(t))\psi_i(t, x(t))dt$.

Then, given $\epsilon > 0$, there exists a Lipschitzian trajectory x_ϵ, a reparameterization of x, such that $x(a) = x_\epsilon(a)$, $x(b) = x_\epsilon(b)$ and $\mathcal{I}(x_\epsilon) \leq \mathcal{I}(x) + \epsilon$. □

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- The class of Lagrangians $L(t, x, x') = \sum_{i=1}^m L_i(x, x')\psi_i(t, x)$ does not exhibit (LP) for any boundary conditions; it includes the autonomous Lagrangians.
- Condition 2. is used only to prove that $\int_a^b L_i(x(t), x'(t))dt$ are finite. The Theorem can be proved under the more general condition:

  2′. $\psi_i(t, x) \geq 0$ and $\int_a^b L_i(x(t), x'(t))dt < +\infty$, for any i.
- We cannot drop condition 2′.: setting $m = 1$, $\psi_1(t, x) = [x^3 - t]^2$ and $L_1(x, x') = x'^6$, we obtain the Lagrangian of Manià, $\psi_1 \geq 0$ and $\int_0^1 L_1(\bar{x}(t), \bar{x}'(t))dt = \int_0^1 1/(3^6 t^4)dt = +\infty$.

A multi-dimensional variational problem without (LP)

Let $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a radial Lagrangian with respect to the gradient, i.e. there exists a function $h : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$ such that $L(u, \xi) = h(u, |\xi|)$.
Consider the action

$$\mathcal{I}(u) = \int_{S[a,b]} L(u(x), \nabla u(x))dx$$

where $S[a, b] = \{x \in \mathbb{R}^N : 0 < a \leq |x| \leq b\}$.
We denote with $\mathbf{Lip}_r(S[a, b])$ and $\mathbf{W}_r^{1,1}(S[a, b])$ respectively the sets

$$\{u \in \mathbf{Lip}(S[a, b]) : u \text{ radial}, u|_{\partial B(0,a)} = A, u|_{\partial B(0,b)} = B\},$$
$$\{u \in \mathbf{W}^{1,1}(S[a, b]) : u \text{ radial}, u|_{\partial B(0,a)} = A, u|_{\partial B(0,b)} = B\}.$$

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- Let L be continuous and convex with respect to the gradient.

  Then, $$\inf_{\mathbf{W}_r^{1,1}(S[a,b])} \mathcal{I} = \inf_{\mathbf{Lip}_r(S[a,b])} \mathcal{I}.$$

(LP) *for higher-order Lagrangians*

The Lavrentiev phenomenon occurs as well for problems of the Calculus of Variations of order $\nu + 1$, with $\nu \in \mathbb{N}$:

- minimize $$\mathcal{I}(x) = \int_a^b L(t, x(t), \cdots, x^{(\nu+1)}(t))dt$$

  on a set $\mathbf{X}$ of admissible trajectories $x : [a, b] \to \mathbb{R}^N$ satisfying the boundary conditions $x(a) = A$, $x(b) = B$, $\cdots$, $x^{(\nu)}(a) = A^{(\nu)}$, $x^{(\nu)}(b) = B^{(\nu)}$.

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$\mathcal{I}$ exhibits the Lavrentiev phenomenon (LP) whenever

$$\inf_{\mathbf{W}_*^{\nu+1,1}(a,b)} \mathcal{I} < \inf_{\mathbf{W}_*^{\nu+1,\infty}(a,b)} \mathcal{I}.$$

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- Autonomous higher-order Lagrangians can present (LP):

$$\mathcal{I}(x) = \int_0^1 |x''(t)|^7 [3x(t) - 3|x'(t) - 1|^2 - 2|x'(t) - 1|^3]^2 dt,$$

with boundary conditions $x(0) = 0$, $x(1) = 5/3$, $x'(0) = 1$, $x'(1) = 2$ [A.V. Sarychev, 1997].

A class of higher-order Lagrangians without (LP)

Theorem [A. F.]. Let $x : [a,b] \to \mathbb{R}^N$ be a trajectory in $\mathbf{W}^{\nu+1,1}[a,b]$. Assume that:

1. $L_1(w,\xi), \cdots, L_m(w,\xi)$: $\mathrm{Im}_\delta[x^{(\nu)}] \times \mathbb{R}^N \to \mathbb{R}$ are continuous and convex in ξ;
2. $\psi_1, \cdots, \psi_m : [a,b] \times \mathrm{T}^\nu_\delta[x] \to [0,+\infty)$ are continuous and $\psi_i(t, x(t), x'(t), \cdots, x^{(\nu)}(t)) \neq 0$, for any t in $[a,b]$, $i = 1, \cdots, m$;
3. $\mathcal{I}(x) = \displaystyle\int_a^b \sum_{i=1}^m L_i(x^{(\nu)}(t), x^{(\nu+1)}(t)) \psi_i(t, x(t), x'(t), \cdots, x^{(\nu)}(t)) dt$.

Then, given $\epsilon > 0$, there exist a trajectory x_ϵ in $\mathbf{W}^{\nu+1,\infty}(a,b)$ such that $x_\epsilon(a) = x(a)$, $x_\epsilon(b) = x(b)$, $\cdots$, $x_\epsilon^{(\nu)}(a) = x^{(\nu)}(a)$, $x_\epsilon^{(\nu)}(b) = x^{(\nu)}(b)$ and $\mathcal{I}(x_\epsilon) \leq \mathcal{I}(x) + \epsilon$. $\square$

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- For strictly positive ψ_i, the class of Lagrangians $L(t,x,\cdots,x^{\nu+1}) = \sum_{i=1}^m L_i(x^{(\nu)},x^{(\nu+1)})\psi_i(t,x,\cdots,x^{(\nu)})$, does not exhibit (LP) for any boundary conditions.
- Condition 2. is used only to prove that $\int_a^b L_i(x^{(\nu)}(t),x^{(\nu+1)}(t))dt$ are finite. The Theorem can be proved under the more general condition:
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"$+\infty$-values" phenomenon

Theorem [A. F.]. Let $\nu \in \mathbb{N} \cup \{0\}$. Assume that:

1. $L_1(w,\xi), \cdots, L_m(w,\xi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are continuous and convex in ξ;
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If $\mathcal{I}$ exhibits the Lavrentiev phenomenon, then $\mathcal{I}$ takes the value $+\infty$ in any neighbourhood in $\mathbf{W}_*^{\nu+1,1}[a,b]$ of a minimizer. □

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- The Theorem applies to the actions of Manià and Sarychev, for instance.
- In case we know a priori that the action does not assume the values $+\infty$ on the admissible trajectories, (LP) does not occur.

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