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Homogenization of Composite Ferromagnetic Materials

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The objective of this paper is to perform, by means of *Γ-convergence* and *two-scale convergence*, a rigorous derivation of the homogenized GIBBS-LANDAU free energy functional associated to a composite periodic ferromagnetic material, i.e. a ferromagnetic material in which the heterogeneities are periodically distributed inside the media. We thus describe the Γ-limit of the GIBBS-LANDAU free energy functional, as the period over which the heterogeneities are distributed inside the ferromagnetic body shrinks to zero.

1. Introduction

The study of composites and their homogenization is a subject with a long history, which has attracted the interest and the efforts of some of the most illustrious names in science (cfr. [21,24] for historical details).

Nowadays, nonhomogeneous and *periodic* ferromagnetic materials are the subject of a growing interest. Actually such periodic configurations often combine the attributes of the constituent materials, while sometimes, their properties can be strikingly different from the properties of the different constituents [26]. These periodic configurations can be therefore used to achieve physical and chemical properties difficult to achieve with homogeneous materials. To predict the magnetic behavior of such composite materials is of prime importance for applications [26].

From a mathematical point of view, the study of composite materials, and more generally of media which involve microstructures, is the main source of inspiration for the *Mathematical Theory of Homogenization* which, roughly speaking, is a mathematical procedure which aims at understanding heterogeneous materials with highly oscillating heterogeneities (at the microscopic level) via an effective model [28].

The main objective of this paper is to perform, in the framework of DE GIORGI's notion of Γ*-convergence* [14] and ALLAIRE [3] and NGUETSENG [29] notion of *two-scale convergence*, a mathematical homogenization study of the GIBBS-LANDAU free energy functional associated to a composite periodic ferromagnetic material, i.e. a ferromagnetic material in which the heterogeneities are periodically distributed inside the ferromagnetic media.

Compared to earlier works related to the subject (see for instance [12,13,16,31]) we consider here the full GIBBS-LANDAU functional for mixtures of different materials in three-dimensional space. The result is achieved by computing the Γ-limit of the exchange energies, and by proving that the remaining terms continuously converge as $\varepsilon \to 0$. Concerning the family of exchange energies, the asymptotic limit is derived by means of a two-scale convergence approach which gives a rather short method to compute the homogenized exchange energy (avoiding the use of [5]). Concerning the remaining terms in the free energy, the homogenization of the magnetostatic self-energy requires some original computation and attention. Indeed, the identification of the asymptotic limit is obtained via the introduction of a suitable notion of weighted two-scale convergence and the proof of two compactness results which permit to rigorously justify some arguments used in [31] to treat a similar result.

(a) The Landau-Lifshitz micromagnetic theory of single-crystal ferromagnetic materials

According to LANDAU and LIFSHITZ micromagnetic theory of ferromagnetic materials (see [6,9,10,20,25]), the states of a rigid *single-crystal* ferromagnet, occupying a region $\Omega \subseteq \mathbb{R}^3$, and subject to a given external magnetic field $\boldsymbol{h}_a$, are described by a vector field, the magnetization $\mathbf{M}$, verifying the so-called *fundamental constraint of micromagnetic theory*: A ferromagnetic body is always locally saturated, i.e. there exists a positive constant M_s such that

$$|\mathbf{M}| = M_s(T) \;\text{ a.e. in } \Omega. \tag{1.1}$$

The *saturation magnetization* M_s depends on the specific material and on the temperature T, and vanishes above a temperature (characteristic of each crystal type) known as the CURIE point. Since we will assume that the specimen is at a fixed temperature below the CURIE point of the material, the value M_s will be regarded as a material dependent function (and therefore as a constant function when working on single-crystal ferromagnets). Due to the constraint (1.1) in the sequel we express the magnetization $\mathbf{M}$ under the form $\mathbf{M} := M_s(T)\boldsymbol{m}$ where $\boldsymbol{m} : \Omega \to \mathbb{S}^2$ is a vector field which takes its values on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$.

Even though the magnitude of the magnetization vector is constant in space, in general it is not the case for its direction, and the observable states can be mathematically characterized as local minimizers of the GIBBS-LANDAU free energy functional associated to the single-crystal ferromagnetic particle (using the notation of [6,25])

$$\begin{aligned}\mathcal{G}_{\mathcal{L}}(\boldsymbol{m}) &:= \int_\Omega a_{\text{ex}} |\nabla \boldsymbol{m}|^2 \, \mathrm{d}\tau + \int_\Omega \varphi_{\text{an}}(\boldsymbol{m}) \, \mathrm{d}\tau - \frac{\mu_0}{2} \int_\Omega \boldsymbol{h}_{\text{d}}[M_s \boldsymbol{m}] \cdot M_s \boldsymbol{m} \, \mathrm{d}\tau - \mu_0 \int_\Omega \boldsymbol{h}_a \cdot M_s \boldsymbol{m} \, \mathrm{d}\tau \\ &:= \mathcal{E}(\boldsymbol{m}) + \mathcal{A}(\boldsymbol{m}) + \mathcal{W}(\boldsymbol{m}) + \mathcal{Z}(\boldsymbol{m}). \end{aligned} \tag{1.2}$$

The first term, $\mathcal{E}(\boldsymbol{m})$, called *exchange energy*, penalizes spatial variations of $\boldsymbol{m}$. The factor a_{ex} in the term is a phenomenological positive material constant which summarizes the effect of (usually very) short-range exchange interactions.

The second term, $\mathcal{A}(\boldsymbol{m})$, called the *anisotropy energy*, models the existence of preferred directions for the magnetization (the so-called *easy axes*), which usually depend on the crystallographic structure of the material. The anisotropy energy density $\varphi_{\text{an}} : \mathbb{S}^2 \to \mathbb{R}^+$ is assumed to be a non-negative even and globally Lipschitz continuous function, that vanishes only on a finite set of unit vectors (the *easy axes*).

The third term, $\mathcal{W}(\boldsymbol{m})$, is called the *magnetostatic self-energy*, and is the energy due to the (dipolar) magnetic field, also known in literature as the stray field, $\boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}]$ generated by $\boldsymbol{m}$. From the mathematical point of view, assuming Ω to be open, bounded and with a Lipschitz boundary, a given magnetization $\boldsymbol{m} \in L^2(\Omega, \mathbb{R}^3)$ generates the stray field $\boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}] = \nabla u_{\boldsymbol{m}}$ where the potential $u_{\boldsymbol{m}}$ solves:

$$\Delta u_{\boldsymbol{m}} = -\operatorname{div}(\boldsymbol{m}\chi_\Omega) \;\text{ in }\; \mathcal{D}'(\mathbb{R}^3). \tag{1.3}$$

In (1.3) we have indicated with $\boldsymbol{m}\chi_\Omega$ the extension of $\boldsymbol{m}$ to $\mathbb{R}^3$ that vanishes outside Ω. LAX-MILGRAM theorem guarantees that equation (1.3) possesses a unique solution in the BEPPO-LEVI space:

$$BL^1(\mathbb{R}^3) = \left\{ u \in \mathcal{D}'(\mathbb{R}^3) \,:\, \frac{u(\cdot)}{\sqrt{1+|\cdot|^2}} \in L^2(\mathbb{R}^3),\; \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3) \right\}. \tag{1.4}$$

Moreover the stray field is a positive semidefinite operator of norm one, i.e., once denoted by $(\cdot,\cdot)_\Omega$ the inner product in $L^2(\Omega)$, one has:

$$\mathcal{W}(\boldsymbol{m}) := -\frac{\mu_0}{2}(\boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}], \boldsymbol{m})_\Omega \geq 0 \quad , \quad \|\boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}]\|_\Omega \leq \|\boldsymbol{m}\|_\Omega \quad \forall \boldsymbol{m} \in L^2(\Omega). \tag{1.5}$$

The physical constant μ_0 denoting the vacuum permeability.

Finally, the fourth term $\mathcal{Z}(\boldsymbol{m})$, is called the *interaction energy* (or *Zeeman energy*), and models the tendency of a specimen to have its magnetization aligned with the constant in space external applied field $\boldsymbol{h}_a \in \mathbb{R}^3$, assumed to be unaffected by variations of $\boldsymbol{m}$.

The competition of those four terms explain most of the striking pictures of the magnetization that one can see in most ferromagnetic material [19], in particular the so-called *domain structure*, that is large regions of uniform or slowly varying magnetization (the *magnetic domains*) separated by very thin transition layers (the *domain walls*).

(b) The Gibbs-Landau energy functional associated to composite ferromagnetic materials

Physically speaking, when considering a ferromagnetic body composed of several magnetic materials (i.e. a non single-crystal ferromagnet) a new mathematical model has to be introduced. In fact, as far as the ferromagnet is no more a single crystal, the material depending functions $a_{\mathrm{ex}}, M_s(T)$ and φ_{an} are no longer constant on the region Ω occupied by the ferromagnet. Moreover one has to describe the local interactions of two grains with different magnetic properties at their touching interface [1].

From a mathematical point of view, this latter requirement is usually taken into account in two different ways. Either one adds to the model a surface energy term which penalizes jumps of the magnetization direction $\boldsymbol{m}$ at the interface of both grains, or, and we stick on this later on, one simply considers a *strong coupling*, meaning that the direction of the magnetization does not jump through an interface. We insist on the fact that only the direction is continuous at an interface while the magnitude M_s is obviously discontinuous. Therefore, the natural mathematical setting for the problem turns out to be characterized by the assumption that the magnetization direction $\boldsymbol{m}$ is in the "weak" SOBOLEV metric space $(H^1(\Omega, \mathbb{S}^2), d_{L^2(\Omega,\mathbb{S}^2)})$, i.e. on the metric subspace of $H^1(\Omega, \mathbb{R}^3)$ constituted by the functions constrained to take values on the unit sphere of $\mathbb{R}^3$ and endowed with the $L^2(\Omega)$ metric. It is in this framework that we will conduct our work from now on.

We start by recalling the basic idea of the mathematical theory of homogenization. Let $\Omega \subset \mathbb{R}^3$ be the region occupied by the composite material. If we assume that the heterogeneities are regularly distributed, we can model the material as periodic. As illustrated in Fig.1, this means that we can think of the material as being built up of small identical cubes, the side length of which being called ε. Let $Q = [0,1]^3$ be the unit cube of $\mathbb{R}^3$. We let for $y \in Q$, $a_{\mathrm{ex}}(y), M_s(y), \varphi_{\mathrm{an}}(y, \boldsymbol{m})$ be the periodic repetitions of the functions that describe how the exchange constant a_{ex}, the

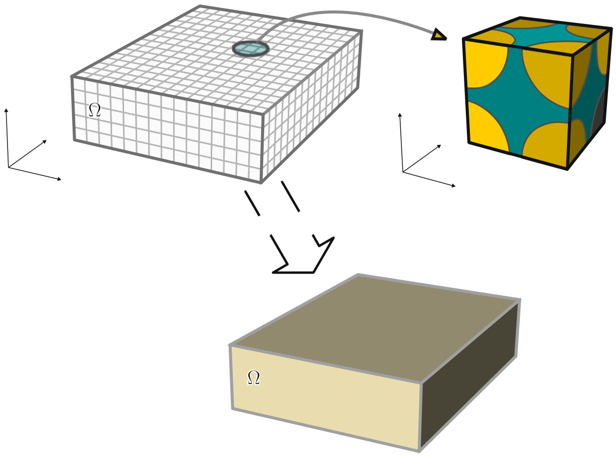

Figure 1. If we assume that the heterogeneities are evenly distributed inside the ferromagnetic media Ω, we can model the material as periodic. As illustrated in the figure, this means that we can think of the material as being built up of small identical cubes Q_ε, the side length of which we call ε.

saturation magnetization M_s and the anisotropy density energy $\varphi_{\mathrm{an}}(y,\boldsymbol{m})$ vary over the representative cell Q (see Fig. 1). Substituting x/ε for y, we obtain the «two-scale» functions $a_\varepsilon(x) := a_{\mathrm{ex}}(x/\varepsilon)$, $M_\varepsilon(x) := M_s(x/\varepsilon)$ and $\varphi_\varepsilon(x,\boldsymbol{m}) := \varphi_{\mathrm{an}}(x/\varepsilon,\boldsymbol{m})$ that oscillate periodically with period ε as the variable x runs through Ω, describing the oscillations of the material dependent parameters of the composite. At every scale ε, the energy associated to the ε-heterogeneous ferromagnet, will be given by the following generalized GIBBS-LANDAU energy functional

$$\mathcal{G}_{\mathcal{L}}^{\varepsilon}(\boldsymbol{m}) := \int_\Omega a_\varepsilon |\nabla \boldsymbol{m}|^2 \,\mathrm{d}\tau + \int_\Omega \varphi_\varepsilon(\cdot,\boldsymbol{m})\,\mathrm{d}\tau - \frac{\mu_0}{2}\int_\Omega \boldsymbol{h}_{\mathrm{d}}[M_\varepsilon \boldsymbol{m}]\cdot M_\varepsilon \boldsymbol{m}\,\mathrm{d}\tau - \mu_0 \int_\Omega \boldsymbol{h}_a \cdot M_\varepsilon \boldsymbol{m}\,\mathrm{d}\tau$$
$$:= \mathcal{E}_\varepsilon(\boldsymbol{m}) + \mathcal{A}_\varepsilon(\boldsymbol{m}) + \frac{\mu_0}{2}\mathcal{W}_\varepsilon(\boldsymbol{m}) + \mu_0 \mathcal{Z}_\varepsilon(\boldsymbol{m}). \tag{1.6}$$

The asymptotic Γ-convergence analysis of the family of functionals $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon\in\mathbb{R}^+}$ as ε tends to 0, is the object of the present paper.

(c) Statement of the main result

The purpose of this paper is to analyze, by the means of both Γ-convergence and two-scale convergence techniques, the asymptotic behavior, as $\varepsilon \to 0$, of the family of GIBBS-LANDAU free energy functionals $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon\in\mathbb{R}^+}$ expressed by (1.6). Let us make the statement more precise.

We consider the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ and, for every $s \in \mathbb{S}^2$, the tangent space of $\mathbb{S}^2$ at a point s will be denoted by $T_s(\mathbb{S}^2)$. The class of admissible maps we are interested in is defined as

$$H^1(\Omega,\mathbb{S}^2) := \left\{ \boldsymbol{m} \in H^1(\Omega,\mathbb{R}^3) \,:\, \boldsymbol{m}(x) \in \mathbb{S}^2 \text{ for } \tau\text{-a.e. } x \in \Omega \right\},$$

where we have denoted by τ the Lebesgue measure[1] on $\mathbb{R}^3$. We consider $H^1(\Omega,\mathbb{S}^2)$ as a metric space endowed with the metric structure induced by the classical $L^2(\Omega,\mathbb{R}^3)$ metric. We recall that a function $u:\mathbb{R}^3 \to \mathbb{R}$ is said to be Q-periodic if $u(\cdot) = u(\cdot + e_i)$ for every e_i in the canonical basis (e_1,e_2,e_3) of $\mathbb{R}^3$.

For the energy densities appearing in the family $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon\in\mathbb{R}^+}$ we assume the following hypotheses:

[1]We denote the Lebesgue measure by τ (as in [9]) to reserve the letter μ for the vacuum permeability.

[$\boldsymbol{H_1}$] The *exchange parameter* a_{ex} is supposed to be a Q-periodic measurable function belonging to $L^\infty(Q)$, bounded from below and above by two positive constants $c_{\mathrm{ex}} > 0, C_{\mathrm{ex}} > 0$, i.e. $0 < c_{\mathrm{ex}} \leqslant a_{\mathrm{ex}}(y) \leqslant C_{\mathrm{ex}}$ for τ-a.e. $y \in Q$. This hypothesis guarantees that the exchange energy density, which has the form $g(y,\xi) := a_{\mathrm{ex}}(y)|\xi|^2$, $\xi \in \mathbb{R}^{3\times 3}$, falls into the category of CARATHÉODORY integrands satisfying, for τ-a.e. $y \in Q$, the standard quadratic growth condition

$$\forall \xi \in \mathbb{R}^{3\times 3} \quad c_{\mathrm{ex}}|\xi|^2 \leqslant g(y,\xi) \leqslant C_{\mathrm{ex}}(1+|\xi|^2). \tag{1.7}$$

Then we set $a_\varepsilon(x) := a_{\mathrm{ex}}(x/\varepsilon)$.

[$\boldsymbol{H_2}$] The *anisotropy density energy* $\varphi_{\mathrm{an}} : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^+$ is supposed to be a Q-periodic measurable function belonging to $L^\infty(Q)$ with respect to the first variable, and globally Lipschitz with respect to the second one (uniformly with respect to the first variable), i.e. $\exists \kappa_L > 0$ such that

$$\operatorname*{ess\,sup}_{y\in Q} |\varphi_{\mathrm{an}}(y,s_1) - \varphi_{\mathrm{an}}(y,s_2)| \leqslant \kappa_L |s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{S}^2. \tag{1.8}$$

We then set $\varphi_\varepsilon(x,s) := \varphi_{\mathrm{an}}(x/\varepsilon, s)$. The hypotheses assumed on φ_{an} are sufficiently general to treat the most common classes of crystal anisotropy energy densities arising in applications. As a sake of example, for uniaxial anisotropy, the energy density reads as

$$\varphi_{\mathrm{an}}(y,\boldsymbol{m}(x)) = \kappa(y)[1 - (\boldsymbol{u}(y)\cdot \boldsymbol{m}(x))^2], \tag{1.9}$$

the spatially dependent unit vector $\boldsymbol{u}(\cdot)$ being the easy axis of the crystal. For cubic type anisotropy, the energy density reads as:

$$\varphi_{\mathrm{an}}(y,\boldsymbol{m}(x)) = \kappa(y)\sum_{i=1}^{3}[(\boldsymbol{u}_i(y)\cdot \boldsymbol{m}(x))^2 - (\boldsymbol{u}_i(y)\cdot \boldsymbol{m}(x))^4], \tag{1.10}$$

the mutually orthogonal unit vectors $\boldsymbol{u}_i(\cdot)$ being the three *easy-axes* of the cubic crystal. Note that the anisotropy depends on the material both in strength $\kappa(y)$ and in the direction $\boldsymbol{u}_i(y)$.

[$\boldsymbol{H_3}$] The *saturation magnetization* M_s is supposed to be a Q-periodic measurable function belonging to $L^\infty(Q)$, and we set $M_\varepsilon(\cdot) = M_s(\cdot/\varepsilon)$.

The main result of this paper is the following:

Theorem 1.1. *Let* $(\mathcal{G}^\varepsilon_{\mathcal{L}})_{\varepsilon\in\mathbb{R}^+}$ *be a family of* GIBBS-LANDAU *free energy functionals satisfying the hypotheses* $[\boldsymbol{H_1}], [\boldsymbol{H_2}]$ *and* $[\boldsymbol{H_3}]$. *Then* $(\mathcal{G}^\varepsilon_{\mathcal{L}})_{\varepsilon\in\mathbb{R}^+}$ *is* ***equicoercive*** *in the metric space* $(H^1(\Omega,\mathbb{S}^2), d_{L^2(\Omega,\mathbb{S}^2)})$. *Moreover* $(\mathcal{G}^\varepsilon_{\mathcal{L}})_{\varepsilon\in\mathbb{R}^+}$ *Γ-converges in* $(H^1(\Omega,\mathbb{S}^2), d_{L^2(\Omega,\mathbb{S}^2)})$ *to the functional* $\mathcal{G}_{\mathrm{hom}} : H^1(\Omega,\mathbb{S}^2) \to \mathbb{R}^+$ *defined by*

$$\mathcal{G}_{\mathrm{hom}}(\boldsymbol{m}) := \mathcal{E}_{\mathrm{hom}}(\boldsymbol{m}) + \mathcal{A}_{\mathrm{hom}}(\boldsymbol{m}) + \frac{\mu_0}{2}\mathcal{W}_{\mathrm{hom}}(\boldsymbol{m}) + \mu_0 \mathcal{Z}_{\mathrm{hom}}(\boldsymbol{m}). \tag{1.11}$$

The four terms that appear in (1.11) have the following expressions: The homogenized ***exchange energy*** *is given by*

$$\mathcal{E}_{\mathrm{hom}}(\boldsymbol{m}) := \int_\Omega A_{\mathrm{hom}} \nabla \boldsymbol{m}(x) : \nabla \boldsymbol{m}(x)\,\mathrm{d}x, \tag{1.12}$$

where A_{hom} *is the «classical» homogenized tensor* A_{hom} *given by the average*

$$A_{\mathrm{hom}} := \langle a_{\mathrm{ex}}(I + \nabla\boldsymbol{\varphi})^T (I + \nabla\boldsymbol{\varphi})\rangle_Q, \; \boldsymbol{\varphi} := (\varphi_1, \varphi_2, \varphi_3), \tag{1.13}$$

where for every $j \in \mathbb{N}_3$ *the component* φ_j *is the unique (up to a constant) solution of the following scalar unit cell problem*

$$\varphi_j := \operatorname*{argmin}_{\varphi\in H^1_\#(Q)} \int_Q a_{\mathrm{ex}}(y)[|\nabla\varphi(y) + e_j|^2]\,\mathrm{d}y. \tag{1.14}$$

The homogenized ***anisotropy energy*** *is given by*

$$\mathcal{A}_{\text{hom}}(\boldsymbol{m}) := \int_{\Omega\times Q} \varphi_{\text{an}}(y, \boldsymbol{m}(x))\,\mathrm{d}y\,\mathrm{d}x, \tag{1.15}$$

*while the homogenized **magnetostatic self-energy** is given by*

$$\mathcal{W}_{\text{hom}}(\boldsymbol{m}) := -\langle M_s\rangle_Q^2 \int_\Omega \boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}]\cdot\boldsymbol{m}\,\mathrm{d}\tau + \int_{\Omega\times Q} |\nabla_y v_{\boldsymbol{m}}(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y, \tag{1.16}$$

where, for every $x\in\Omega$, the scalar function $v_{\boldsymbol{m}}:\Omega\times Q\to\mathbb{R}$, is the unique solution of the cell problem:

$$\boldsymbol{m}(x)\cdot\int_Q M_s(y)\nabla_y\psi(y)\,\mathrm{d}y = -\int_Q \nabla_y v_{\boldsymbol{m}}(x,y)\cdot\nabla_y\psi(y)\,\mathrm{d}y,\ \int_Q v_{\boldsymbol{m}}(x,y)\,\mathrm{d}y = 0 \tag{1.17}$$

for all $\psi\in H^1_\#(Q)$.

*Finally, the homogenized **interaction energy** is given by*

$$\mathcal{Z}_{\text{hom}}(\boldsymbol{m}) = -\langle M_s\rangle_Q \int_\Omega \boldsymbol{h}_a\cdot\boldsymbol{m}\,\mathrm{d}\tau. \tag{1.18}$$

The paper is organized as follows: The equicoercivity of the family $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon\in\mathbb{R}^+}$ is established in Section 2; the Γ-limit of the exchange energy family of functionals $(\mathcal{E}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is computed in Section 3; in Section 4 it is shown that the family of magnetostatic self-energies $(\mathcal{W}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ continuously converges to $\mathcal{W}_{\text{hom}}$, while in Section 5 it is established the continuous convergence of the family of anisotropy energies $(\mathcal{A}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ to $\mathcal{A}_{\text{hom}}$ and the continuous convergence of the family of interaction energies $(\mathcal{Z}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ to the functional $\mathcal{Z}_{\text{hom}}$. The proof of Theorem 1.1 is completed in Section 6. In the Appendix we give a brief survey of the main mathematical concepts and results in homogenization theory, that we need in the sequel.

2. The equicoercivity of the composite Gibbs-Landau free energy functionals

This section is devoted to the proof of the equicoercivity of the family of GIBBS-LANDAU free energy functionals $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon\in\mathbb{R}^+}$ expressed by (1.6). Equicoercivity has an important role in homogenization theory. In fact, the metric space in which to work, must be able to guarantee the equicoercivity of the family of functionals under consideration, i.e. the validity of the Fundamental Theorem of Γ-convergence (see Appendix A).

Proposition 2.1. *The family $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon\in\mathbb{R}^+}$ of* GIBBS-LANDAU *energy functionals is equicoercive on the metric space $(H^1(\Omega,\mathbb{S}^2), d_{L^2(\Omega,\mathbb{S}^2)})$.*

Proof. According to the hypotheses $[H_1]$, $[H_2]$ and $[H_3]$, there exist positive constants c_{ex}, C_{ex}, C_s, C_{an} such that for every $y\in Q$ and every $\boldsymbol{m}\in H^1(\Omega,\mathbb{S}^2)$

$$0 < c_{\text{ex}} \leqslant a_{\text{ex}}(y) \leqslant C_{\text{ex}} \quad,\quad 0\leqslant M_s\leqslant C_s \quad,\quad 0\leqslant \varphi_{\text{an}}(y,\boldsymbol{m})\leqslant C_{\text{an}}. \tag{2.1}$$

Next we observe that since all energy terms except for $\mathcal{Z}_\varepsilon$ are non negative, and $c_{\text{ex}}^{-1}|\mathcal{Z}_\varepsilon| \leqslant \mu_0\frac{C_s}{c_{\text{ex}}}|\boldsymbol{h}_a||\Omega|$, by defining

$$\mathcal{G}_\star^\varepsilon := \frac{1}{c_{\text{ex}}}\mathcal{G}_{\mathcal{L}}^\varepsilon + \left(1+\mu_0\frac{C_s}{c_{\text{ex}}}|\boldsymbol{h}_a|\right)|\Omega|, \tag{2.2}$$

we have $\mathcal{G}_\star^\varepsilon(\boldsymbol{m}) \geqslant \|\boldsymbol{m}\|_{H^1(\Omega)}$ for every $\boldsymbol{m}$ in $H^1(\Omega,\mathbb{S}^2)$. Moreover, the equicoercivity of $\mathcal{G}_\star^\varepsilon$ is equivalent to the one of $\mathcal{G}_{\mathcal{L}}^\varepsilon$, so that we can focus on $\mathcal{G}_\star^\varepsilon$. Since $\|\boldsymbol{m}\|_\Omega^2 = |\Omega|$, there exists a constant

$C_\star > 0$ (depending on the previous constants, on $\boldsymbol{h}_a$ and Ω) such that that for every $\varepsilon > 0$

$$\|\boldsymbol{m}\|^2_{H^1(\Omega)} \leqslant \mathcal{G}^\varepsilon_\star(\boldsymbol{m}) \leqslant C_\star \|\boldsymbol{m}\|^2_{H^1(\Omega)}. \tag{2.3}$$

In particular for every constant in space magnetization $\boldsymbol{u}$ one has $\mathcal{G}^\varepsilon_\star(\boldsymbol{u}) \leqslant C_\star|\Omega|$ and hence

$$\inf_{H^1(\Omega,\mathbb{S}^2)} \mathcal{G}^\varepsilon_\star = \inf_{\{\boldsymbol{m}\in H^1(\Omega,\mathbb{S}^2)\,:\,\mathcal{G}^\varepsilon_\star(\boldsymbol{u})\leqslant C_\star|\Omega|\}} \mathcal{G}^\varepsilon_\star. \tag{2.4}$$

We then observe that due to (2.3), the set on which the infimum is taken in (2.4) is included in the set

$$K(\Omega,\mathbb{S}^2) := \left\{\boldsymbol{m} \in H^1(\Omega,\mathbb{S}^2) \,:\, \|\boldsymbol{m}\|^2_{H^1(\Omega)} \leqslant C_\star|\Omega|\right\} \tag{2.5}$$

and therefore $\inf_{H^1(\Omega,\mathbb{S}^2)} \mathcal{G}^\varepsilon_\star = \inf_{K(\Omega,\mathbb{S}^2)} \mathcal{G}^\varepsilon_\star$. To finish we recall that due to RELLICH-KONDRACHOV theorem, $K(\Omega,\mathbb{S}^2)$ is a compact subset of $(H^1(\Omega,\mathbb{S}^2), d_{L^2(\Omega,\mathbb{S}^2)})$. □

3. The Γ-limit of exchange energy functionals $\mathcal{E}_\varepsilon$

The fundamental constraint of micromagnetic theory, i.e. the fact that the domain of definition of the family $\mathcal{E}_\varepsilon$ is a manifold value Sobolev space, plays a fundamental role in the homogenization process. In fact, although the unconstrained problem has been fully investigated (see [7,23,27]), it is not possible to get full information about the manifold constrained Γ-limit by just looking at the unconstrained one.

(a) The tangential homogenization theorem

This problem was tackled by BABADJIAN and MILLOT (see [5]) who showed that the dependence of the Γ-limit from the tangent bundle of the manifold is taken into account via the so-called *tangentially homogenized energy density*. The following result is a consequence of [5] but we give an alternate proof, based on 2-scale convergence, which is adequate for our purposes. It is stated for a general (smooth) manifold $\mathcal{M}$ which is the boundary of a convex bounded domain $\Theta_{\mathcal{M}}$, though the case $\mathcal{M} = \mathbb{S}^2$ allows for a more precise result that we give afterwards.

Proposition 3.1. *Let $\mathcal{M} = \partial\Theta_{\mathcal{M}}$ be the boundary of a smooth convex bounded domain $\Theta_{\mathcal{M}} \subset \mathbb{R}^3$ and $a_{\rm ex} : \mathbb{R}^3 \to \mathbb{R}^+$ be a Q periodic function satisfying hypothesis [H_1]. Then the family*

$$\mathcal{E}^\varepsilon_{\mathcal{M}}(\boldsymbol{m}) := \int_\Omega a_{\rm ex}(x/\varepsilon)\,|\nabla\boldsymbol{m}|^2\,\mathrm{d}\tau \tag{3.1}$$

defined in the metric space $(H^1(\Omega,\mathcal{M}), d_{L^2(\Omega,\mathcal{M})})$ Γ-converges to the functional

$$\mathcal{E}_{\mathcal{M},\mathrm{hom}}(\boldsymbol{m}) := \int_\Omega Tg_{\mathrm{hom}}(\boldsymbol{m},\nabla\boldsymbol{m})\mathrm{d}\tau, \tag{3.2}$$

where for every $s \in \mathcal{M}$ and $\xi \in [T_s(\mathcal{M})]^3$,

$$Tg_{\mathrm{hom}}(s,\xi) = \inf_{\boldsymbol{\varphi}\in H^1_\#(Q,T_s(\mathcal{M}))} \int_Q a_{\rm ex}(y)\,|\xi + \nabla\boldsymbol{\varphi}(y)|^2\,\mathrm{d}y \tag{3.3}$$

*is the **tangentially homogenized energy density**.*

Proof. The proof will be done in the next three subsections. More precisely we observe that since $(H^1(\Omega,\mathcal{M}), d_{L^2(\Omega,\mathcal{M})})$ is a separable metric space, there exists a subsequence of $\mathcal{E}^\varepsilon_{\mathcal{M}}$ which Γ-converges. We call $\mathcal{E}_{\mathcal{M}}$ this Γ-limit, without explicitly denoting the dependence of the Γ-limit from the extracted subsequence (as far as no confusion may arise).

We next prove (cfr. subsection (ii)) that $\mathcal{E}_{\mathcal{M}} \leqslant \mathcal{E}_{\mathcal{M},\mathrm{hom}}$, while in subsection (iii) we prove that $\mathcal{E}_{\mathcal{M}} \geqslant \mathcal{E}_{\mathcal{M},\mathrm{hom}}$. Hence, $\mathcal{E}_{\mathcal{M}}$ does not depends on the extracted subsequence and the URYSOHN property of Γ-convergence (cfr. [8]) assures that $\Gamma\text{-}\lim_{\varepsilon\to 0^+} \mathcal{E}^\varepsilon_{\mathcal{M}} = \mathcal{E}_{\mathcal{M},\mathrm{hom}}$. □

(i) The role of the tangent bundle.

Let us emphasize why the tangent bundle $[\mathcal{T}(\mathcal{M})]^3 := \cup_{s\in\mathcal{M}}[T_s(\mathcal{M})]^3$ plays a role. In order to understand this, it is convenient to develop a minimizer $\boldsymbol{m}_\varepsilon$ of $\mathcal{E}_\varepsilon$ under the so-called multiscale expansion

$$\boldsymbol{m}_\varepsilon(x) = \boldsymbol{m}_0(x) + \varepsilon\boldsymbol{m}_1(x, x/\varepsilon) + o(\varepsilon), \tag{3.4}$$

where $\boldsymbol{m}_0, \boldsymbol{m}_1$ are respectively a minimizer of the Γ-limit of $\mathcal{E}_\varepsilon$ and the null average first order corrector. Clearly, due to the constraint $\boldsymbol{m}_\varepsilon(x) \in \mathcal{M}$ for a.e. $x \in \Omega$, we get

$$0 \equiv \boldsymbol{n}(\boldsymbol{m}_\varepsilon) \cdot \nabla\boldsymbol{m}_\varepsilon, \tag{3.5}$$

where we have denoted by $\boldsymbol{n}$ the local normal field defined around $\boldsymbol{m}_\varepsilon(x) \in \mathcal{M}$. By passing to the two-scale limit in both terms of (3.5), we formally reach the equality $0 \equiv \boldsymbol{n}(\boldsymbol{m}_0) \cdot (\nabla\boldsymbol{m}_0 + \nabla_y\boldsymbol{m}_1) = \boldsymbol{n}(\boldsymbol{m}_0) \cdot \nabla_y\boldsymbol{m}_1$, which shows that $\boldsymbol{n}(\boldsymbol{m}_0(x)) \cdot \boldsymbol{m}_1(x, y)$ does not depend on y. Then, passing to the average over Q we get $\boldsymbol{m}_1(x, y) \in T_{\boldsymbol{m}_0(x)}(\mathcal{M})$. The rigorous formulation of the previous idea is the object of the next result.

Proposition 3.2. *Let $\mathcal{M}$ be a connected smooth submanifold of $\mathbb{R}^3$, and let $(\boldsymbol{m}_\varepsilon)$ be a sequence in $H^1(\Omega, \mathcal{M})$ that converges weakly to a limit $\boldsymbol{m} \in H^1(\Omega, \mathbb{R}^3)$. Then*

$$\boldsymbol{m} \in H^1(\Omega, \mathcal{M}) \quad \textit{and} \quad \boldsymbol{m}_\varepsilon \twoheadrightarrow \boldsymbol{m}.$$

Moreover there exists a null average function $\boldsymbol{v} \in L^2[\Omega; H^1_\#(Q)/\mathbb{R}]$ such that, up to the extraction of a subsequence:

$$\nabla\boldsymbol{m}_\varepsilon \twoheadrightarrow \nabla\boldsymbol{m} + \nabla_y\boldsymbol{v} \quad \textit{and} \quad \boldsymbol{v}(x, y) \in T_{\boldsymbol{m}(x)}(\mathcal{M}) \textit{ for a.e. } (x, y) \in \Omega \times Q.$$

Proof. In view of Proposition A.4, we only need to prove that $\boldsymbol{v}(x, y) \in T_{\boldsymbol{m}_0(x)}(\mathcal{M})$ for a.e. $(x, y) \in \Omega \times Q$. To this end, let us denote by $\boldsymbol{n}(\boldsymbol{m})$ the normal vector at $\boldsymbol{m} \in \mathcal{M}$ and observe that it is sufficient to prove that the scalar function $\boldsymbol{n}(\boldsymbol{m}(x)) \cdot \boldsymbol{v}(x, y)$ does not depend on the y variable, i.e. that in the sense of distributions on $\Omega \times Y$ one has

$$\int_{\Omega\times Q} [\boldsymbol{n}(\boldsymbol{m}(x)) \cdot \boldsymbol{v}(x, y)] \operatorname{div}_y \boldsymbol{\varphi}(x, y) \mathrm{d}y\mathrm{d}x = 0 \qquad \forall\boldsymbol{\varphi} \in \mathcal{D}[\Omega; C^\infty_\#(Q)]. \tag{3.6}$$

Indeed, as far as $\boldsymbol{n}(\boldsymbol{m}(x)) \cdot \boldsymbol{v}(x, y)$ is independent from the y variable, since by assumption $\langle\boldsymbol{v}(x, \cdot)\rangle_Q = 0$ for a.e. $x \in \Omega$, one has $\boldsymbol{n}(\boldsymbol{m}(x)) \cdot \boldsymbol{v}(x, y) = \boldsymbol{n}(\boldsymbol{m}(x)) \cdot \langle\boldsymbol{v}(x, \cdot)\rangle_Q = 0$ and therefore $\boldsymbol{v}(x, y) \in T_{\boldsymbol{m}(x)}(\mathcal{M})$ for a.e. $(x, y) \in \Omega \times Q$.

To prove (3.6) we note that since $\boldsymbol{m}_\varepsilon \to \boldsymbol{m}$ in $L^2(\Omega)$, one also has $\boldsymbol{n}(\boldsymbol{m}_\varepsilon) \to \boldsymbol{n}(\boldsymbol{m})$ in $L^2(\Omega)$. Therefore the family $\boldsymbol{n}(\boldsymbol{m}_\varepsilon)$ strongly two-scale converges to $\boldsymbol{n}(\boldsymbol{m})$ and moreover $\mathbf{0} = \boldsymbol{n}(\boldsymbol{m}_\varepsilon) \cdot \nabla\boldsymbol{m}_\varepsilon \to \boldsymbol{n}(\boldsymbol{m}) \cdot \nabla\boldsymbol{m}$. Hence, due to Proposition A.6 we get

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \int_\Omega [\boldsymbol{n}(\boldsymbol{m}_\varepsilon(x)) \cdot \nabla\boldsymbol{m}_\varepsilon(x)] \cdot \boldsymbol{\varphi}(x, x/\varepsilon)\mathrm{d}x \\
&= \int_{\Omega\times Q} \boldsymbol{n}(\boldsymbol{m}(x)) \cdot [\nabla\boldsymbol{m}(x) + \nabla_y\boldsymbol{v}(x, y)] \cdot \boldsymbol{\varphi}(x, y) \,\mathrm{d}y\mathrm{d}x \\
&= \int_{\Omega\times Q} [\boldsymbol{n}(\boldsymbol{m}(x)) \cdot \nabla_y\boldsymbol{v}(x, y)] \cdot \boldsymbol{\varphi}(x, y) \,\mathrm{d}y\mathrm{d}x \\
&= -\int_{\Omega\times Q} [\boldsymbol{n}(\boldsymbol{m}(x)) \cdot \boldsymbol{v}(x, y)] \operatorname{div}_y \boldsymbol{\varphi}(x, y) \,\mathrm{d}x,
\end{aligned}$$

i.e. the desired relation (3.6). □

(ii) Tangentially homogenized energy density – upper bound

Let us denote by $\Pi_{\mathcal{M}}$ the nearest point projection on $\mathcal{M}$. Since $\Pi_{\mathcal{M}}$ is a (Lipschitz) non-expansive map, one has $\Pi_{\mathcal{M}}[\boldsymbol{u}] \in H^1(\Omega, \mathcal{M})$ for every $\boldsymbol{u} \in H^1(\Omega, \Theta^c_{\mathcal{M}})$, and moreover (see [4])

$$|\nabla \Pi_{\mathcal{M}}[\boldsymbol{u}]| \leqslant |\nabla \boldsymbol{u}| \quad \tau\text{-a.e. in } \Omega. \tag{3.7}$$

Let us now consider every test function $\boldsymbol{m}_1 \in H^1[\Omega; C^\infty_\#(Q, T_{\boldsymbol{m}_0}(\mathcal{M}))]$, the family $\boldsymbol{m}_\varepsilon(x) := \boldsymbol{m}_0(x) + \varepsilon \boldsymbol{m}_1(x, x/\varepsilon)$ belongs to $H^1(\Omega, \Theta^c_{\mathcal{M}})$. In this hypothesis one has

$$\Pi_{\mathcal{M}}[\boldsymbol{m}_\varepsilon] \to \boldsymbol{m}_0 \quad \text{in } (H^1(\Omega, \mathcal{M}), d_{L^2(\Omega,\mathcal{M})}).$$

Therefore, taking into account the estimate (3.7) and the fact that $\nabla_y \boldsymbol{m}_1$ is an admissible test function (see [3], Remark 1.11), we get (passing to the two-scale limit):

$$\begin{aligned}
\mathcal{E}_{\mathcal{M}}(\boldsymbol{m}_0) &\leqslant \liminf_{\varepsilon \to 0} \mathcal{E}^\varepsilon_{\mathcal{M}}(\Pi_{\mathcal{M}}[\boldsymbol{m}_\varepsilon]) \\
&\leqslant \lim_{\varepsilon \to 0} \int_\Omega a_{\mathrm{ex}}\left(\frac{x}{\varepsilon}\right) \left| \nabla \boldsymbol{m}_0(x) + \nabla_y \boldsymbol{m}_1\left(x, \frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x \\
&= \int_\Omega \left[\int_Q a_{\mathrm{ex}}(y) |\nabla \boldsymbol{m}_0(x) + \nabla_y \boldsymbol{m}_1(x,y)|^2 \mathrm{d}y \right] \mathrm{d}x.
\end{aligned}$$

Since $\boldsymbol{m}_1 \in H^1[\Omega; C^\infty_\#(Q, T_{\boldsymbol{m}_0}(\mathcal{M}))]$ is an arbitrary test function, passing to the infimum we finish with the following upper bound for the manifold constrained homogenized functional:

$$\mathcal{E}_{\mathcal{M}}(\boldsymbol{m}_0) \leqslant \int_\Omega \left[\inf_{\boldsymbol{\varphi} \in C^\infty_\#(Q, T_s(\mathcal{M}))} \mathcal{I}[\xi, \boldsymbol{\varphi}] \right]_{(s,\xi) := (\boldsymbol{m}_0(x), \nabla \boldsymbol{m}_0(x))} \mathrm{d}x, \tag{3.8}$$

with

$$\mathcal{I}[\xi, \boldsymbol{\varphi}] := \int_Q a_{\mathrm{ex}}(y) |\xi + \nabla \boldsymbol{\varphi}(y)|^2 \mathrm{d}y. \tag{3.9}$$

Since the functional $\mathcal{I}[\xi, \cdot] : C^\infty_\#(Q, T_s(\mathcal{M})) \to \mathbb{R}^+$ is continuous with respect to the $H^1_\#(Q, T_s(\mathcal{M}))$ norm and $C^\infty_\#(Q, T_s(\mathcal{M}))$ is dense in $H^1_\#(Q, T_s(\mathcal{M}))$, the infimum in (3.8) can be taken over $H^1_\#(Q, T_s(\mathcal{M}))$, which gives

$$\mathcal{E}_{\mathcal{M}}(\boldsymbol{m}_0) \leqslant \mathcal{E}_{\mathcal{M},\mathrm{hom}}(\boldsymbol{m}_0)\,. \tag{3.10}$$

(iii) Tangentially homogenized energy density – lower bound

For the lower bound, we argue as follows. Let $\boldsymbol{m}_0 \in H^1(\Omega, \mathcal{M})$ and a family $(\boldsymbol{m}_\varepsilon)_{\varepsilon > 0}$ such that

$$\boldsymbol{m}_\varepsilon \to \boldsymbol{m}_0 \quad \text{in } (H^1(\Omega, \mathcal{M}), d_{L^2(\Omega,\mathcal{M})}).$$

Assume furthermore that $\liminf_{\varepsilon \to 0} \|\boldsymbol{m}_\varepsilon\|_{H^1} < +\infty$ (otherwise the lower bound is trivially satisfied) so that one can extract a subsequence from $(\boldsymbol{m}_\varepsilon)_{\varepsilon > 0}$ that weakly converges to $\boldsymbol{m}_0$ in H^1 and applying Proposition 3.2, there exists $\boldsymbol{m}_1 \in L^2[\Omega, H^1_\#(Q)/\mathbb{R}]$ such that

$$\nabla \boldsymbol{m}_\varepsilon \twoheadrightarrow \nabla \boldsymbol{m}_0 + \nabla_y \boldsymbol{m}_1$$

with $\boldsymbol{m}_1(x,y) \in T_{\boldsymbol{m}_0(x)}(\mathcal{M})$ for a.e. $(x,y) \in \Omega \times Q$. Let now $\boldsymbol{\phi} \in \mathcal{D}[\Omega; C^\infty_\#(Q, \mathbb{R}^3)]$ be an arbitrary test function, passing to the $\liminf$ in

$$\int_\Omega a_{\mathrm{ex}}\left(\frac{x}{\varepsilon}\right) \left| \nabla \boldsymbol{m}_\varepsilon(x) - \nabla \boldsymbol{m}_0(x) - \nabla_y \boldsymbol{\phi}\left(x, \frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x \geq 0$$

we infer, since $\boldsymbol{\phi}$ is admissible,

$$\begin{aligned}
\liminf_{\varepsilon \to 0} \int_\Omega a_{\mathrm{ex}}\left(\frac{x}{\varepsilon}\right) |\nabla \boldsymbol{m}_\varepsilon(x)|^2 \mathrm{d}x \geq &- \int_\Omega \int_Q a_{\mathrm{ex}}(y) |\nabla \boldsymbol{m}_0(x) + \nabla_y \boldsymbol{\phi}(x,y)|^2 \mathrm{d}y \mathrm{d}x \\
&+ 2 \int_\Omega \int_Q a_{\mathrm{ex}}(y) (\nabla \boldsymbol{m}_0(x) + \nabla_y \boldsymbol{m}_1(x,y)) \cdot (\nabla \boldsymbol{m}_0(x) + \nabla_y \boldsymbol{\phi}(x,y)) \mathrm{d}y \mathrm{d}x.
\end{aligned}$$

Eventually, applying this identity with $\phi = \phi_n$ where the sequence $(\phi_n)_{n\geq 0}$ converges to $\boldsymbol{m}_1$ in $L^2[\Omega, H^1_\#(Q)]$ and passing to the limit $n \to +\infty$ leads to

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \mathcal{E}^\varepsilon_{\mathcal{M}}(\boldsymbol{m}_\varepsilon) &\geq \int_\Omega \int_Q a_{\rm ex}\,(y)\,|\nabla \boldsymbol{m}_0(x) + \nabla_y \boldsymbol{m}_1\,(x,y)|^2\,\mathrm{d}y\mathrm{d}x \\
&\geq \int_\Omega \left[\inf_{\boldsymbol{\varphi}\in H^1_\#(Q, T_{\boldsymbol{m}_0(x)}(\mathcal{M}))} \int_Q a_{\rm ex}\,(y)\,|\nabla \boldsymbol{m}_0(x) + \nabla_y \boldsymbol{\varphi}\,(y)|^2\,\mathrm{d}y \right] \mathrm{d}x \\
&= \mathcal{E}_{\mathcal{M},\mathrm{hom}}(\boldsymbol{m}_0)\,,
\end{aligned}$$

from which we deduce

$$\mathcal{E}_{\mathcal{M}}(\boldsymbol{m}_0) \geqslant \mathcal{E}_{\mathcal{M},\mathrm{hom}}(\boldsymbol{m}_0)\,, \tag{3.11}$$

which together with (3.10) proves proposition 3.1.

(b) The tangentially homogenized Exchange Energy $\mathcal{E}_{\mathrm{hom}}$

Quite remarkably, as we prove below, when $\mathcal{M} := \mathbb{S}^2$ the formula giving the Γ limit of the energy does not depend on the tangent plane to $\mathcal{M}$. This gives a simpler expression which turns out to be the *unconstrained* homogenization formula. Indeed we consider the family of exchange energy functionals, all defined in $H^1(\Omega, \mathbb{S}^2)$, given by $(\mathcal{E}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$. Since $[\boldsymbol{H_1}]$ holds, Proposition 3.1 ensures that the family $(\mathcal{E}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ Γ-converges in the metric space $(H^1(\Omega;\mathbb{S}^2), d_{L^2(\Omega,\mathbb{S}^2)})$, i.e. with respect to the topology induced on $H^1(\Omega, \mathbb{S}^2)$ by the strong $L^2(\Omega, \mathbb{R}^3)$ topology, to the functional

$$\mathcal{E}_{\mathrm{hom}} : H^1(\Omega, \mathbb{S}^2) \to \mathbb{R}^+ \quad , \quad \boldsymbol{m} \mapsto \mathcal{E}_{\mathrm{hom}}(\boldsymbol{m}) = \int_\Omega T g_{\mathrm{hom}}(\boldsymbol{m}, \nabla \boldsymbol{m})\mathrm{d}\tau, \tag{3.12}$$

where for every $s \in \mathbb{S}^2$ and every $\xi \in [T_s(\mathbb{S}^2)]^3$,

$$T g_{\mathrm{hom}}(s,\xi) = \inf_{\boldsymbol{\varphi}\in H^1_\#(Q, T_s(\mathbb{S}^2))} \int_Q a_{\rm ex}(y)\,|\xi + \nabla\boldsymbol{\varphi}(y)|^2\,\mathrm{d}y\,. \tag{3.13}$$

Let us now consider the problem that defines the **classical homogenization problem** (see [7, 23,27]), namely

$$g_{\mathrm{hom}}(\xi) := \inf_{\boldsymbol{\varphi}\in H^1_\#(Q, \mathbb{R}^3)} \int_Q a_{\rm ex}(y)\,|\xi + \nabla\boldsymbol{\varphi}(y)|^2\,\mathrm{d}y, \tag{3.14}$$

in which the constraint to belong to the tangent plane has been removed. Our aim is to prove that the natural extension of g_{hom} to the tangent bundle $[\mathcal{T}(\mathbb{S}^2)]^3 := \cup_{s\in\mathbb{S}^2}[T_s(\mathbb{S}^2)]^3$ coincides with the tangentially homogenized energy density $T g_{\mathrm{hom}}(s,\xi)$. To this end we observe that in the «classical» problem (3.14), the function space among which the minimization takes place is bigger than the one involved in the original problem (3.13), so that $g_{\mathrm{hom}}(\xi) \leqslant T g_{\mathrm{hom}}(s,\xi)$ for every $(s,\xi) \in [\mathcal{T}(\mathbb{S}^2)]^3$.

To prove that $g_{\mathrm{hom}}(\xi) \equiv T g_{\mathrm{hom}}(s,\xi)$ on $[\mathcal{T}(\mathbb{S}^2)]^3$ it is thus sufficient to show that for every $(s,\xi) \in [\mathcal{T}(\mathbb{S}^2)]^3$ there exists a function $\boldsymbol{\psi}_\xi \in H^1_\#(Q, T_s(\mathbb{S}^2))$ such that

$$\int_Q a_{\rm ex}(y)\,|\xi + \nabla\boldsymbol{\psi}_\xi(y)|^2\,\mathrm{d}y \leqslant g_{\mathrm{hom}}(\xi)\,.$$

Having this goal in mind, we observe that if $\boldsymbol{\varphi}_\xi$ is the unique (up to a constant) solution of the minimization problem arising in (3.14), denoting by $\boldsymbol{\psi}_\xi := \boldsymbol{\varphi}_\xi - (\boldsymbol{\varphi}_\xi \cdot s)s$ the nearest point projection of φ_ξ on $[T_s(\mathbb{S}^2)]^3$, one has $\psi_\xi \in H^1_\#(Q, T_s(\mathbb{S}^2))$ and

$$\begin{aligned}|\xi + \nabla\varphi_\xi|^2 &= |\xi + \nabla\psi_\xi|^2 + |s \otimes \nabla(\varphi_\xi \cdot s)|^2 + 2(\xi + \nabla\psi_\xi) : s \otimes \nabla(\varphi_\xi \cdot s)\\ &= |\xi + \nabla\psi_\xi|^2 + |\nabla(\varphi_\xi \cdot s)|^2 + 2(\xi + \nabla\psi_\xi)\nabla(\varphi_\xi \cdot s) \cdot s\\ &= |\xi + \nabla\psi_\xi|^2 + |\nabla(\varphi_\xi \cdot s)|^2\\ &\geqslant |\xi + \nabla\psi_\xi|^2.\end{aligned}$$

Multiplying by a_{ex} and integrating over Q immediately leads to

$$\int_Q a_{\mathrm{ex}}(y)|\xi + \nabla\psi_\xi(y)|^2 \mathrm{d}y \leqslant \int_Q a_{\mathrm{ex}}(y)|\xi + \nabla\varphi_\xi(y)|^2 \mathrm{d}y = g_{\mathrm{hom}}(\xi)\,. \tag{3.15}$$

which is the desired inequality from which we may deduce $g_{\mathrm{hom}}(\xi) = Tg_{\mathrm{hom}}(s, \xi)$. In particular $Tg_{\mathrm{hom}}(s, \xi)$ does not depend on s, and is given by (3.14).

Remark 3.1. *The fact that tangential homogenization energy density $Tg_{\mathrm{hom}}(s, \xi)$ reduces to the «classical» one (i.e. g_{hom}), which does not depend from the s-variable, is a quite remarkable fact. Indeed, it is possible to build elementary examples where the dependence on the s-variable in the tangential homogenization energy density is explicit (cfr. [5]). The independence from the s-variable in our framework, is mainly due to the very particular situation that for every $y \in \mathbb{R}^3$, the Carathéodory function $g(y, \cdot) = a_{\mathrm{ex}}(y)|\cdot|^2$ is invariant under the rotation group of the manifold under consideration (the 2-sphere $\mathbb{S}^2$).*

To complete the proof concerning the exchange energy part stated in Theorem 1.1, it is sufficient to recall that g_{hom} is a quadratic form in ξ (see [7,23]), i.e. that there exists a symmetric and positive definite matrix $A_{\mathrm{hom}} \in \mathbb{R}^{3\times3}$ such that for every $\xi \in \mathbb{R}^{3\times3}$

$$g_{\mathrm{hom}}(\xi) = A_{\mathrm{hom}}\xi : \xi \tag{3.16}$$

with A_{hom} expressed as

$$A_{\mathrm{hom}} := \langle a_{\mathrm{ex}}(\cdot)(I + \nabla\varphi(\cdot))^T(I + \nabla\varphi(\cdot))\rangle_Q\,, \quad \varphi := (\varphi_1, \varphi_2, \varphi_3)\,, \tag{3.17}$$

where for every $j = 1, 2, 3$ the component φ_j is the unique (up to a constant) solution of the *scalar unit cell problem*

$$\varphi_j := \underset{\varphi \in H^1_\#(Q,\mathbb{R}^3)}{\operatorname{argmin}} \int_Q a_{\mathrm{ex}}(y)|\nabla\varphi(y) + e_j|^2 \mathrm{d}x. \tag{3.18}$$

4. The periodic homogenization of the demagnetizing field

This Section is devoted to show that the family of magnetostatic self-energies $(\mathcal{W}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ continuously converges to $\mathcal{W}_{\mathrm{hom}}$. To this end, let us first recall some essential facts concerning the demagnetizing field operator.

(a) The Beppo-Levi space and the variational formulation for the demagnetizing field

From the mathematical point of view, assuming Ω to be open, bounded and with a Lipschitz boundary, a given magnetization $\boldsymbol{m} \in L^2(\Omega, \mathbb{R}^3)$ generates the stray field $\boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}] = \nabla u_{\boldsymbol{m}}$ where the potential $u_{\boldsymbol{m}}$ solves:

$$\Delta u_{\boldsymbol{m}} = -\operatorname{div}(\boldsymbol{m}\chi_\Omega) \;\text{ in }\; \mathcal{D}'(\mathbb{R}^3). \tag{4.1}$$

In (4.1) we have denoted by $\boldsymbol{m}\chi_\Omega$ the extension of $\boldsymbol{m}$ to $\mathbb{R}^3$ that vanishes outside Ω.

Once introduced the weight $\omega(x) = (1+|x|^2)^{-1/2}$, and the weighted Lebesgue space $L^2_\omega(\mathbb{R}^3) := \{u \in \mathcal{D}'(\mathbb{R}^3) : u\omega \in L^2(\mathbb{R}^3)\}$, we define the BEPPO-LEVI space

$$BL^1(\mathbb{R}^3) := \left\{ u \in \mathcal{D}'(\mathbb{R}^3) : u \in L^2_\omega(\mathbb{R}^3) \text{ and } \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3) \right\}. \tag{4.2}$$

Using the POINCARÉ-HARDY type inequality $\|u\omega\|_{\mathbb{R}^3} \leqslant 2\|\nabla u\|_{\mathbb{R}^3}$, it is well known that $BL^1(\mathbb{R}^3)$ equipped with the norm $\|u\|_{BL^1(\mathbb{R}^3)} := \|\nabla u\|_{\mathbb{R}^3}$ is a Hilbert space.

After that, it is straightforward to show, by the means of LAX-MILGRAM theorem, the existence and uniqueness of the solution of the variational formulation associated to equation (4.1): namely to prove the existence of a potential $u_{\boldsymbol{m}} \in BL^1(\mathbb{R}^3)$ such that for all $\varphi \in BL^1(\mathbb{R}^3)$

$$(u_{\boldsymbol{m}}, \varphi)_{BL^1(\mathbb{R}^3)} := \int_{\mathbb{R}^3} \nabla u_{\boldsymbol{m}} \cdot \nabla\varphi \mathrm{d}\tau = -\int_{\mathbb{R}^3} \boldsymbol{m} \cdot \nabla\varphi \mathrm{d}\tau =: F_{\boldsymbol{m}}[\varphi]. \tag{4.3}$$

Thus, for every $\boldsymbol{m} \in L^2(\mathbb{R}^3)$ there exists a unique $u_{\boldsymbol{m}} \in BL^1(\mathbb{R}^3)$ such that (4.3) holds. Moreover, the following stability estimate holds:

$$\|u_{\boldsymbol{m}}\|_{BL^1(\mathbb{R}^3)} \equiv \|\nabla u_{\boldsymbol{m}}\|^2_{\Omega} \leqslant \sup_{\substack{\varphi \in BL^1(\mathbb{R}^3) \\ \|\nabla\varphi\|_\Omega = 1}} |F_{\boldsymbol{m}}[\varphi]| \leqslant \|\boldsymbol{m}\|_{L^2(\mathbb{R}^3)}. \tag{4.4}$$

The quantity $\mathbf{h}_d[\boldsymbol{m}] := \nabla u_{\boldsymbol{m}}$ is what is referred to as the demagnetizing (or stray) field, and it can be viewed as a linear and continuous operator from $L^2(\mathbb{R}^3, \mathbb{R}^3)$ into $L^2(\mathbb{R}^3, \mathbb{R}^3)$.

(b) Weighted two-scale convergence

The aim of the section is to make use of the notion of two-scale convergence, to characterize the behavior of the demagnetizing field operator under two-scale convergence. More precisely, we suppose to have a bounded sequence $(\boldsymbol{m}_\varepsilon)$ in $L^2(\mathbb{R}^3)$ which two scale converges to some $\boldsymbol{m}_\infty(x,y) \in L^2$ and we want to understand if the two-scale limit of the sequence $\mathbf{h}_d[\boldsymbol{m}_\varepsilon]$ exists, and in the affirmative case to characterize in some analytic sense such a limit.

This problem has already been treated in [31] but without justifying the use of two-scale compactness results in weighted space. This is something that requires some work and that is why we start this subsection by proving two compactness results concerning two-scale convergence in the weighted spaces $L^2_\omega(\mathbb{R}^3)$ and $BL^1(\mathbb{R}^3)$.

The first one is a «weighted» variant of the compactness result stated in Proposition A.3, and shows that a notion of two-scale convergence in $L^2_\omega(\mathbb{R}^3)$ makes sense.

Proposition 4.1. *Let (u_ε) be bounded sequence in $L^2_\omega(\mathbb{R}^3)$. There exists a function $u \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \times Q)$ such that $\langle u\rangle_Q \in L^2_\omega(\mathbb{R}^3)$ and, up to the extraction of a subsequence,*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} u_\varepsilon(x)\varphi(x, x/\varepsilon)\mathrm{d}x = \int_{\mathbb{R}^3\times Q} u(x,y)\varphi(x,y)\mathrm{d}y\mathrm{d}x \tag{4.5}$$

for all $\varphi \in \mathcal{D}[\mathbb{R}^3; C^\infty_\#(Q)]$. In this case we say that the sequence (u_ε) L^2_ω-two-scale converges to u.

Proof. Since (u_ε) is bounded in the Hilbert space $L^2_\omega(\mathbb{R}^3)$, there exists an element $u_\infty \in L^2_\omega(\mathbb{R}^3)$ and a sequence $(u_{\varepsilon(n)}) \subseteq (u_\varepsilon)$ such that

$$u_{\varepsilon(n)} \rightharpoonup u_\infty \quad \text{weakly in } L^2_\omega(\mathbb{R}^3). \tag{4.6}$$

This implies that for every bounded domain $\Omega \subseteq \mathbb{R}^3$, one has $u_{\varepsilon(n)} \rightharpoonup u_\infty$ weakly in $L^2(\Omega)$. We now consider a sequence of bounded domain $(\Omega_i)_{i\in\mathbb{N}}$ covering $\mathbb{R}^3$. Let us start with the index $i=1$, i.e. with Ω_1. According to the two-scale compactness result (see Proposition A.3), there exists a subsequence $u_{\varepsilon(n_{k_1})}$ and an element $u_1 \in L^2(\Omega_1 \times Q)$ such that

$$u_{\varepsilon(n_{k_1})} \twoheadrightarrow u_1 \;\text{ in } L^2(\Omega_1 \times Q). \tag{4.7}$$

Now we consider $i=2$, i.e. Ω_2. Since $u_{\varepsilon(n_{k_1})} \rightharpoonup u_\infty$ weakly in $L^2_\omega(\mathbb{R}^3)$, it is possible to extract a further subsequence $(u_{\varepsilon(n_{k_2})})$ from $u_{\varepsilon(n_{k_1})}$ such that $u_{\varepsilon(n_{k_2})} \twoheadrightarrow u_2$ in $L^2(\Omega_2 \times Q)$ for some suitable $u_2 \in L^2(\Omega_2 \times Q)$. Moreover, due to the unicity of the two-scale limit, one has

$$u_{1|(\Omega_1\cap\Omega_2)\times Q} \equiv u_{2|(\Omega_1\cap\Omega_2)\times Q} \tag{4.8}$$

whenever $\Omega_1 \cap \Omega_2 \neq \emptyset$. Proceeding in this way, we find for every $i \in \mathbb{N}$ a subsequence $u_{\varepsilon(n_{k_i})}$ such that

$$n_{k_i} \subseteq n_{k_{i-1}} \qquad \text{and} \qquad u_{\varepsilon(n_{k_i})} \twoheadrightarrow u_i \tag{4.9}$$

for some $u_i \in L^2(\Omega_i \times Q)$. We then define the **diagonal sequence** of indices defined by

$$n_{k_\infty(1)} := n_{k_1(1)}, n_{k_\infty(2)} := n_{k_2(2)}, \ldots, n_{k_\infty(i)} = n_{k_i(i)}, \ldots. \tag{4.10}$$

From (4.9) we get that for every $i \in \mathbb{N}$, up to the first $i-1$ terms, the sequence of indices n_{k_∞} is included in n_{k_i}, and this means that for every $i \in \mathbb{N}$

$$u_{\varepsilon(n_{k_\infty})} \twoheadrightarrow u_i \;\text{ in } L^2(\Omega_i \times Q). \tag{4.11}$$

By observing again that $u_{i|(\Omega_i\cap\Omega_j)\times Q} \equiv u_{j|(\Omega_i\cap\Omega_j)\times Q}$ if $\Omega_i \cap \Omega_j \neq \emptyset$, from the «*principe du recollement des morceaux*» (cfr. [32]) there exists a unique distribution $u \in L^2_{\text{loc}}(\mathbb{R}^3 \times Q)$ such that $u_{|\Omega_i \times Q} \equiv u_i$, and therefore

$$\lim_{k_\infty\to\infty} \int_{\mathbb{R}^3} u_{\varepsilon(n_{k_\infty})}(x)\varphi(x, x/\varepsilon)\mathrm{d}x = \int_{\mathbb{R}^3\times Q} u(x,y)\varphi(x,y)\mathrm{d}y\mathrm{d}x \tag{4.12}$$

for every $\varphi \in \mathcal{D}[\mathbb{R}^3; C^\infty_\#(Q)]$. Moreover since

$$u_{\varepsilon(n_{k_\infty})} \rightharpoonup u_\infty \quad \text{in } L^2_\omega(\mathbb{R}^3) \tag{4.13}$$

and

$$\forall i \in \mathbb{N} \quad u_{\varepsilon(n_{k_\infty})} \rightharpoonup \langle u(x,y)\rangle_Q \quad \text{in } L^2(\Omega_i) \tag{4.14}$$

we get also $\langle u(x,y)\rangle_Q \equiv u_\infty \in L^2_\omega(\mathbb{R}^3)$. This completes the proof. □

Exactly with the same diagonal argument, it is possible to prove the weighted variant of the compactness result stated in Proposition A.4.

Proposition 4.2. *Let (u_ε) be bounded sequence in $BL^1(\mathbb{R}^3)$ weakly convergent to u_∞. Then u_ε L^2_ω-two-scale converges to $u_\infty \in L^2_\omega(\mathbb{R}^3)$ and there exists a function $v \in L^2[\mathbb{R}^3; H^1_\#(Q)/\mathbb{R}]$ such that, up to the extraction of a subsequence:*

$$\nabla u_\varepsilon \twoheadrightarrow \nabla u_\infty + \nabla_y v. \tag{4.15}$$

Proof. We start by observing that since $u_{\varepsilon(n)} \rightharpoonup u_\infty$ weakly in $BL^1(\mathbb{R}^3)$, $u_{\varepsilon(n)} \rightharpoonup u_\infty$ in $L^2_\omega(\mathbb{R}^3)$, and therefore, according to the previous proposition, there exists a function $u \in L^2_\omega(\mathbb{R}^3 \times Q)$ such that, up to a subsequence,

$$\begin{aligned} &u_{\varepsilon(n)} \twoheadrightarrow u(x,y) && \text{in } L^2_\omega(\mathbb{R}^3 \times Q) \\ &u_{\varepsilon(n)} \rightharpoonup u_\infty(x) \equiv \langle u(x,y)\rangle_Q && \text{in } L^2_\omega(\mathbb{R}^3). \end{aligned} \tag{4.16}$$

We now consider a sequence of bounded domain $(\Omega_i)_{i\in\mathbb{N}}$. Proceeding as in the proof of the previous Proposition 4.1, one proves that for every $i \in \mathbb{N}$ there exists a subsequence $u_{\varepsilon(n_{k_i})}$ such that

$$n_{k_i} \subseteq n_{k_{i-1}} \text{ and } u_{\varepsilon(n_{k_i})} \twoheadrightarrow u_\infty(x) \equiv \langle u(x,y)\rangle_Q \equiv u(x,y) \quad \text{in } L^2(\Omega_i \times Q). \tag{4.17}$$

We then define the diagonal sequence of indices defined by the position $n_{k_\infty(i)} = n_{k_i(i)}$. From (4.17) we get that for every $i \in \mathbb{N}$, up to the first $i-1$ terms, the sequence of indices n_{k_∞} is included in n_{k_i}, and this means that for every $i \in \mathbb{N}$

$$u_{\varepsilon(n_{k_\infty})} \twoheadrightarrow u_\infty(x) \equiv \langle u(x,y)\rangle_Q \equiv u(x,y) \quad \text{in } L^2(\Omega_i \times Q). \tag{4.18}$$

Thus $u \equiv u_\infty \in L^2_\omega(\mathbb{R}^3)$ in $\mathbb{R}^3$.

Next, we observe that since $u_{\varepsilon(n)} \rightharpoonup u_\infty$ weakly in $BL^1(\mathbb{R}^3)$ we have $\nabla u_{\varepsilon(n)} \rightharpoonup \nabla u_\infty$ and $(\nabla u_{\varepsilon(n)})$ bounded in $L^2(\mathbb{R}^3)$. Thus, according to the classical two-scale compactness result (see Proposition A.4) there exists a function $\boldsymbol{\kappa}_\infty \in L^2(\mathbb{R}^3 \times Q)$ such that, up to a subsequence,

$$\nabla u_\varepsilon \twoheadrightarrow \boldsymbol{\kappa}_\infty \quad \text{in } L^2(\mathbb{R}^3, \mathbb{R}^3). \tag{4.19}$$

Thus, for any test function $[\varphi \otimes \boldsymbol{\psi}_\#](x,y) := \varphi(x)\boldsymbol{\psi}_\#(y) \in \mathcal{D}[\mathbb{R}^3; C^\infty_\#(Q)]$ with $\operatorname{div}_y \boldsymbol{\psi}_\#(y) = 0$ one has

$$\int_{\mathbb{R}^3} u_\varepsilon(x) \operatorname{div}_x[\varphi \otimes \boldsymbol{\psi}_\#](x, x/\varepsilon)\mathrm{d}x = -\int_{\mathbb{R}^3} \nabla u_\varepsilon(x) \cdot [\varphi \otimes \boldsymbol{\psi}_\#](x, x/\varepsilon)\mathrm{d}x. \tag{4.20}$$

Passing to the two-scale convergence on both sides we get that for a.e. $x \in \mathbb{R}^3$ and for every $\boldsymbol{\psi}_\# \in C^\infty_\#(Q)$ such that $\operatorname{div}_y \boldsymbol{\psi}_\#(y) = 0$ in Q.

$$\int_Q [\boldsymbol{\kappa}_\infty(x,y) - \nabla u_\infty(x)] \cdot \boldsymbol{\psi}_\#(y)\mathrm{d}y = 0. \tag{4.21}$$

Since the orthogonal complement of the divergence-free functions is the space of gradients, for a.e. $x \in \mathbb{R}^3$ there exists a unique function $v(x,\cdot) \in H^1_\#(Q)/\mathbb{R}$ such that $\nabla_y v(x,y) \equiv \boldsymbol{\kappa}_\infty(x,y) - \nabla u_\infty(x)$. Thus $\nabla_y v \in L^2(\mathbb{R}^3 \times Q)$ and $v(x,\cdot) \in H^1_\#(Q)/\mathbb{R}$, i.e. $v \in L^2[\mathbb{R}^3, H^1_\#(Q)/\mathbb{R}]$. This completes the proof. □

(c) The two-scale limit of the demagnetizing field

We are now ready to prove the two-scale convergence of the demagnetizing field operator.

Proposition 4.3. *Let $(\boldsymbol{m}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$ be a bounded family in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ that two-scale converges to $\boldsymbol{m}(x,y)$. Then the two-scale limit of $(\mathbf{h}_d[\boldsymbol{m}_\varepsilon])_{\varepsilon \in \mathbb{R}^+} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ exists and is given by*

$$\mathbf{h}_d(x,y) = \mathbf{h}_d[\langle \boldsymbol{m}(x,\cdot)\rangle_Q] + \nabla_y v_{\boldsymbol{m}}(x,y) \tag{4.22}$$

where for every $x \in \mathbb{R}^3$ the scalar function $v_{\boldsymbol{m}}(x,\cdot)$ is the unique solution in $H^1_\#(Q)$ of the cell problem

$$\Delta_y v_{\boldsymbol{m}}(x,y) = -\operatorname{div}_y \boldsymbol{m}(x,y) \quad \textit{in } H^1_\#(Q)/\mathbb{R} \tag{4.23}$$

and therefore of the variational cell problem

$$\int_Q \boldsymbol{m}(x,y) \cdot \nabla_y \psi(y)\mathrm{d}y = -\int_Q \nabla_y v_{\boldsymbol{m}}(x,y) \cdot \nabla_y \psi(y)\mathrm{d}y, \quad \int_Q v_{\boldsymbol{m}}(x,y)\mathrm{d}y = 0 \tag{4.24}$$

for all $\psi \in H^1_\#(Q)$.

Proof. Since $(\boldsymbol{m}_\varepsilon)$ is bounded in $L^2(\mathbb{R}^3)$, due to the stability estimate (4.4), the sequence of magnetostatic potentials $(u^\varepsilon_{\boldsymbol{m}})$ solution of the problem $\Delta u^\varepsilon_{\boldsymbol{m}} = -\operatorname{div}(\boldsymbol{m}_\varepsilon)$, remains bounded in $BL^1(\mathbb{R}^3)$. This means that, up to a subsequence, $(u^\varepsilon_{\boldsymbol{m}}) \rightharpoonup u_{\boldsymbol{m}}$ weakly in $BL^1(\mathbb{R}^3)$ for some suitable $u_{\boldsymbol{m}} \in BL^1(\mathbb{R}^3)$. Thus, according to Proposition 4.2, there exist functions $u_{\boldsymbol{m}} \in BL^1(\mathbb{R}^3)$ and

$v_{\boldsymbol{m}} \in L^2[\mathbb{R}^3; H^1_\#(Q)/\mathbb{R}]$ such that

$$(u^\varepsilon_{\boldsymbol{m}}) \twoheadrightarrow u_{\boldsymbol{m}} \ \text{ in } L^2_\omega \quad \text{and} \quad \nabla u^\varepsilon_{\boldsymbol{m}}(x) \twoheadrightarrow \nabla u_{\boldsymbol{m}}(x) + \nabla_y v_{\boldsymbol{m}}(x,y). \tag{4.25}$$

In view of the previous limit relations, $u^\varepsilon_{\boldsymbol{m}}$ is expected to behave as $u_{\boldsymbol{m}}(x) + \varepsilon v_{\boldsymbol{m}}(x, x/\varepsilon)$. This suggest to use, in the variational formulation of the magnetostatic problem expressed by equation (4.3), test functions having the form $\varphi(x) + \varepsilon\psi(x, x/\varepsilon)$, with $\varphi \in \mathcal{D}(\mathbb{R}^3)$ and $\psi \in \mathcal{D}[\mathbb{R}^3; C^\infty_\#(Q)]$. This yields

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla u^\varepsilon_{\boldsymbol{m}} \cdot \left(\nabla\varphi + \varepsilon\nabla\psi\left(x, \frac{x}{\varepsilon}\right) + \nabla_y\psi\left(x, \frac{x}{\varepsilon}\right)\right) \mathrm{d}x \\ = -\int_{\mathbb{R}^3} \boldsymbol{m}_\varepsilon \cdot \left(\nabla\varphi + \varepsilon\nabla\psi\left(x, \frac{x}{\varepsilon}\right) + \nabla_y\psi\left(x, \frac{x}{\varepsilon}\right)\right) \mathrm{d}x.\end{aligned}$$

From the second of the two limit relations (4.25), we get

$$\begin{aligned}\int_{\mathbb{R}^3\times Q} (\nabla u_{\boldsymbol{m}}(x) + \nabla_y v_{\boldsymbol{m}}(x,y)) \cdot (\nabla\varphi + \nabla_y\psi(x,y))\mathrm{d}y\mathrm{d}x \\ = -\int_{\mathbb{R}^3\times Q} \boldsymbol{m}(x,y) \cdot (\nabla\varphi + \nabla_y\psi(x,y))\mathrm{d}y\mathrm{d}x.\end{aligned} \tag{4.26}$$

In particular, by choosing $\psi \equiv 0$ we get

$$\begin{aligned}-\int_{\mathbb{R}^3} \langle \boldsymbol{m}(x,y)\rangle_Q \cdot \nabla\varphi(x)\mathrm{d}x = \int_{\mathbb{R}^3\times Q} (\nabla u_{\boldsymbol{m}}(x) + \nabla_y v_{\boldsymbol{m}}(x,y)) \cdot \nabla\varphi\mathrm{d}y\mathrm{d}x \\ = \int_{\mathbb{R}^3} \nabla u_{\boldsymbol{m}}(x) \cdot \nabla\varphi(x),\end{aligned} \tag{4.27}$$

where the last equality follows from the fact that $\langle\nabla_y v_{\boldsymbol{m}}(x,y)\rangle_Q = 0$. Thus, we reach the conclusion that the weak limit $u_{\boldsymbol{m}}$ satisfies the variational formulation (4.27), i.e. is a solution of the «homogenized» equation

$$u_{\boldsymbol{m}}(x) = -\operatorname{div}\langle \boldsymbol{m}(x,y)\rangle_Q \quad \text{in } BL^1(\mathbb{R}^3). \tag{4.28}$$

On the other hand by choosing $\varphi \equiv 0$ and $\psi(x,y) = \psi_1(x)\psi_2(y)$ in (4.26) we get

$$\begin{aligned}\int_{\mathbb{R}^3} \langle(\nabla u_{\boldsymbol{m}}(x) + \nabla_y v_{\boldsymbol{m}}(x,y)) \cdot \nabla_y\psi_2(y)\rangle_Q \psi_1(x)\mathrm{d}x \\ = -\int_{\mathbb{R}^3} \langle \boldsymbol{m}(x,y) \cdot \nabla_y\psi_2(y)\rangle_Q \psi_1(x)\mathrm{d}x\,,\end{aligned}$$

and hence the so-called cell problem

$$\begin{aligned}-\int_Q \boldsymbol{m}(x,y) \cdot \nabla_y\psi_2(y)\mathrm{d}y = \int_Q (\nabla u_{\boldsymbol{m}}(x) + \nabla_y v_{\boldsymbol{m}}(x,y)) \cdot \nabla_y\psi_2(y)\mathrm{d}y \\ = \int_Q \nabla_y v_{\boldsymbol{m}}(x,y) \cdot \nabla_y\psi_2(y)\mathrm{d}y\,,\end{aligned} \tag{4.29}$$

where, again, the last equality follows from the fact that $\langle\nabla_y\psi_2(y)\rangle_Q = 0$. Note that the variational formulation (4.29) can be more concisely expressed in the form

$$\Delta_y v_{\boldsymbol{m}}(x,y) = -\operatorname{div}_y \boldsymbol{m}(x,y) \quad \text{in } H^1_\#(Q)/\mathbb{R}, \tag{4.30}$$

and the well-posedness of the previous variational problem is again a direct consequence of LAX-MILGRAM theorem. □

(d) The continuous limit of magnetostatic self-energy functionals $\mathcal{W}_\varepsilon$

In what follows we will make use of Proposition 4.3, to prove the following

Proposition 4.4. *The family of magnetostatic self-energies*

$$\mathcal{W}_\varepsilon : \boldsymbol{m} \in L^2(\Omega, S^2) \mapsto -(\boldsymbol{h}_{\mathrm{d}}[M_\varepsilon \boldsymbol{m}], M_\varepsilon \boldsymbol{m})_\Omega \tag{4.31}$$

continuously converges to the functional

$$\mathcal{W}_{\mathrm{hom}} : \boldsymbol{m} \in L^2(\Omega, S^2) \mapsto -\langle M_s \rangle_Q^2 (\boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}], \boldsymbol{m})_\Omega + \|\nabla_y v_{\boldsymbol{m}}\|^2_{\Omega\times Q} \tag{4.32}$$

where for every $x \in \Omega$ the scalar function $v_{\boldsymbol{m}} : \Omega \times Q \to \mathbb{R}$ is the unique solution of the following variational cell problem:

$$\boldsymbol{m}(x) \cdot \int_Q M_s(y) \nabla_y \psi(y) \mathrm{d}y = -\int_Q \nabla_y v_{\boldsymbol{m}}(x, y) \cdot \nabla_y \psi(y) \mathrm{d}y, \tag{4.33}$$

$$\int_Q v_{\boldsymbol{m}}(x, y) \mathrm{d}y = 0, \tag{4.34}$$

for all $\psi \in H^1_\#(Q)$.

Proof. We know (see Proposition A.2) that $|M_\varepsilon \boldsymbol{m}|^2 \equiv |M_\varepsilon|^2 \rightharpoonup \langle |M_\varepsilon|^2 \rangle_Q$ weakly* in $L^\infty(\Omega)$. In particular, by choosing $|\boldsymbol{m}|^2 \in L^1(\Omega)$ as a test function we get

$$\begin{aligned} \|M_\varepsilon\|^2_\Omega = (|M_\varepsilon \boldsymbol{m}|^2, |\boldsymbol{m}|^2)_\Omega \to (\langle |M_\varepsilon|^2 \rangle_Q, |\boldsymbol{m}|^2)_\Omega &= |\Omega| \langle |M_\varepsilon|^2 \rangle_Q \\ &= \|M_s(y) \boldsymbol{m}(x)\|^2_{\Omega\times Q} \end{aligned}$$

and therefore $M_\varepsilon(x) \boldsymbol{m}(x) \twoheadrightarrow M_s(y) \boldsymbol{m}(x)$ strongly.

Next, since $\boldsymbol{h}_{\mathrm{d}}[M_\varepsilon \boldsymbol{m}] \cdot M_\varepsilon \boldsymbol{m}$ is bounded in $L^2(\Omega)$, from Proposition A.6 and Proposition 4.3, we get

$$\begin{aligned} &\lim_{\varepsilon\to 0} -(\boldsymbol{h}_{\mathrm{d}}[M_\varepsilon \boldsymbol{m}], M_\varepsilon \boldsymbol{m})_\Omega \\ &\qquad = -\langle M_s \rangle_Q^2 (\boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}], \boldsymbol{m})_\Omega - \int_{\Omega\times Q} \nabla_y v_{\boldsymbol{m}}(x, y) \cdot M_s(y) \boldsymbol{m}(x) \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{4.35}$$

Now, we observe that for every $x \in \Omega$ the scalar function $v_{\boldsymbol{m}}(x, \cdot)$ is the unique solution of the variational cell problem (4.24), therefore setting $\psi(\cdot) := v_{\boldsymbol{m}}(x, \cdot)$ in (4.24) we get

$$-\,\boldsymbol{m}(x) \cdot \int_Q M_s(y) \nabla_y v_{\boldsymbol{m}}(x, y) \mathrm{d}y = \int_Q |\nabla_y v_{\boldsymbol{m}}(x, y)|^2 \mathrm{d}y \tag{4.36}$$

and therefore

$$\lim_{\varepsilon\to 0} -(\boldsymbol{h}_{\mathrm{d}}[M_\varepsilon \boldsymbol{m}], M_\varepsilon \boldsymbol{m})_\Omega = -\langle M_s \rangle_Q^2 (\boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}], \boldsymbol{m})_\Omega + \|\nabla_y v_{\boldsymbol{m}}\|^2_{\Omega\times Q} =: \mathcal{W}_{\mathrm{hom}}(\boldsymbol{m}). \tag{4.37}$$

Now we show that the family $\mathcal{W}_\varepsilon$ continuously converges to $\mathcal{W}_{\mathrm{hom}}$. This amounts to prove that

$$\lim_{(\boldsymbol{m},\varepsilon)\to(\boldsymbol{m_0},0^+)} \mathcal{W}_\varepsilon(\boldsymbol{m}) = \mathcal{W}_{\mathrm{hom}}(\boldsymbol{m}_0). \tag{4.38}$$

To this end, we split:

$$|\mathcal{W}_\varepsilon(\boldsymbol{m}) - \mathcal{W}_{\mathrm{hom}}(\boldsymbol{m}_0)| \leqslant |\mathcal{W}_\varepsilon(\boldsymbol{m}) - \mathcal{W}_\varepsilon(\boldsymbol{m}_0)| + |\mathcal{W}_\varepsilon(\boldsymbol{m}_0) - \mathcal{W}_{\mathrm{hom}}(\boldsymbol{m}_0)|. \tag{4.39}$$

From (4.37) we already know that $|\mathcal{W}_\varepsilon(\boldsymbol{m}_0) - \mathcal{W}_{\mathrm{hom}}(\boldsymbol{m}_0)| \to 0$ when $(\boldsymbol{m}, \varepsilon) \to (\boldsymbol{m_0}, 0^+)$. Therefore to finish it is sufficient to prove that

$$\lim_{(\boldsymbol{m},\varepsilon)\to(\boldsymbol{m_0},0^+)} |\mathcal{W}_\varepsilon(\boldsymbol{m}) - \mathcal{W}_\varepsilon(\boldsymbol{m}_0)| = 0, \tag{4.40}$$

and this is a consequence of the following estimate (uniform with respect to ε):

$$\begin{aligned} |\mathcal{W}_\varepsilon(\boldsymbol{m}) - \mathcal{W}_\varepsilon(\boldsymbol{m}_0)| &\leqslant |(\boldsymbol{h}_{\mathrm{d}}[M_\varepsilon \boldsymbol{m}], M_\varepsilon(\boldsymbol{m} - \boldsymbol{m}_0))_\Omega + (\boldsymbol{h}_{\mathrm{d}}[M_\varepsilon(\boldsymbol{m} - \boldsymbol{m}_0)], M_\varepsilon \boldsymbol{m}_0)_\Omega| \\ &\leqslant 2\|M_s\|_\infty |\Omega|^{1/2} \|\boldsymbol{m} - \boldsymbol{m}_0\|_\Omega. \end{aligned}$$

□

5. The homogenized anisotropy and interaction energies

This section is devoted to the proof of the continuous convergence of the family of anisotropy energy functionals $\mathcal{A}_\varepsilon$ and of the family of interaction energy functionals $\mathcal{Z}_\varepsilon$, respectively to $\mathcal{A}_{\mathrm{hom}}$ and $\mathcal{Z}_{\mathrm{hom}}$, whose expression is given by (1.15) and (1.18).

(a) The continuous limit of the anisotropy energy functionals $\mathcal{A}_\varepsilon$

Proposition 5.1. *If the anisotropy energy density $\varphi_{\mathrm{an}} : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^+$ is Q-periodic with respect to the first variable and globally Lipschitz with respect to the second one (uniformly with respect to the first variable) then the family $\mathcal{A}_\varepsilon$ of anisotropy energies continuously converges to the homogenized anisotropy energy*

$$\mathcal{A}_{\mathrm{hom}} : \boldsymbol{m} \in L^2(\Omega, \mathbb{S}^2) \mapsto \int_{\Omega \times Q} \varphi_{\mathrm{an}}(y, \boldsymbol{m}(x)) \mathrm{d}y \mathrm{d}x. \tag{5.1}$$

Proof. We have to prove that for every $\boldsymbol{m}_0 \in L^2(\Omega, \mathbb{S}^2)$ one has

$$\lim_{(\boldsymbol{m},\varepsilon)\to(\boldsymbol{m}_0,0^+)} \mathcal{A}_\varepsilon(\boldsymbol{m}) = \mathcal{A}_{\mathrm{hom}}(\boldsymbol{m}_0). \tag{5.2}$$

For every $\boldsymbol{m}, \boldsymbol{m}_0 \in L^2(\Omega, \mathbb{S}^2)$ we write

$$|\mathcal{A}_\varepsilon(\boldsymbol{m}) - \mathcal{A}_{\mathrm{hom}}(\boldsymbol{m}_0)| \leqslant |\mathcal{A}_\varepsilon(\boldsymbol{m}) - \mathcal{A}_\varepsilon(\boldsymbol{m}_0)| + |\mathcal{A}_\varepsilon(\boldsymbol{m}_0) - \mathcal{A}_{\mathrm{hom}}(\boldsymbol{m}_0)|. \tag{5.3}$$

Due to generalized RIEMANN-LEBESGUE lemma (cfr. Proposition A.2) one has $\varphi_{\mathrm{an}}(x/\varepsilon, \boldsymbol{m}_0(x)) \rightharpoonup \langle \varphi_{\mathrm{an}}(y, \boldsymbol{m}_0(x)) \rangle_Q$ weakly* in $L^\infty(\Omega)$; therefore in particular $|\mathcal{A}_\varepsilon(\boldsymbol{m}_0) - \mathcal{A}_{\mathrm{hom}}(\boldsymbol{m}_0)| \to 0$ as $(\boldsymbol{m}, \varepsilon) \to (\boldsymbol{m}_0, 0^+)$. To finish the proof it remains to prove that

$$\lim_{(\boldsymbol{m},\varepsilon)\to(\boldsymbol{m}_0,0^+)} |\mathcal{A}_\varepsilon(\boldsymbol{m}) - \mathcal{A}_\varepsilon(\boldsymbol{m}_0)| = 0. \tag{5.4}$$

But this is an immediate consequence of the the global and uniform Lipschitz continuity of φ_{an} and CAUCHY-SCHWARZ inequality; indeed one has

$$\begin{aligned}\int_{\Omega \times Q} |\varphi_{\mathrm{an}}(x/\varepsilon, \boldsymbol{m}(x)) - \varphi_{\mathrm{an}}(x/\varepsilon, \boldsymbol{m}_0(x))| \mathrm{d}y \mathrm{d}x &\leqslant c_L \int_\Omega |\boldsymbol{m}(x) - \boldsymbol{m}_0(x)| \mathrm{d}x \\ &\leqslant c_L |\Omega|^{1/2} \|\boldsymbol{m} - \boldsymbol{m}_0\|_\Omega.\end{aligned}$$

This concludes the proof. □

Corollary 5.1. (Uniaxial anisotropy energy density). *If $\varphi_{\mathrm{an}}(y, \boldsymbol{m}) = \kappa(y)|\boldsymbol{m}(x) \wedge \boldsymbol{u}(y)|^2$ then*

$$\mathcal{A}_{\mathrm{hom}}(\boldsymbol{m}) = \int_\Omega \langle \kappa \rangle_Q - \langle \kappa \boldsymbol{u} \otimes \boldsymbol{u} \rangle_Q : \boldsymbol{m} \otimes \boldsymbol{m} \mathrm{d}\tau.$$

(b) The continuous limit of interaction energy functionals $\mathcal{Z}_\varepsilon$

The convergence of $(\mathcal{Z}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$ to $\mathcal{Z}_\varepsilon$ is straightforward. Indeed this energy term is expressed by the product, with respect to the $L^2(\Omega)$ scalar product, of the constant function $\boldsymbol{h}_a$ and the weakly converging sequence $(M_\varepsilon \boldsymbol{m})_{\varepsilon \in \mathbb{R}^+} \rightharpoonup \langle M_s \rangle_Q$ weakly* in $L^\infty(\Omega)$ (cfr. Proposition A.2). Therefore repeating the same argument given in the previous subsection:

$$\mathcal{Z}_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma_{\mathrm{cont}}} \mathcal{Z}_{\mathrm{hom}} \quad \text{with} \quad \mathcal{Z}_{\mathrm{hom}}(\boldsymbol{m}) := -\langle M_s \rangle_Q \int_\Omega \boldsymbol{h}_a \cdot \boldsymbol{m} \mathrm{d}\tau.$$

6. Proof of Theorem 1.1 completed

It is now easy to complete the proof of Theorem 1.1. Indeed the equicoercivity of the family of GIBBS-LANDAU free energy functionals $(\mathcal{G}^\varepsilon_{\mathcal{L}})_{\varepsilon \in \mathbb{R}^+}$ expressed by (1.6) has been proved in Section 2. It is therefore sufficient to recall the stability properties of the Γ-limit under the sum of a continuously convergent family of functionals. In fact, what has been proved in the previous subsections, can be summarized by the following convergence scheme

$$\mathcal{E}_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma} \mathcal{E}_{\text{hom}} \quad , \quad \mathcal{W}_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma_{\text{cont}}} \mathcal{W}_{\text{hom}} \quad , \quad \mathcal{A}_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma_{\text{cont}}} \mathcal{A}_{\text{hom}} \quad , \quad \mathcal{Z}_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma_{\text{cont}}} \mathcal{Z}_{\text{hom}}. \tag{6.1}$$

Thus, Proposition A.1 completes the proof.

7. Conclusions

We have given in this paper a complete theory for periodic microstructured magnetic materials. Obtained through a process of $\Gamma-$convergence the model derives rigorously the energy terms from the parameters of each constituent of the sample and the mixing geometry of the different materials in the unit periodic cell. We believe that the result applies to most of magnetic composites that are nowadays considered, e.g. those obtained from a mixing of hard and soft phases [17,33] or the multilayer magnetic materials [18,30]. In this latter case, the formula obtained further simplifies since the exchange coefficient can be analytically computed. We leave the exploration of potential applications to a forthcoming work.

Data accessibility. The results of this paper are not based on experimental data.

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Authors contributions. The authors contributed equally to conception and design of this research, to construction and verification of the proofs of results, and to the writing and critical revision of the paper. All authors gave final approval for publication.

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A. Γ-convergence, Two-scale convergence and related results

The purpose of this section is to fix some notations and to give a survey of the concepts and results that are used throughout this work. All results are stated without proof as they can be readily found in the references given below.

(a) Γ-convergence of a family of functionals

We start by recalling De Giorgi's notion of Γ-convergence and some of its basic properties (see [11,14]). Throughout this part we indicate with (X,d) a metric space and, for every $m\in X$, with $\mathfrak{C}_d(m)$ the subset of all sequences of elements of X which converge to m.

Definition A.1. (Γ-convergence of a family of functionals) *Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of functionals defined on X with values on $\overline{\mathbb{R}}$. The functional $\mathcal{F}: X\to\overline{\mathbb{R}}$ is said to be the Γ-lim of $(\mathcal{F}_n)_{n\in\mathbb{N}}$ with respect to the metric d, if for every $m\in X$ we have:*

$$\forall (m_n)\in\mathfrak{C}_d(m) \quad \mathcal{F}(m)\leqslant \liminf_{n\to\infty}\mathcal{F}_n(m_n) \tag{A 1}$$

and

$$\exists (\bar{m}_n)\in\mathfrak{C}_d(m) \quad \mathcal{F}(m)=\lim_{n\to\infty}\mathcal{F}_n(\bar{m}_n). \tag{A 2}$$

In this case we write $\mathcal{F}=\Gamma\text{-}\lim_{n\to\infty}\mathcal{F}_n$.

If $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is a family of functionals, we say that $\mathcal{F}: X\to\overline{\mathbb{R}}$ is the Γ-lim of $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ as $\varepsilon\to 0$, if for every $\varepsilon_n\downarrow 0$ one has $\mathcal{F}=\Gamma\text{-}\lim_{n\to\infty}\mathcal{F}_{\varepsilon_n}$. In this case we write $\mathcal{F}=\Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon$.

One of the most important properties of Γ-convergence, and the reason why this kind of variational convergence is so important in the asymptotic analysis of variational problems, is that under appropriate compactness hypotheses it implies the convergence of (almost) minimizers of a family of equicoercive functionals to the minimum of the Γ-limit functional. To this end we first recall the notion of equicoerciveness (sometimes referred to as equi-mildly coerciveness [7]):

Definition A.2. *A family $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of functionals defined on X with values on $\overline{\mathbb{R}}$ is said to be* equicoercive *in X, if there exists a compact subset $K \subseteq X$ such that*

$$\inf_X \mathcal{F}_\varepsilon = \inf_K \mathcal{F}_\varepsilon \quad \forall \varepsilon \in \mathbb{R}^+.$$

We then have (cfr. [7]):

Theorem A.1. (Fundamental Theorem of Γ-convergence) *If $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is a family of equicoercive functionals Γ-converging on X to the functional $\mathcal{F}$. Then $\mathcal{F}$ is coercive and lower semicontinuous (therefore there exists a minimizer for $\mathcal{F}$ on X) and we have the convergence of minima values*

$$\min_{m\in X} \mathcal{F}(m) = \lim_{\varepsilon\to 0} \inf_{m\in X} \mathcal{F}_\varepsilon(m). \tag{A 3}$$

Moreover, given $\varepsilon_n \downarrow 0$ and $(m_n)_{n\in\mathbb{N}}$ a converging sequence such that

$$\lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}(m_n) = \lim_{n\to\infty} \big(\inf_{m\in X} \mathcal{F}_{\varepsilon_n}(m) \big), \tag{A 4}$$

its limit is a minimizer for $\mathcal{F}$ on X. If (A 4) holds, the sequence $(m_n)_{n\in\mathbb{N}}$ is said to be a sequence of almost-minimizers for $\mathcal{F}$.

Let us recall now that given two families of functional $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ and $(\mathcal{G}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ Γ-converging respectively to $\mathcal{F}$ and $\mathcal{G}$, it is in general not the case (see [11]) that Γ-$\lim_{\varepsilon\to 0}(\mathcal{F}_\varepsilon + \mathcal{G}_\varepsilon) = \mathcal{F} + \mathcal{G}$. A sufficient condition for that property to hold is that at least one of the two families of functionals satisfies a stronger type of convergence:

Definition A.3. *We say that a family of functionals $(\mathcal{G}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is* continuously convergent *in X to a functional $\mathcal{G} : X \to \overline{\mathbb{R}}$, and we will write $\mathcal{G}_\varepsilon \xrightarrow{\Gamma_{\text{cont}}} \mathcal{G}$, if for every $m_0 \in X$*

$$\lim_{(m,\varepsilon)\to(m_0,0)} \mathcal{G}_\varepsilon(m) = \mathcal{G}(m_0).$$

We then have (see [11] for a proof):

Proposition A.1. *Let $\mathcal{F} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon$. Suppose that the family of functionals $(\mathcal{G}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ continuously converges to $\mathcal{G}$, and that $\mathcal{G}_\varepsilon$ and $\mathcal{G}$ are everywhere finite on X. Then $\mathcal{G} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon$ and*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} (\mathcal{F}_\varepsilon + \mathcal{G}_\varepsilon) = \mathcal{F} + \mathcal{G}.$$

In particular if $\mathcal{Z} : X \to \mathbb{R}$ is a continuous functional then $\Gamma\text{-}\lim_{\varepsilon\to 0}(\mathcal{F}_\varepsilon + \mathcal{Z}) = \mathcal{F} + \mathcal{Z}$ and $\mathcal{Z}$ is called a continuous perturbation of the Γ-limit.

(b) Two-scale convergence

The aim of this section is to present in a schematic way the main properties of two-scale convergence, a notion that is first due to NGUETSENG [29], developed as a methodology by ALLAIRE [3] and further investigated by many others (see [2] and references therein for instance).

We denote by $C^\infty_\#(Q)$ the set of infinitely differentiable real functions over $\mathbb{R}^3$ that are Q-periodic and define $H^1_\#(Q)$ as the closure of $C^\infty_\#(Q)$ in $H^1_{\text{loc}}(\Omega)$. Obviously any element of $H^1_\#(Q)$ has the same trace on the opposite faces of Q.

A generalized version of the RIEMANN-LEBESGUE lemma holds for the weak limit of rapidly oscillating functions. For a proof we refer the reader to [15].

Proposition A.2. *Let $\Omega \subset \mathbb{R}^3$ be any open set. Let $1 \leqslant p < \infty$ and $t > 0$ be a positive real number. Let $u \in L^p(Q)$ be a Q-periodic function. Set $u_\varepsilon(x) := u(x/\varepsilon)$ τ-a.e. on Ω. Then, if $p < \infty$, as $\varepsilon \to 0$*

$$u_\varepsilon \rightharpoonup \langle u \rangle_Q := \frac{1}{|Q|} \int_Q u \, \mathrm{d}\tau \quad \textit{weakly in } L^p(\Omega).$$

If $p = \infty$, one has

$$u_\varepsilon \rightharpoonup \langle u \rangle_Q := \frac{1}{|Q|} \int_Q u \, \mathrm{d}\tau \quad \textit{weakly}^* \textit{ in } L^\infty(\Omega).$$

Definition A.4. *Let Ω be an open set $\Omega \subset \mathbb{R}^3$, and let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a fixed sequence of positive real numbers (when it is clear from the context we will omit the subscript k) converging to 0. The sequence of functions $(u_\varepsilon) \in L^2(\Omega)$ is said to two-scale converge to a limit $u \in L^2(\Omega \times Q)$, if for any function $\varphi \in \mathcal{D}[\Omega; C^\infty_\#(Q)]$ we have*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\varphi(x, x/\varepsilon) \, \mathrm{d}x = \int_{\Omega\times Q} u(x,y)\varphi(x,y) \, \mathrm{d}y \, \mathrm{d}x \tag{A 5}$$

*In this case we write $u_\varepsilon \twoheadrightarrow u$. We say that (u_ε) in $L^2(\Omega)$ **strongly** two-scale converges to a limit $u \in L^2(\Omega \times Q)$ if $u_\varepsilon \twoheadrightarrow u$ and moreover*

$$\|u\|_{\Omega\times Q} = \lim_{\varepsilon\to 0} \|u_\varepsilon\|_\Omega.$$

The importance of this new notion of convergence relies on the following compactness results.

Proposition A.3. *For each bounded sequence (u_ε) in $L^2(\Omega)$, there exists an $u \in L^2(\Omega \times Q)$ such that, up to a subsequence, $u_\varepsilon \twoheadrightarrow u$.*

Moreover, for bounded sequences in $H^1(\Omega)$ we have the following result:

Proposition A.4. *Let (u_ε) be a sequence in $H^1(\Omega)$ that converges weakly to a limit $u \in H^1(\Omega)$. Then $u_\varepsilon \twoheadrightarrow u$ and there exists a function $v \in L^2[\Omega; H^1_\#(Q)/\mathbb{R}]$ such that, up to a subsequence:*

$$\nabla u_\varepsilon \twoheadrightarrow \nabla u + \nabla_y v.$$

Next we recall that if the sequence (u_ε) is bounded in $L^2(\Omega)$, it is possible to enlarge the class of test functions used in the definition of two-scale convergence.

Proposition A.5. *Let (u_ε) be a bounded sequence in $L^2(\Omega)$ which two-scale converges to $u \in L^2(\Omega \times Q)$. Then*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\varphi(x, x/\varepsilon) \, \mathrm{d}x = \int_{\Omega\times Q} u(x,y)\varphi(x,y) \, \mathrm{d}y \, \mathrm{d}x$$

for every $\varphi \in L^2[\Omega; C_\#(Q)]$.

Finally we recall a simple criteria that permits to «bypass» the problem concerning the convergence of the product of two $L^2(\Omega)$-weakly convergence sequences (cfr. [3,22]).

Proposition A.6. *Let (u_ε) and (v_ε) be sequences in $L^2(\Omega)$ that respectively two-scale converge to u and v in $L^2(\Omega \times Q)$. If at least one of them **strongly** two-scale converges, then*

$$u_\varepsilon v_\varepsilon \twoheadrightarrow uv.$$

In particular, if $(u_\varepsilon v_\varepsilon)$ is bounded in $L^2(\Omega)$, from the previous proposition, we have

$$\int_\Omega u_\varepsilon(x) v_\varepsilon(x)\varphi(x, x/\varepsilon) \, \mathrm{d}x = \int_{\Omega\times Q} u(x,y)v(x,y)\varphi(x,y) \, \mathrm{d}y \, \mathrm{d}x$$

for every $\varphi \in L^2[\Omega; C_\#(Q)]$.

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