Asymptotic Problems in Radiative Transfer Theory

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Contents:

- Diffusion Approximation
- Intermediate Models
- AP Schemes
- Radiative Hydrodynamics

Warm Up: Simple Models

Behavior as $\epsilon \to 0$ of the solutions of

$$\partial_t f_\epsilon + \frac{v}{\epsilon} \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon^2} \sigma(\rho_\epsilon)(\rho_\epsilon - f_\epsilon),$$

$$\rho_\epsilon(t,x) = \int f_\epsilon(t,x,v)\,\mathrm{d}v = \langle f_\epsilon \rangle$$

$$0 < \sigma_\star \leq \sigma(z) \leq \sigma^\star, \qquad \text{continuous,}$$

$$t \geq 0, \quad x \in \mathbb{R}^N, \quad v \in \mathbb{S}^{N-1},$$

$$\int_{\mathbb{S}^{N-1}} \mathrm{d}v = 1, \qquad \int_{\mathbb{S}^{N-1}} v\,\mathrm{d}v = 0, \quad \int_{\mathbb{S}^{N-1}} v \otimes v\,\mathrm{d}v \text{ positive definite.}$$

"Grey Radiative Transfer Eq." (no frequency, no coupling with matter $\rho \simeq aT^4$).

Strategy #1: Hilbert Expansion

Plug the ansatz

$$f_\epsilon = F_0 + \epsilon F_1 + \epsilon^2 F_2 + \ldots$$

into

$$\partial_t f_\epsilon + \frac{v}{\epsilon} \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon^2} \sigma(\rho_\epsilon)(\rho_\epsilon - f_\epsilon),$$

- ϵ^{-2} terms: $\langle F_0 \rangle - F_0 = 0$ so that $F_0 = \rho_0(t,x) \mathbb{1}(v)$ (hydrodynamic behavior).
- ϵ^{-1} terms: $\sigma(\rho_0)\big(\langle F_1 \rangle - F_1\big) = v \cdot \nabla_x \rho_0$ which yields ($\langle v \rangle = 0$)

$$F_1(t,x,v) = -\frac{v \cdot \nabla_x \rho_0(t,x)}{\sigma(\rho_0(t,x))}.$$

- ϵ^0 terms: smthg with vanishing average $= \partial_t \rho_0 + v \cdot \nabla_x F_1$ which

...

... yields

$$\partial_t \rho_0 + \int_{\mathbb{S}^{N-1}} v \cdot \nabla_x \Big(- \frac{v \cdot \nabla_x \rho_0(t,x)}{\sigma(\rho_0(t,x))} \Big) \,\mathrm{d}v = 0$$
$$= \partial_t \rho_0 - \mathrm{div}_x \Big(\int_{\mathbb{S}^{N-1}} v \otimes v \,\mathrm{d}v \; \frac{\nabla_x \rho_0(t,x)}{\sigma(\rho_0(t,x))} \Big) = 0.$$

Non linear diffusion equation: Rosseland Approximation.

Possible proof: Expand $f_\epsilon = \rho - \epsilon v \partial_x \rho + \epsilon^2 f_2 + \epsilon g_\epsilon$ and estimate the remainder g_ϵ...

Strategy #2: Entropy and Moment Equations

$$\partial_t f_\epsilon + \frac{v}{\epsilon} \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon^2} \sigma(\rho_\epsilon)(\rho_\epsilon - f_\epsilon),$$

leads to ($\langle 1 \rangle = 1$)

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} |f_\epsilon|^2 \,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\epsilon^2} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \sigma(\rho_\epsilon)\, |f_\epsilon - \rho_\epsilon|^2 \,\mathrm{d}v\,\mathrm{d}x = 0$$

so that $f_\epsilon = \rho_\epsilon + \epsilon r_\epsilon$ with r_ϵ bounded in L^2.

Set $(\rho_\epsilon, J_\epsilon, \mathbb{P}_\epsilon)(t,x) = \int_{\mathbb{S}^{N-1}} (1, \frac{v}{\epsilon}, v \otimes v)\ f_\epsilon(t,x,v)\,\mathrm{d}v.$

We get
$$\begin{cases} \partial_t \rho_\epsilon + \mathrm{div}_x J_\epsilon = 0, \\ \quad \text{and} \quad \epsilon^2 \partial_t J_\epsilon + \mathrm{Div}_x \mathbb{P}_\epsilon = -\sigma(\rho_\epsilon) J_\epsilon. \end{cases}$$

As $\epsilon \to 0$, $\mathbb{P}_\epsilon = \int_{\mathbb{S}^{N-1}} v \otimes v \,\mathrm{d}v\ \rho_\epsilon + \mathcal{O}(\epsilon)$ yields the Limit System

$$\mathrm{Div}_x \mathbb{P} = \nabla_x \rho = -\sigma(\rho) J \qquad \text{and} \qquad \partial_t \rho + \mathrm{div}_x J = 0.$$

Mathematical Tools

Nonlinearities require a compactness argument

⋆ Semi-group techniques Bardos-Golse-Perthame '87... but it needs some monotonicity property.

⋆ Average Lemma Bardos-Golse-Perthame-Sentis, '88 ;

⋆ Compensated-Compactness Argument (Div-Curl Lemma)

Murat-Tartar '78 (Homogenization & Conservation Laws) ;
Marcati-Milani '90, Lions-Toscani '98, G.-Poupaud '01

Average Lemma

If $f(x,v) \in L^2(\mathbb{R}^N \times \mathcal{V})$ and $v \cdot \nabla_x f \in L^2(\mathbb{R}^N \times \mathcal{V})$ then

$$\int_{\mathcal{V}} f\psi(v)\,\mathrm{d}v \in H^{1/2}(\mathbb{R}^N).$$

Crucial Assumption: $\forall \xi \in \mathbb{S}^{N-1}$, $\Big|\{v \in \mathcal{V} \text{ such that } v \cdot \xi = 0\}\Big| = 0.$

Div-Curl Lemma

Let $U_n = (u_n^1, \ldots, u_n^N) \rightharpoonup U$, $V_n \rightharpoonup V$ in $L^2(\Omega)$ with furthermore $\mathrm{div} U_n = \sum \partial_i u_n^i$ and $\mathrm{curl} V_n = \left[\partial_j v_n^i - \partial_i v_n^j\right]_{ij}$ compact in H^{-1} then

$$U_n \cdot V_n = \sum_{i=1}^{N} u_n^i v_n^i \rightharpoonup U \cdot V \text{ in } \mathcal{D}'.$$

Crucial Assumption: $\forall \xi \in \mathbb{S}^{N-1}$, $\Big|\{v \in \mathcal{V} \text{ such that } v \cdot \xi \neq 0\}\Big| > 0.$
Thus it works for discrete velocity models $v \in \{v^1, \ldots, v^M\}$, $\mathrm{d}v = \sum_{i=1}^{M} \omega_i \delta_{v=v^i}$. (cf. $\langle v \otimes v \rangle > 0$).

How does it work ?

Using $f_\epsilon = \rho_\epsilon + \epsilon r_\epsilon$ we rewrite the Moment equations as

$$\begin{cases} \operatorname{div}_{t,x}(\rho_\epsilon, J_\epsilon) = 0, \\ \left(\int_{\mathbb{S}^{N-1}} v \otimes v \,\mathrm{d}v\right) \nabla_x \rho_\epsilon = -\epsilon^2 \partial_t J_\epsilon - \sigma(\rho_\epsilon) J_\epsilon - \epsilon \mathrm{Div}_x(R_\epsilon) \end{cases}$$

so that

$$\operatorname{div}_{t,x}(\rho_\epsilon, J_\epsilon) \text{ and } \operatorname{curl}_{t,x}(\rho_\epsilon, 0, \dots, 0) = \begin{pmatrix} 0 & -\nabla_x \rho_\epsilon^T \\ \nabla_x \rho_\epsilon & 0 \end{pmatrix}$$

belong to a compact set of $H^{-1}_{\mathrm{loc}}((0,T) \times \mathbb{R}^N)$.

Coupling to Homogeneization

We seek reduced models for routine computations that take into account heterogeneities of the medium

[Allaire with Bal, Capdeboscq, Sieiss ; G.-Poupaud, G.-Mellet]

$$\epsilon\partial_t f + v\cdot\nabla_x f = \frac{1}{\epsilon}\Big(\int \sigma(x, x/\epsilon, v, v_\star) f(v_\star)\,\mathrm{d}v_\star - \int \sigma(x, x/\epsilon, v_\star, v)\,\mathrm{d}v_\star\ f(v)\Big)$$

Set $T = v\cdot\nabla_y - Q$, $y = x/\epsilon$ and expand $f_\epsilon = \sum \epsilon^j f^{(j)}(t, x, x/\epsilon, v)$

- $Tf^{(0)} = 0$ is solved by $\rho(t,x)M(x,y,v)$
- $Tf^{(1)} = v\cdot\nabla_x f_0 = vM\cdot\nabla_x\rho + v\cdot\nabla_x M\rho$. If $\int vM\,\mathrm{d}v\,\mathrm{d}y = 0$ then $f^{(1)} = -\chi\cdot\nabla_x\rho + \lambda\rho$ where

$$T\chi = -vM, \qquad T\lambda = v\cdot\nabla_x M$$

Eventually, we get

$$\partial_t \rho - \nabla_x \cdot (D(x)\nabla_x \rho - U(x)\rho) = 0,$$
$$D(x) = \int\int v \otimes \chi(x,y,v)\,\mathrm{d}v\,\mathrm{d}y, \qquad U(x) = \int\int v\lambda(x,y,v)\,\mathrm{d}v\,\mathrm{d}y$$

Convection term related to the space dependance of the equilibrium function.

Degond-G.-Poupaud, 2003 ;
Chalub-Markowich-Perthame-Schmeiser, 2004

Treatment of the homogeneization aspect relies on double scale technics

$$\int_\Omega u_\epsilon \psi(x, x/\epsilon)\,\mathrm{d}x \underset{\epsilon \to 0}{\longrightarrow} \int_{(0,1)^N} \int_\Omega U\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Intermediate Models

Assuming $\mathcal{V} \subset (-1,+1)$, $\int_{\mathcal{V}} \mathrm{d}v = 1$, $\int_{\mathcal{V}} v \,\mathrm{d}v = 0$, $\int_{\mathcal{V}} v^2 \,\mathrm{d}v = d > 0$, solutions of

$$\epsilon \partial_t f_\epsilon + v \partial_x f_\epsilon = \frac{1}{\epsilon}\Big(\int_{\mathcal{V}} f_\epsilon \,\mathrm{d}v - f_\epsilon \Big)$$

converge to $\rho(t,x)$ solution of

$$\partial_t \rho - d \partial^2_{xx} \rho = 0.$$

One seeks intermediate models for $0 < \epsilon \ll 1$:

$\star$ heat eq. propagates at infinite speed instead of $\mathcal{O}(1/\epsilon)$,

$\star$ $\rho - \epsilon v \partial_x \rho$ does not preserve non-negativeness, nor the flux limited condition

$$\Big| \int_{\mathcal{V}} \frac{v}{\epsilon} f_\epsilon \,\mathrm{d}v \Big| \leq \frac{1}{\epsilon} \int_{\mathcal{V}} f_\epsilon \,\mathrm{d}v.$$

Minimum Entropy Principle Closure

The Moment System

$$\begin{cases} \partial_t \rho_\epsilon + \mathrm{div}_x J_\epsilon = 0 \\ \epsilon^2 \partial_t J_\epsilon + \mathrm{Div}_x \mathbb{P}_\epsilon = -J_\epsilon \end{cases}$$

is closed by imposing [Levermore'97, Dubroca-Feugeas'99, Fort'97]

$$\mathbb{P}_\epsilon = \int_{\mathcal{V}} v^2 f_\epsilon^\star \, \mathrm{d}v$$

where $f_\epsilon^\star$ minimizes

$$\int_{\mathcal{V}} f \ln f \, \mathrm{d}v, \text{ with the constraints } \int_{\mathcal{V}} (1,\ v/\epsilon) f \, \mathrm{d}v = (\rho_\epsilon,\ J_\epsilon)$$

One obtains $\mathbb{P}_\epsilon = \rho_\epsilon \psi(\epsilon J_\epsilon / \rho_\epsilon)$, a (strictly) hyperbolic system which is globally well-posed for small enough initial data, and consistent to the diffusion eq. as ϵ goes to 0 ($\psi(0) = d$).

On the Entropy-Based Model

Theorem. [Coulombel-Golse-G.'06] Let $\overline{\rho} > 0$. There exist $\delta > 0$, $C > 0$ such that, for any $\epsilon \in]0,1]$, and for any (ρ_0, J_0) with $\|\rho_0 - \overline{\rho}\|_{H^2(\mathbb{R})} \leq \delta$ and $\|\epsilon\, J_0\|_{H^2(\mathbb{R})} \leq \delta$, there exists a unique global solution $(\widehat{\rho_\epsilon}, \widehat{J_\epsilon})$ to the "Levermore System" with initial data (ρ_0, J_0), and that satisfies $(\widehat{\rho_\epsilon} - \overline{\rho}, \widehat{J_\epsilon}) \in \mathcal{C}(\mathbb{R}^+; H^2(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; H^1(\mathbb{R}))$. For r solution to the heat equation with initial data ρ_0, we have

$$\|\widehat{\rho_\epsilon} - r\|_{L^2(\mathbb{R}^+ \times \mathbb{R})} \leq C\,\epsilon, \qquad \|f^\star_\epsilon - f_\epsilon\|_{L^2((0,T)\times\mathbb{R}\times\mathbb{R})} \leq C\,\epsilon.$$

Arguments:

- Use Relaxation (it looks like $y' = y^2 - \lambda y$)
- Strong Coupling: "Kawashima-Shizuta Condition"
- Adapt Hanouzet-Natalini'03 analysis... and make it uniform wrt ϵ!
- Junca-Rascle'02's trick for the convergence to the heat eq.

Nonlocal Models for Hydrodynamic Regimes: Derivation and Numerical Schemes

Origin and Motivation of the Problem

• Nonlocal Model for Temperature : More or less heuristics models arising in plasmas physics (FIC)

Luciani-Mora-Virmont, Phys. Rev. Lett. 89

Epperlein-Short, Phys. Fluid B, 91

Where does it come from? properties of the solution? how can we compute the solution?

• Nonlocal terms : convolution kernel → pseudo-differential op.

...That certainly shares some features with the Gyroaverage operators arising in tokamaks' theory... or nonlocal electrostatic models in biology!

Towards Nonlocal Versions of the Heat Eq.

Idea: Replace the heat eq.

$$(H) \qquad \partial_t \rho = \nabla_x \cdot (K \nabla_x \rho)$$

by

$$\partial_t \rho = \nabla_x \cdot \Big(\int_{\mathbb{R}^N} G_\epsilon(x-y) \cdot K \nabla_x \rho(t,y)\, \mathrm{d}y \Big).$$

ϵ : Scaling Parameter such that as $\epsilon \to 0$ we recover (H).

Questions :

- Expression of the kernel G_ϵ?
- Computation of the solution (loss of sparsity)?

Diffusion Asymptotics

$$\epsilon\partial_t f_\epsilon + v \cdot \partial_x f_\epsilon = \frac{1}{\epsilon}\,(\rho_\epsilon - f_\epsilon),$$
$$v \in (-1, +1), \qquad \int \mathrm{d}v = 1, \qquad \rho_\epsilon = \int f_\epsilon \,\mathrm{d}v.$$

Hilbert Expansion: $f_\epsilon = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \dots$

- $\mathcal{O}(1/\epsilon)$ term: $(\rho^{(0)} - f^{(0)}) = 0$ yields $f^{(0)} = \rho^{(0)}(t, x)$,
- $\mathcal{O}(1)$ term: $(\rho^{(1)} - f^{(1)}) = v \cdot \partial_x \rho^{(0)}(t, x)$ yields $f^{(1)} = -v \cdot \partial_x \rho^{(0)}$,
- $\mathcal{O}(\epsilon)$ term: $\partial_t f^{(0)} + v \cdot \partial_x f^{(1)} =$ smthg with vanishing average hence

$$\partial_t \rho - \partial_x \cdot \Big(\underbrace{\int v \otimes v \,\mathrm{d}v}_{= K (= 1/3)} \ \partial_x \rho \Big) = 0$$

A Modified Hilbert Expansion

We set

Leading Term $F_\epsilon^{(0)} = \varrho_\epsilon(t,x)$ Corrector $F_\epsilon^{(1)} = -v \cdot \partial_x \varrho_\epsilon$

(solution of $\int F_\epsilon^{(1)}\,\mathrm{d}v - F_\epsilon^{(1)} = v \cdot \partial_x F_\epsilon^{(0)}$)

Equation for ϱ_ϵ:

- Define $G_\epsilon^{(1)}$ solution of

$$\epsilon v \cdot \partial_x G_\epsilon^{(1)} + G_\epsilon^{(1)} = F_\epsilon^{(1)}$$

- And require the conservation relation

$$\partial_t \int F_\epsilon^{(0)}\,\mathrm{d}v + \partial_x \int v G_\epsilon^{(1)}\,\mathrm{d}v = 0.$$

Nonlocal Equation

Convolution Operator:

Integrating the eq. for $G_\epsilon^{(1)}$ along characteristics yields

$$\int_{-1}^{+1} v G_\epsilon^{(1)}\,\mathrm{d}v = -\int_{\mathbb{R}} k_\epsilon(x-y)\;\partial_x \varrho_\epsilon(t,y)\,\mathrm{d}y$$

with

$$k_\epsilon(z) = \frac{z^2}{\epsilon}\int_{|z|}^{\infty} \frac{e^{-\tau/\epsilon}}{\tau^3}\,\mathrm{d}\tau.$$

Fourier Operator:

$$\widehat{G_\epsilon^{(1)}}(t,\xi,v) = \frac{iv\cdot\xi}{1+i\epsilon v\cdot\xi}\;\widehat{\varrho}_\epsilon(t,\xi).$$

so that

$$\partial_t \widehat{\varrho_\epsilon} = -\Psi_\epsilon(\xi)\widehat{\varrho_\epsilon}(t,\xi), \qquad \Psi_\epsilon(\xi) = \frac{1}{\epsilon^2}\Big(1 - \frac{\arctan(\epsilon|\xi|)}{\epsilon|\xi|}\Big) \xrightarrow[\epsilon\to 0]{} \frac{\xi^2}{3}.$$

Properties of the solution

Theorem. For any $\rho_{\mathrm{Init}} \in L^2(\mathbb{R}^N)$, the nonlocal eq. has a unique solution $\varrho_\epsilon \in C^0(\mathbb{R}^+; L^2(\mathbb{R}^N))$, which converges to ρ, solution of the heat eq. in $C^0([0,T]; L^2(\mathbb{R}^N))$ as ϵ goes to 0 (with rate $\mathcal{O}(\epsilon)$ if $\rho_{\mathrm{Init}} \in H^2(\mathbb{R}^N)$).

Approximation

Idea : replace $\Psi_\epsilon(\xi)$ by a rationale function $P_\epsilon(\xi)/Q_\epsilon(\xi)$ so that

$$\text{NLeq} \longrightarrow \partial_t Q_\epsilon(i\partial_x)\varrho_\epsilon = P_\epsilon(i\partial_x)\varrho_\epsilon$$

Taylor approx. of $\Psi_\epsilon(\xi)$ is definitely useless (polynomial behavior)

Padé approximation with $\deg(Q_\epsilon) = \deg(P_\epsilon)$: $\dfrac{|\xi|^2/3}{1 + 3\epsilon^2|\xi|^2/5}$

+ impose the correct behavior at ∞: $\dfrac{|\xi|^2/3}{1 + \epsilon^2|\xi|^2/3}$

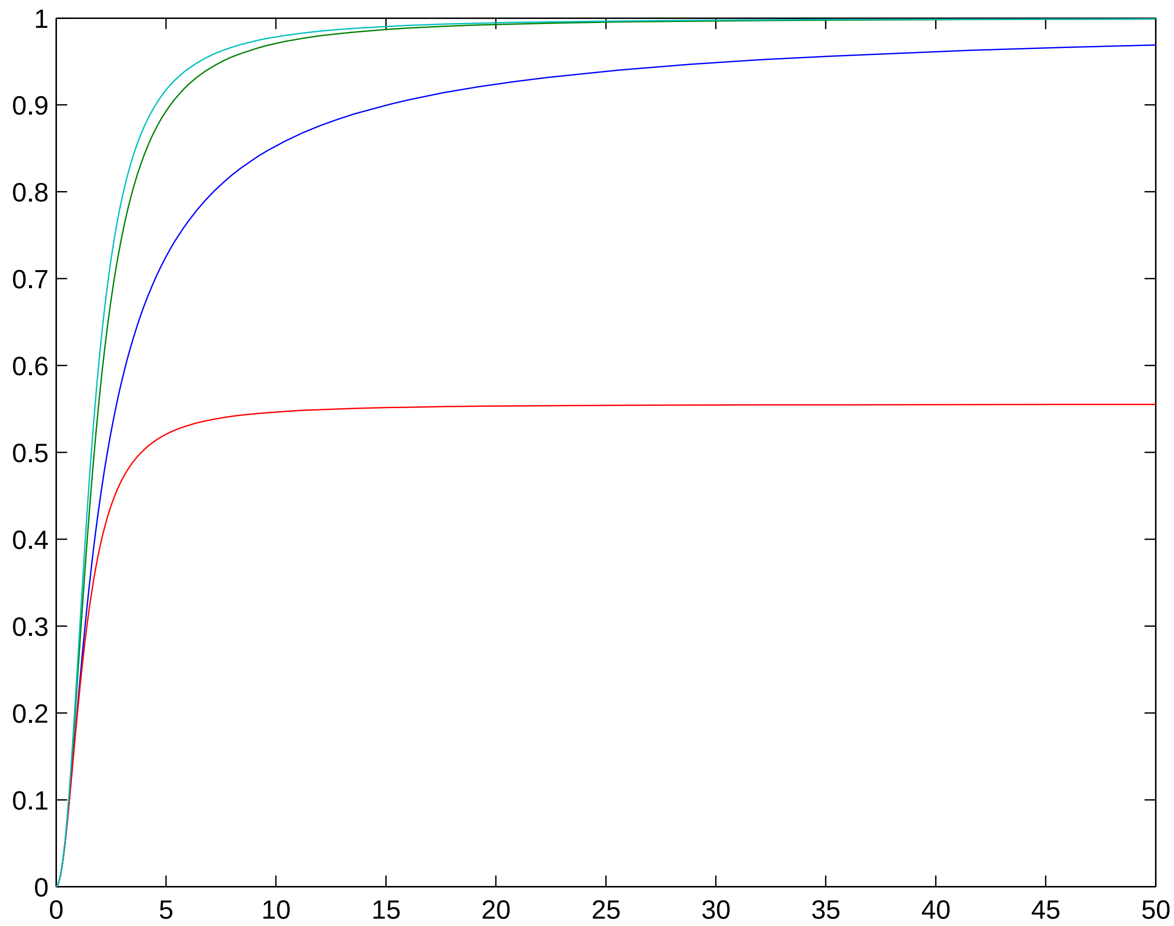
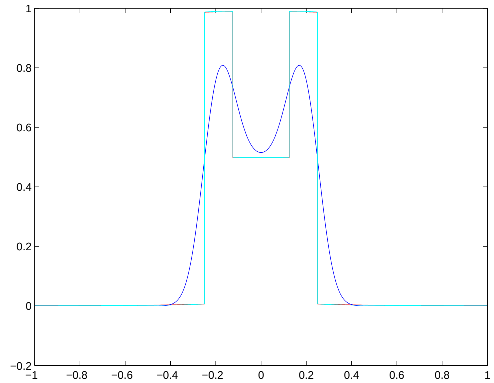
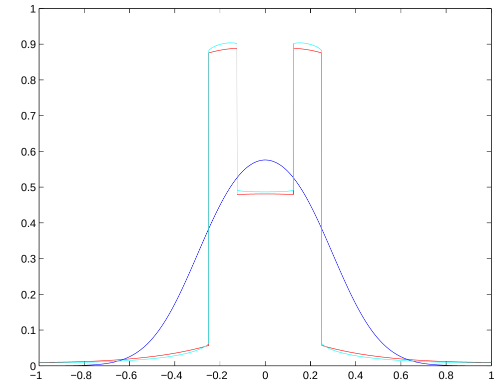


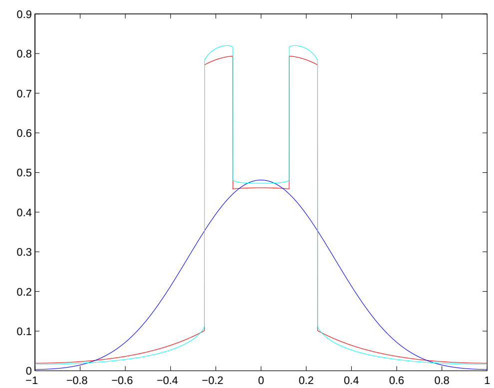
Figure 1: Graph of Ψ (blue curve, $\epsilon = 1$) vs. $(x^2/3)/(1{+}3x^2/5)$ (red), $x^2/3/(1+x^2/3)$ (dark green), $(x^2/3 + 38x^4/245)/(1 + 33x^2/49 + 38x^4/245)$ (green).



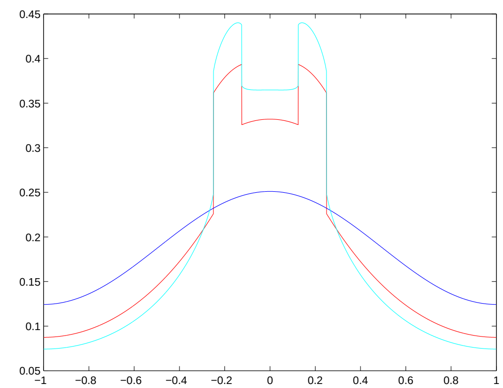
(a) $\epsilon = 0.5,\ T = 0.005$



(b) $\epsilon = 0.5,\ T = 0.05$

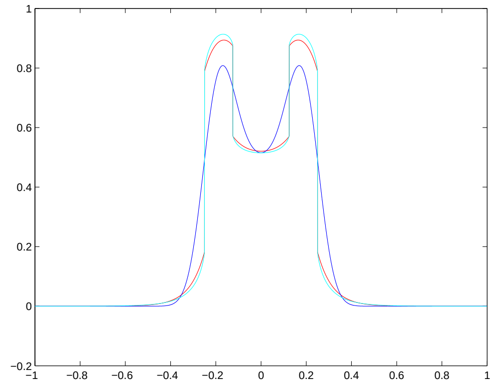


(c) $\epsilon = 0.5,\ T = 0.1$

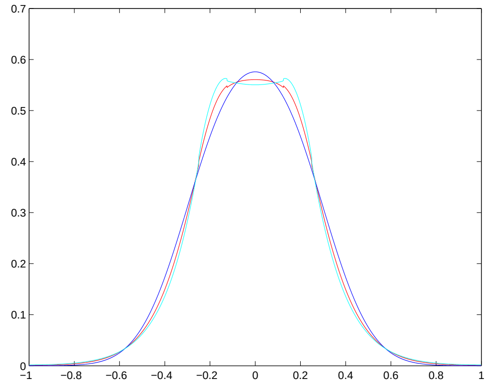


(d) $\epsilon = 0.5,\ T = 0.5$

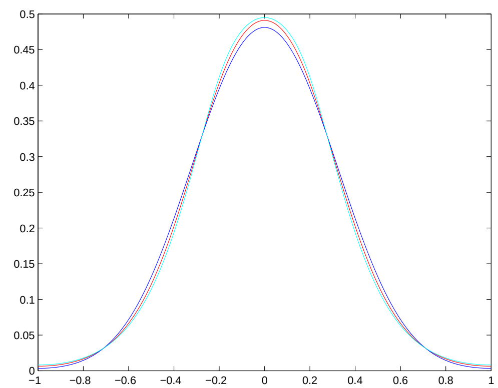
Figure 2: Periodic case/FFT : blue = heat, cyan: $\Psi_\epsilon(\xi)$, red: Padé.



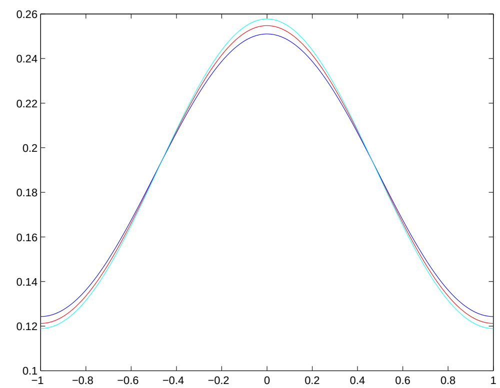
(a) $\epsilon = 0.1$, $T = 0.005$



(b) $\epsilon = 0.1$, $T = 0.05$



(c) $\epsilon = 0.1$, $T = 0.1$



(d) $\epsilon = 0.1$, $T = 0.5$

Figure 3: Periodic case/FFT : blue = heat, cyan: $\Psi_\epsilon(\xi)$, red: Padé.

Numerical Scheme for the Approximated Model

The Approx. Eq : $\partial_t(1 - \frac{\epsilon^2}{3}\partial_{xx}^2)\varrho_\epsilon = \frac{1}{3}\partial_{xx}^2\varrho_\epsilon$ can be treated by usual FD or FE methods...

θ-scheme:

$$\rho_j^{n+1} - \left(\theta\frac{\Delta t}{\Delta x^2} + \frac{\epsilon^2}{\Delta x^2}\right)\frac{\rho_{j+1}^{n+1} - 2\rho_j^{n+1} + \rho_{j-1}^{n+1}}{3}$$
$$= \rho_j^n + \left((1-\theta)\frac{\Delta t}{\Delta x^2} - \frac{\epsilon^2}{\Delta x^2}\right)\frac{\rho_{j+1}^{n} - 2\rho_j^{n} + \rho_{j-1}^{n}}{3}$$

CFL condition: $1 - 2\theta - \dfrac{2\epsilon^2}{\Delta t} \leq \dfrac{3\Delta x^2}{2\Delta t}$.

Energy Dissipation

Similar features as for the periodic case.

Asymptotically-Induced Schemes

$$\partial_t f_\epsilon + \frac{v}{\epsilon}\partial_x f_\epsilon = \frac{1}{\epsilon^2}(\langle f_\epsilon \rangle - f_\epsilon)$$

We have $f_\epsilon = \rho_\epsilon(t,x) + \epsilon r_\epsilon$ so that

$$\underbrace{\partial_t f_\epsilon + v\partial_x r_\epsilon}_{\text{transport-like}} = \underbrace{\frac{1}{\epsilon^2}(\rho_\epsilon - f_\epsilon) - \frac{v}{\epsilon}\partial_x \rho_\epsilon = -\frac{1}{\epsilon} r_\epsilon - \frac{v}{\epsilon}\partial_x \rho_\epsilon}_{\text{Stiff sources with } \langle \cdot \rangle = 0.}$$

Splitting Approach

- Solving $\partial_t f_\epsilon + v\partial_x r_\epsilon = 0$ defines $f^{n+1/2}$, $\rho^{+1/2}$
- Solve ODEs $\partial_t f_\epsilon = \frac{1}{\epsilon^2}(\rho_\epsilon - f_\epsilon) - \frac{v}{\epsilon}\partial_x \rho_\epsilon$

  Since $\langle \text{rhs} \rangle = 0$ we have $\rho^{n+1} = \rho^{n+1/2}$

and we write

$$f^{n+1} = e^{-\Delta t/\epsilon^2} f^{n+1/2} + (1 - e^{-\Delta t/\epsilon^2})\rho^{n+1/2}$$
$$r^{n+1} = e^{-\Delta t/\epsilon^2} r^{n+1/2} - (1 - e^{-\Delta t/\epsilon^2}) v \partial_x \rho^{n+1/2}$$

The scheme is Asymptotic Preserving by construction.

Fully Explicit.

Stability condition: certainly better than $\Delta t/\Delta x^2$...

Cheap scheme adapted for intermediate regimes $0 \leq \epsilon \ll 1$.

Adapts to more complicated models (coupling with hydro, fluid-particles flows...)

... and to the Levermore flux-limited model as well (through relaxation approach combined to a kinetic interpretation)

[G.-Lafitte '05, Carrillo-G.-Lafitte-Vecil '07, G. Lafitte '08]

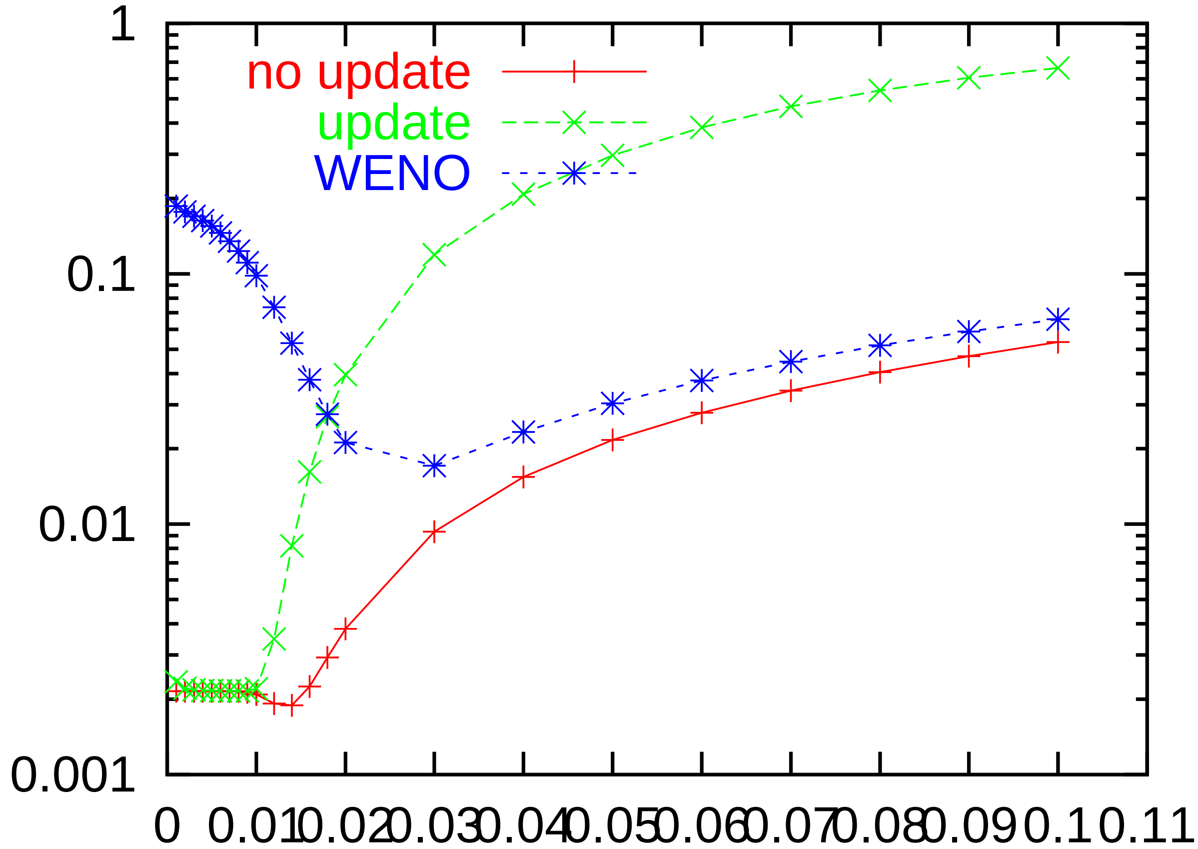


Figure 4: $L^2_{t,x,v}$-error of the distribution function f with respect to the solution of the heat equation with a symmetric initial data and a mesh of 100x100 with respect to ϵ.

Towards more realistic Radiative Transfer Problems: Radiative Hydrodynamics and Non Equilibrium regime

Euler System
$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ \partial_t(nu) + \partial_x(nu^2 + p) = -S_m, \\ \partial_t(nE) + \partial_x((nE+p)u) = -S_e \end{cases}$$

coupled to (scattering vs emission/absorption, Relativistic effects)

$$\epsilon \partial_t f + v \partial_x f = \frac{1}{\epsilon} Q_s + \epsilon Q_a$$
$$Q_s = \sigma_s \Big(\frac{1}{\Lambda^3} \langle \Lambda^2 f \rangle - \Lambda f \Big), \qquad Q_a = \sigma_a \Big(\frac{1}{\Lambda^3} \frac{1}{\pi} \theta^4 - \Lambda f \Big).$$

with $\Lambda = (1 - \epsilon u v)/\sqrt{1 - \epsilon^2 u^2}$ and $S_m = \frac{1}{\epsilon} \langle v Q_s \rangle + \epsilon \langle v Q_a \rangle$, $S_e = \frac{1}{\epsilon^2} \langle Q_s \rangle + \langle Q_a \rangle$.

As $\epsilon \to 0$, f_ϵ becomes proportional to Λ^{-4}, which has a $\mathcal{O}(\epsilon)$ flux.

Non Equilibrium Diffusion Regime

Scattering dominates: relaxation to an isotropic distribution but final model with TWO temperatures $\theta \neq \theta_{rad} (\simeq \rho^{1/4})$.

Ref. : Lowrie-Morel-Hittinger'99, Buet-Després'04

Full Model:

$$\left\{ \begin{array}{l} \partial_t n + \partial_x (nu) = 0, \\ \partial_t (nu) + \partial_x (nu^2 + p) = -\mathcal{P} \dfrac{\partial_x \rho}{3}, \\ \partial_t (nE) + \partial_x (nEu + pu) = -\mathcal{P} \dfrac{1}{3} u \partial_x \rho + \mathcal{P} \sigma_a (\rho - \theta^4), \\ \partial_t \rho - \dfrac{1}{3\sigma_s} \partial^2_{xx} \rho + \dfrac{4}{3} \partial_x (\rho u) - \dfrac{1}{3} u \partial_x \rho = \sigma_a (\theta^4 - \rho). \end{array} \right.$$

Doppler corrections make non conservative $p_{rad} \partial_x u$ terms appear

Non Equilibrium Diffusion Regime

Questions are related to the effects of the Energy Exchanges on the features of the usual Euler system:

- Well posedness of the kinetic/hyperbolic system [Lin'06, Zhong-Jiang'06]

- Asymptotic problems: diffusion regime [G.-Lafitte '06]

- (Smoothing?) effects on the shock profile [Lin-Coulombel-G. '06]

- Stability questions (of constants, of shocks profiles...) [Lin-Coulombel-G. '06, Coulombel-Mascia'0?]

- Numerical Experiments

Radiative Shock Profiles

Simplified Model:
$$\begin{cases} \partial_t n + \partial_x (nu) = 0, \\ \partial_t (nu) + \partial_x (nu^2 + p) = 0, \\ \partial_t (nE) + \partial_x (nEu + pu) = \rho - \theta^4, \\ -\partial_{xx}^2 \rho = \theta^4 - \rho, \end{cases}$$

The last eq. recasts as

$$\rho(t,x) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}\theta^4(t,y)\,\mathrm{d}y, \qquad q = -\partial_x \rho, \qquad \partial_x q = -(\rho - \theta^4)$$

System version of the toy model

$$\partial_t u + \partial_x \frac{u^2}{2} = -\partial_x q = Ku - u, \qquad Ku(t,x) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|} u(t,y)\,\mathrm{d}y$$

Ref. : Kawashima-Nishibata'99

Radiative (Small) Shock Profiles

Theorem. [Lin, Coulombel, G.'07] Let γ satisfy $1 < \gamma < \frac{\sqrt{7}+1}{\sqrt{7}-1} \simeq 2.215$ and let (ρ_-, u_-, e_-) be fixed. Then there exists a positive constant δ (that depends on (ρ_-, u_-, e_-), and γ) such that, for all state (ρ_+, u_+, e_+) verifying:

$$\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \leq \delta$$

and $(\rho_\pm, u_\pm, e_\pm)$ is a shock wave, with speed σ, for the (standard) Euler equations, then there exists a C^2 traveling wave $(\rho, u, e)(x - \sigma t)$ solution of the Radiative Euler eq. Furthermore, there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ if $\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \leq \delta_n$, then the profile is C^{n+2}. The profile can be shown to be asymptotically stable wrt zero mass perturbation.

A smooth profile

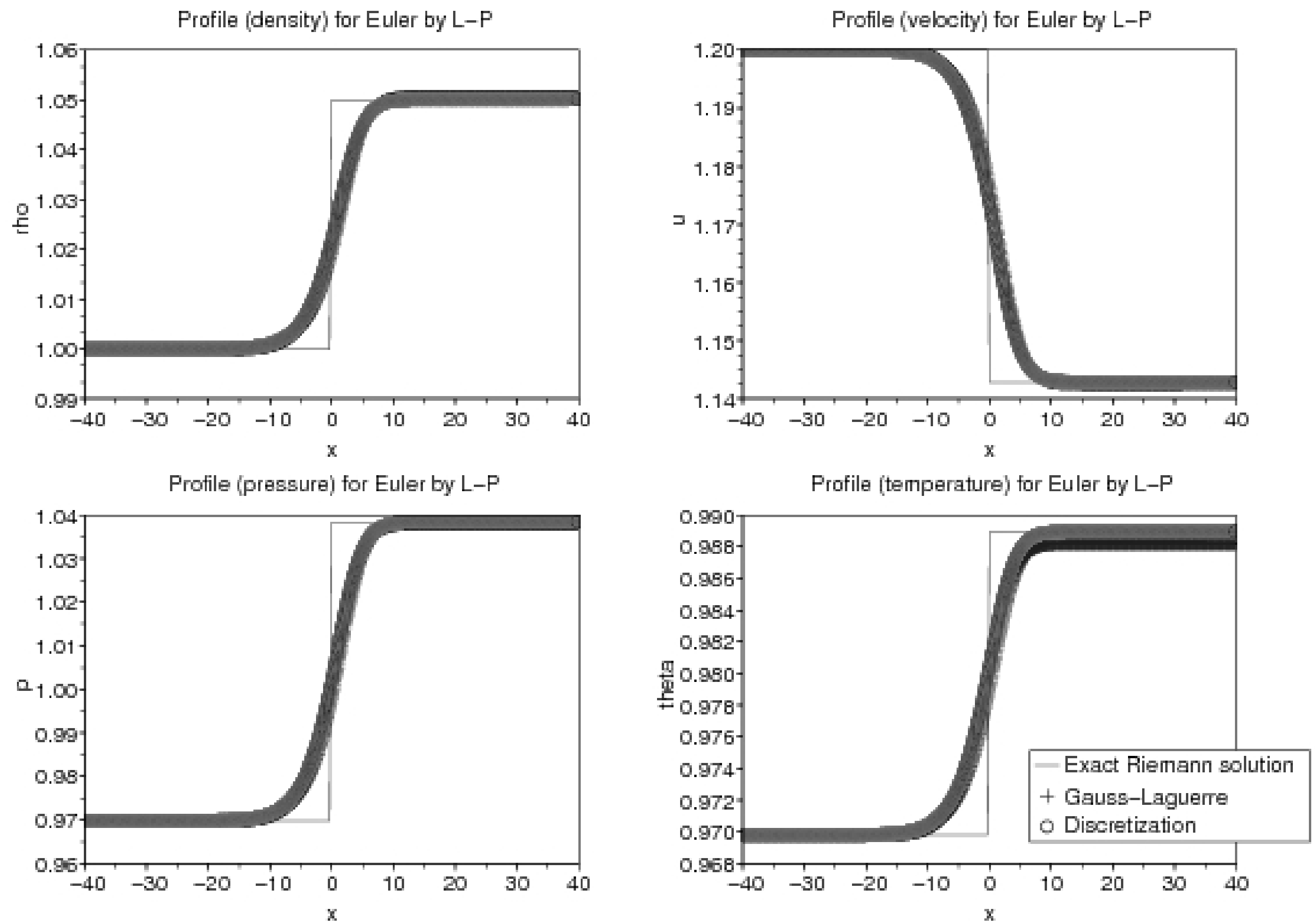


Figure 5: Numerical experiments by Coulombel-Lafitte $\delta \simeq 0.2$

A non-smooth profile (Zeldovich spike)

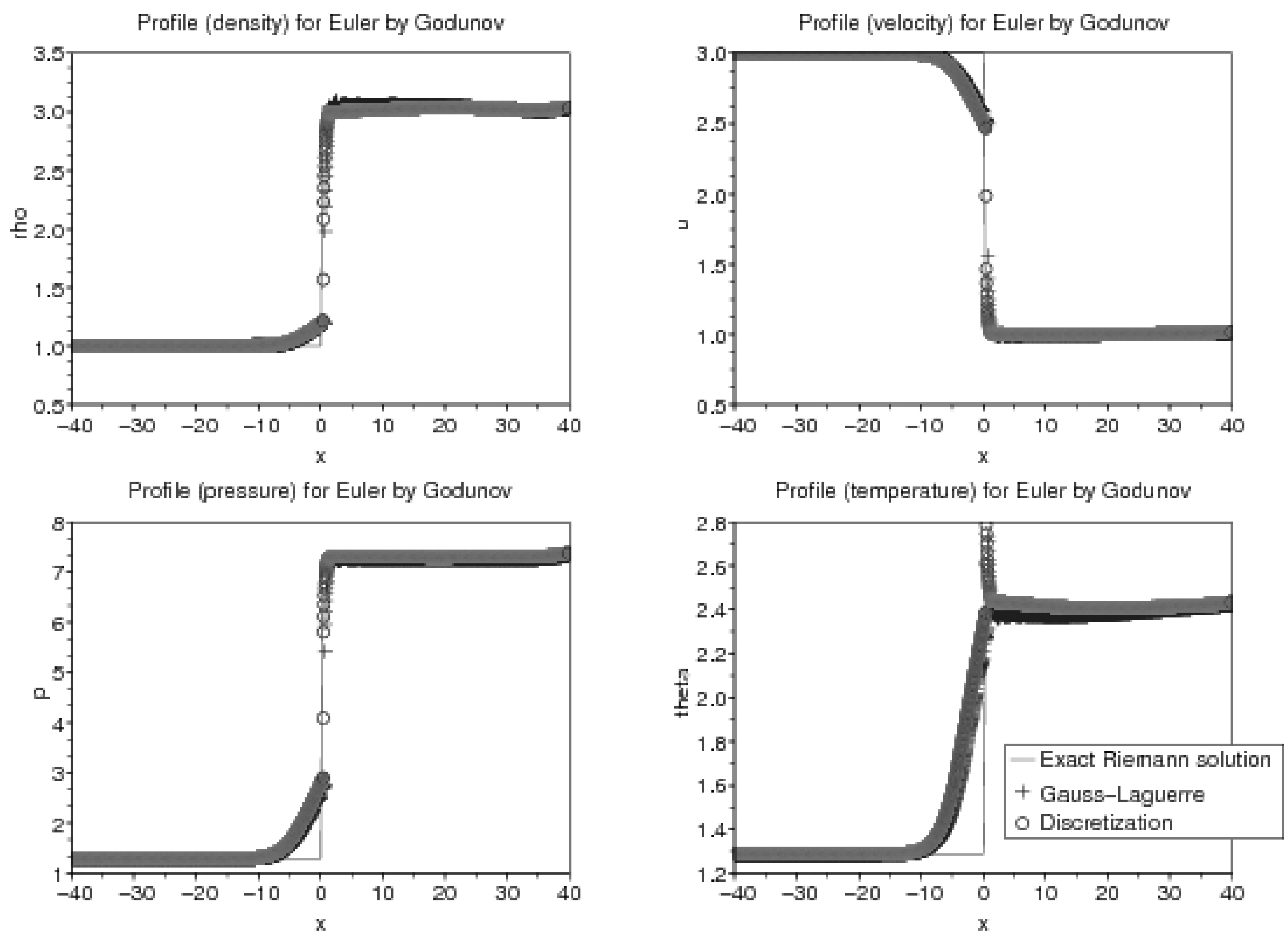


Figure 6: Numerical experiments by Coulombel-Lafitte $\delta \simeq 2$

As a conclusion

- Highly nonlinear, strongly coupled models

- Multiscale features

- Many asymptotic problems

- A large variety of relevant models (with maybe different behavior...)

- Many challenging questions both for analysis and simulations