Traffic plans: book to appear Springer

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Optimal Transportation Networks

Models and Theory

The aim of this book is to give a unified mathematical theory of branched transportation (or irrigation) networks. The only axiom of the theory is a $l \times s^{\alpha}$ cost law ($0 \leq \alpha \leq 1$) for transporting a good with size s on a path with length l. Let us explain first why this assumption is relevant.

The economic justification in graph theory (with fixed geometry)

W.I. ZANGWILL. Minimum Concave Cost Flows in Certain Networks. *Management Science*, 14(7):429–450, 1968.

A similar view is developed in the more recent article [98]: *Although a mathematical model with a linear arc cost function is easier to solve, it may not reflect the actual transportation cost in real operations. In practice, the unit cost for transporting freight usually decreases as the amount of freight increases. The cargo transportation cost in particular is mainly influenced by the cargo type, the loading/unloading activities, the transportation distance, and the amount. In general, each transportation unit cost decreases as the amount of cargo increases, due to economy of scale in practice. Hence, in actual operations the transportation cost function can usually be formulated as a concave cost function.*

Economic justifications: communication networks, pipe-lines: problem of optimal geometry

E.N. GILBERT. Minimum cost communication networks. *Bell System Tech. J.*, 46:2209–2227, 1967.

Minimum Concave Cost Flows: *Much research has been developed around optimizing pipeline design assuming a predetermined geographical layout of the distribution system. There has been less work done, however, on the problem of optimizing the configuration of the network itself. Generally, engineers develop the basic layout through experience and sheer intuition.*

R.P. LEJANO. Optimizing the layout and design of branched pipeline water distribution systems. *Irrigation and Drainage Systems*, 20(1):125–137, 2006.

First biological motivation: plants

Q. XIA. The formation of a tree leaf. *to appear in ESAIM: Control, Opt. Calc. Var.*, 2006.

Fig. 1.1. The structure of the nerves of a leaf (see [96] for a model of leaves based on optimal irrigation transport).

Fig. 1.3. A very old tree (1200 years) spans his branches towards the light. Trees and plants solve the problem of spanning their branches as much as possible in order to maximize the amount of light their leaves receive for photosynthesis. The surface of the branches is minimized for a better resistance to parasites, temperature changes, etc.

B. MAUROY, M. FILOCHE, E. WEIBEL, and B. SAPOVAL. An optimal bronchial tree may be dangerous. *Nature*, 427:633–636, 2004.

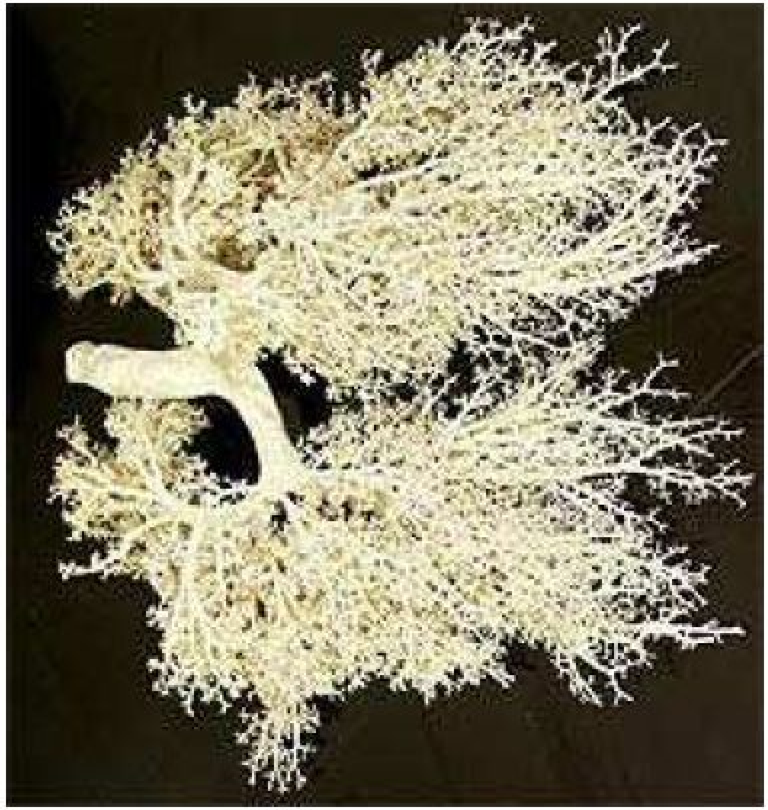


Fig. 1.2. Image: A cast of a set of lungs. They solve the problem of bringing the air entering the trachea onto a surface with very large area (about 500 m^2). A mathematical and physical study of the lungs efficiency is developed in [60]. Copyright: Jonathan Natiuk, stock.xchng

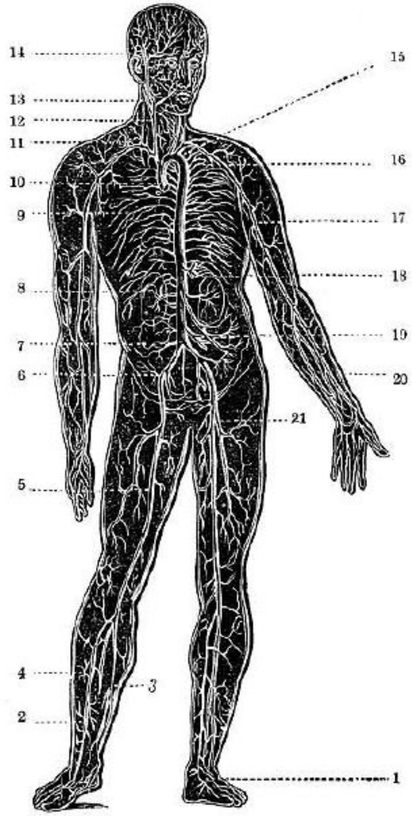


Fig. 1.4. Arteries of the human body. They solve the problem of transporting the blood from the heart to the whole body with very low basal metabolic rate. Attempts to demonstrate scaling laws in Nature have focused on the basal metabolic rate [88, 89]. This rate has been linked to the total blood flow. (From www.mspong.org/cyclopedia/medicine_pics.html).

G.B. WEST, J.H. BROWN, and B.J. ENQUIST. A general model for the origin of allometric scaling laws in biology. *Science*, 276(4):122–126, 1997.

C.D. MURRAY. The physiological principle of minimum work applied to the angle of branching arteries. *J. Gen. Physiology*, 9:835–841, 1926.

Besicovich’s plumbery: the first irrigation fractal

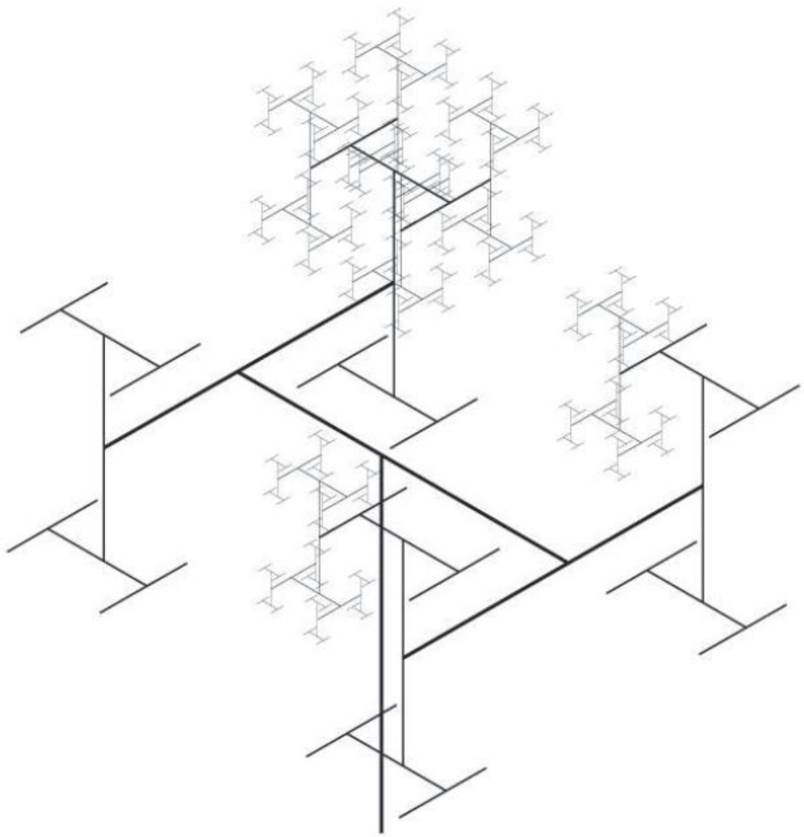


Fig. 1.5. An irrigating tree

A.S. BESICOVITCH. On the definition and value of the area of a surface. *Quart. J. Math.*, 16:86–102, 1945.

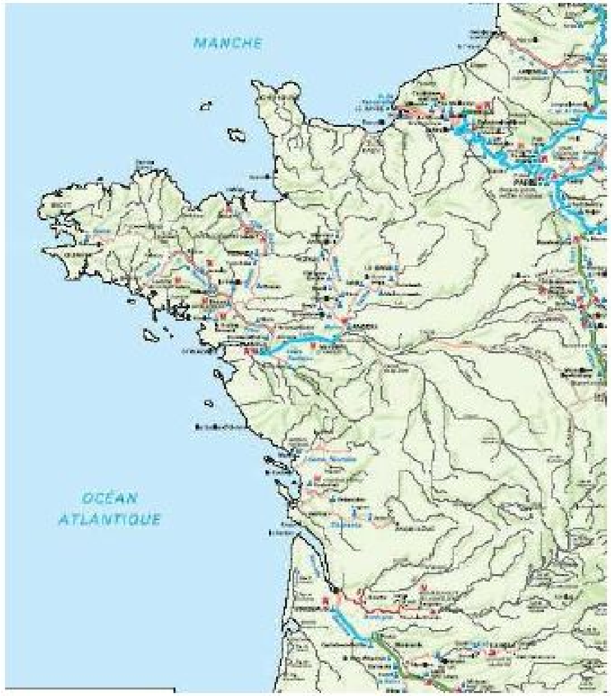


Fig. 2.4. A map of western France's river network. This branched network manages to bring back to the Atlantic ocean water raining over the whole territory. According to [72], such a network evolves towards a locally minimal flow energy configuration

I. RODRIGUEZ-ITURBE and A. RINALDO. *Fractal River Basins*. Cambridge University Press, 1997.

A.N. STRAHLER. Quantitative analysis of watershed geomorphology. *Am. Geophys. Un. Trans.*, 38:913, 1957.

E. TOKUNAGA. Consideration on the composition of drainage networks and their evolution. *Geogr. Rep., Tokyo Metrop. Univ*, 13:1–27, 1978.

J.R. BANAVAR, F. COLAIORI, A. FLAMMINI, A. MARITAN, and A. RINALDO. Scaling, optimality, and landscape evolution. *J. Stat. Physics*, 104(1):1–48, 2001.

Important variant : subway + walk models:

A. BRANCOLINI and G. BUTTAZZO. Optimal networks for mass transportation problems. *ESAIM: COCV*, 11:88–101, 2005.

G. BUTTAZZO. Three optimization problems in mass transportation theory. *preprint*, 2004.

G. BUTTAZZO, A. PRATELLI, S. SOLIMINI, and E. STEPANOV. Mass transportation and urban planning problems. *Forthcoming.*

G. BUTTAZZO, A. PRATELLI, and E. STEPANOV. Optimal pricing policies for public transportation networks. *SIAM J. Opt.*, 16:826–853, 2006.

G. BUTTAZZO and E. STEPANOV. Optimal transportation networks as free Dirichlet regions for the Monge-Kantorovich problem. *Ann. Sc. Norm. Super. Pisa*, 5(4):631–678, 2003.

G. BUTTAZZO and E. STEPANOV. *Minimization problems for average distance functionals*, volume 14, pages 47–83. 2004.

Subway + walk model

Fig. 2.6. A Paris subway map. According to the works of Brancolini, Buttazzo, Paolini, Pratelli, Santambrogio, Stepanov and Solimini described in Section 2.5.2, the urban transportation model involves two or more competing transportation means. In the basic model, the subway, or any fast network, is modeled as a connected set with finite length and very low cost for the users. Users walk to this network and then use it. Thus the problem is a mixed problem involving a Monge-Kantorovich individual transport to and from the fast network ($\alpha = 1$). The fast transportation means has instead a Steiner energy (the length, $\alpha = 0$).

Mathematical side: problem introduced by :

V. CASELLES and J.-M. MOREL. *Irrigation*, pages 81–90. Progress in Nonlinear Differential Equations and their Applications, vol. 51, VARMET 2001, Trieste, June, 2001. Birkhauser, F. Tomarelli and G. Dal Maso edition, 2002.

Q. XIA. Optimal paths related to transport problems. *Commun. Contemp. Math.*, 5:251–279, 2003.

F. MADDALENA, S. SOLIMINI, and J.-M. MOREL. A variational model of irrigation patterns. *Interfaces and Free Boundaries*, 5(4):391–416, 2003.

Q. XIA. Boundary regularity of optimal transport paths. *Preprint*, 2004.

Q. XIA. Interior regularity of optimal transport paths. *Calculus of Variations and Partial Differential Equations*, 20(3):283–299, 2004.

M. BERNOT, V. CASELLES, and J.-M. MOREL. Traffic plans. *Publicacions Matemàtiques*, 49(2):417–451, 2005.

M. BERNOT, V. CASELLES, and J.-M. MOREL. The structure of branched transportation networks. *to appear in Calc. Var. and PDE*, 2006.

S. SOLIMINI and G. DE VILLANOVA. Elementary properties of optimal irrigation patterns. *to appear in Calc. Var. and PDE*, 2005.

F. SANTAMBROGIO. Optimal channel networks, landscape function and branched transport. *to appear in Interface and Free Boundaries*, 2006.

Interpolating from Monge-Kantorovich to Steiner

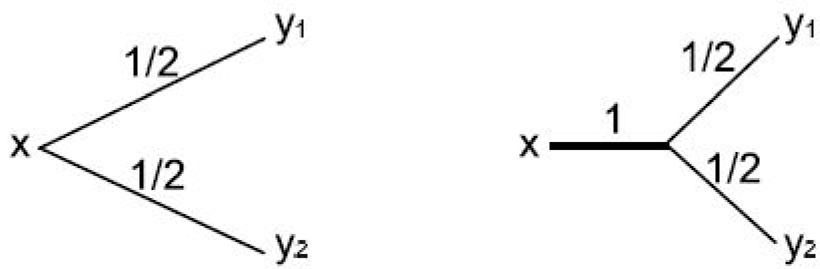
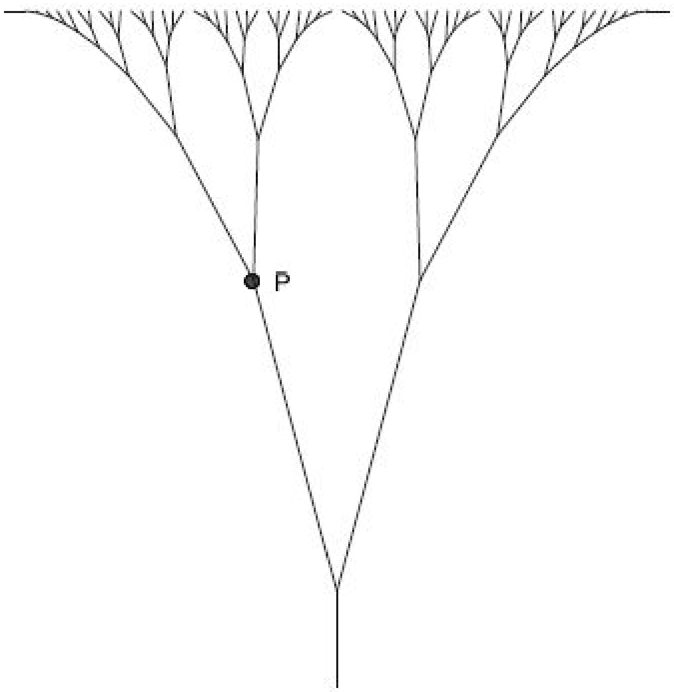


Fig. 2.1. The transport from δ_x to $\frac{1}{2}(\delta_{y_1} + \delta_{y_2})$. Monge-Kantorovich straight solution (left) versus Gilbert's branching one (right).

Optimal network, 0<alpha<1, Dirac to discrete 1D Lebesgue (Bernot)



The Gilbert discrete energy (1967)

$$G = \sum_{e \in E(G)} \varphi(e) \mathcal{H}^1|_e \mathbf{e} \tag{2.1}$$

where $\mathbf{e}$ denotes the unit vector in the direction of e and $\mathcal{H}^1$ the Hausdorff one-dimensional measure. We say that G irrigates (μ^+, μ^-) if its distributional derivative ∂G satisfies

$$\partial G = \mu^- - \mu^+. \tag{2.2}$$

The Gilbert energy of G is defined by

$$M^\alpha(G) = \sum_{e \in E(G)} \varphi(e)^\alpha \mathcal{H}^1(e). \tag{2.3}$$

We call the problem of minimizing $M^\alpha(G)$ among all finite graphs irrigating (μ^+, μ^-) the Gilbert-Steiner problem. The Monge-Kantorovich model corresponds to the limit case $\alpha = 1$ and the Steiner problem to $\alpha = 0$. The

Defining infinite irrigation graphs irrigating measures: Traffic plans

Let K be the set of 1-Lipschitz maps $\gamma : \mathbb{R}^+ \to X$.

metric d of uniform convergence on compact sets.

Let $\gamma \in K$. *We define its stopping time as*

$$T(\gamma) := \inf\{t : \gamma \textit{ constant on } [t, \infty[\}$$

Let us denote by $\mathcal{B}$ the Borel σ-algebra on K.

We define a traffic plan $\boldsymbol{P}$ *as a positive measure on* $(K, \mathcal{B})$

such that $\int_K T(\gamma) d\boldsymbol{P}(\gamma) < \infty$.

Traffic plans as measures on the space of Lipschitz paths

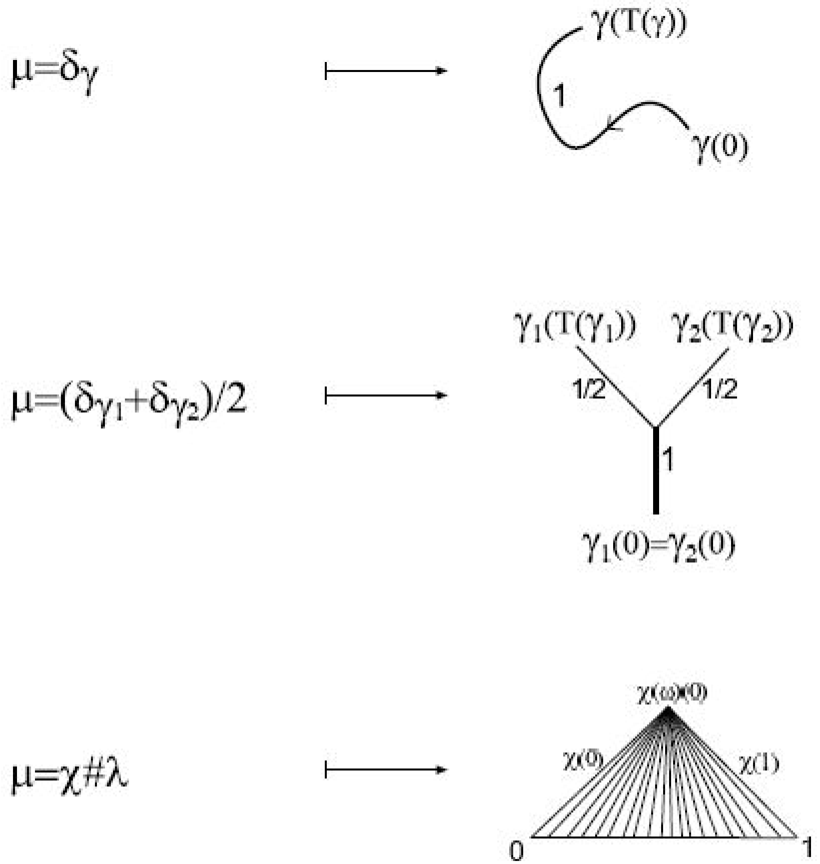


Fig. 2.2. Three traffic plans and their associated embedding : a Dirac measure on γ, a tree with one bifurcation, a spread tree irrigating Lebesgue's measure on the segment $[0,1]\times\{0\}$ of the plane. In the bottom example, to $\omega\in[0,1]$ corresponds $\chi(\omega)\in K$, the straight path from $(1/2,1)$ to $(\omega,0)$.

Other definition: parametric traffic plans as sets of fibers

call parameterized traffic plan *a measurable map* $\chi : \Omega \times \mathbb{R}^+ \to X$ *such that* $t \mapsto \chi(\omega, t)$ *is* 1*-Lipschitz for all* $\omega \in \Omega$ *and* $\int_\Omega T_\chi(\omega) < +\infty$. *Without risk of ambiguity we shall call* fiber *both a path* $\chi(\omega, \cdot)$ *and the range in* $\mathbb{R}^N$ *of* $\chi(\omega, \cdot)$. *Denote by* $|\chi| := |\Omega|$ *the total mass transported by* χ *and by* $\mathbf{P}_\chi$ *the law of* $\omega \mapsto \chi(\omega) \in K$ *defined by* $\mathbf{P}_\chi(E) = |\chi^{-1}(E)|$ *for every Borel set* $E \subset K$. *Then* $\mathbf{P}_\chi$ *is a traffic plan.* *Ve*

From traffic plans to parametric traffic plans

According to Skorokhod theorem we can parameterize any traffic plan $\boldsymbol{P}$ by a measurable function $\chi : \Omega = [0, |\Omega|] \to K$ such that $\boldsymbol{P} = \chi_\sharp \lambda$, where λ is the Lebesgue measure on $[0, |\Omega|]$. We shall set $\chi(\omega, t) := \chi(\omega)(t)$ and consider it as a function of the variable pair (ω, t).

stopping time

$$T_\chi(\omega) := \inf\{t : \chi(\omega) \text{ is constant on } [t, \infty)\}.$$

Handling convergence both ways : weak convergence of measures, or uniform convergence of all paths

Convergence

Definition 3.16. *Let $\mathbf{P}_n$ be a sequence of traffic plans. We shall say that $\mathbf{P}_n$ converges to a traffic plan $\mathbf{P}$ if one of the equivalent relations is satisfied:*

$$\mathbf{P}_n \rightharpoonup \mathbf{P},$$

$$\chi_n(\omega) \to \chi(\omega) \text{ in } K \text{ for almost all } \omega \in \Omega,$$

for some common parameterization of $\mathbf{P}_n$ and $\mathbf{P}$.

Irrigated, irrigating measure, transference plan

Irrigated measures and transference plan in the parametric setting.

$\pi_\chi(E) = |\{\omega : (\chi(\omega,0), \chi(\omega, T_\chi(\omega))) \in E| = |(\pi_0, \pi_\infty)^{-1}(E)|$, and in particular

$$\pi_\chi(A \times B) := |\{\omega : \chi(\omega, 0) \in A,\ \chi(\omega, T_\chi(\omega)) \in B\}|.$$

Informally, $\pi_\chi(A \times B)$ represents the amount of mass transported from A to B through the traffic plan χ. In the same way we can compute the irrigating and irrigated measures of χ by

$$\mu^+(\chi)(A) := |\{\omega : \chi(\omega, 0) \in A\}|,$$

$$\mu^-(\chi)(A) := |\{\omega : \chi(\omega, T_\chi(\omega)) \in A\}|,$$

respectively, where A is any Borel subset of $\mathbb{R}^N$. Observe that $\mu^+(\chi)(A) = \pi_\chi(A \times X)$ and $\mu^-(\chi)(B) = \pi_\chi(X \times B)$. In short, $\mu^- = \pi_{0\sharp}\lambda$ and $\mu^+ = \pi_{\infty\sharp}\lambda$.

The Gilbert energy for traffic plans

the multiplicity of χ *at* x

$$|x|_\chi = \ \boldsymbol{P}(\{\gamma : \exists t, \gamma(t) = x\}) = |x|_{\boldsymbol{P}}.$$

Let $\alpha \in [0,1]$. *We call (Gilbert) energy of a traffic plan* $\boldsymbol{P}$

parameterized by χ *the functional*

$$\mathcal{E}^\alpha(\boldsymbol{P}) = \int_\Omega \int_{\mathbb{R}+} |\chi(\omega,t)|_\chi^{\alpha-1} |\dot{\chi}(\omega,t)| dt d\omega.$$

The energy of a traffic plan can also be written

$$\mathcal{E}^\alpha(\boldsymbol{P}) = \int_K \int_{\mathbb{R}+} |\gamma(t)|_{\boldsymbol{P}}^{\alpha-1} |\dot{\gamma}(t)| dt d\boldsymbol{P}(\gamma).$$

Main semicontinuity lemma (Maddalena-Solimini)

Let χ_n *be a sequence of traffic plans*

converging to χ. *Assume that* $\int_\Omega T(\chi_n(\omega))d\omega \leq C$ *for some* C. *Then, for almost all* ω,

$$\limsup |\chi_n(\omega, t)|_{\chi_n} \leq |\chi(\omega, t)|_\chi.$$

Back to a geometric energy ; lower semicontinuity of energy

Let $\boldsymbol{P}$ be a traffic plan with mass one. Then,

$$\mathcal{E}^\alpha(\boldsymbol{P}) \geq \int_K L(\gamma) d\boldsymbol{P}(\gamma).$$

$$\mathcal{E}^\alpha(\boldsymbol{P}) = \int_\Omega \int_0^\infty |\chi(\omega,t)|_\chi^{\alpha-1} |\dot{\chi}(\omega,t)| \, dt \, d\omega = \int_{\mathbb{R}^N} |x|_\chi^\alpha \, d\mathcal{H}^1(x).$$

for loop-free traffic plans, where $|x|_\chi$ denotes the measure of the set of fibers passing by x.

If $(\boldsymbol{P}_n)_n$ is a sequence in TP_C of traffic plans with mass one such that $\boldsymbol{P}_n \rightharpoonup \boldsymbol{P}$, then

$$\mathcal{E}^\alpha(\boldsymbol{P}) \leq \liminf_n \mathcal{E}^\alpha(\boldsymbol{P}_n).$$

Traffic plans

Theorem *Given a bounded sequence of traffic plans* $(\boldsymbol{P}_n)_n$ *in* TP_C *it is possible to extract a subsequence converging to some* $\boldsymbol{P}$*. In addition,* $\mu^+(\boldsymbol{P}_n) \rightharpoonup \mu^+(\boldsymbol{P})$, $\mu^-(\boldsymbol{P}_n) \rightharpoonup \mu^-(\boldsymbol{P})$ *and* $\pi_{\boldsymbol{P}_n} \rightharpoonup \pi_{\boldsymbol{P}}$.

Definition of optimality for the irrigation and for the « who goes where » problem

Definition *A traffic plan* χ *is said to be* optimal for the irrigation problem, *respectively* optimal for the who goes where problem *if it is of minimal cost in* $\mathrm{TP}(\mu^+(\chi), \mu^-(\chi))$, *respectively in* $\mathrm{TP}(\pi_\chi)$.

All measures can be irrigated for $\alpha > 1 - \frac{1}{N}$

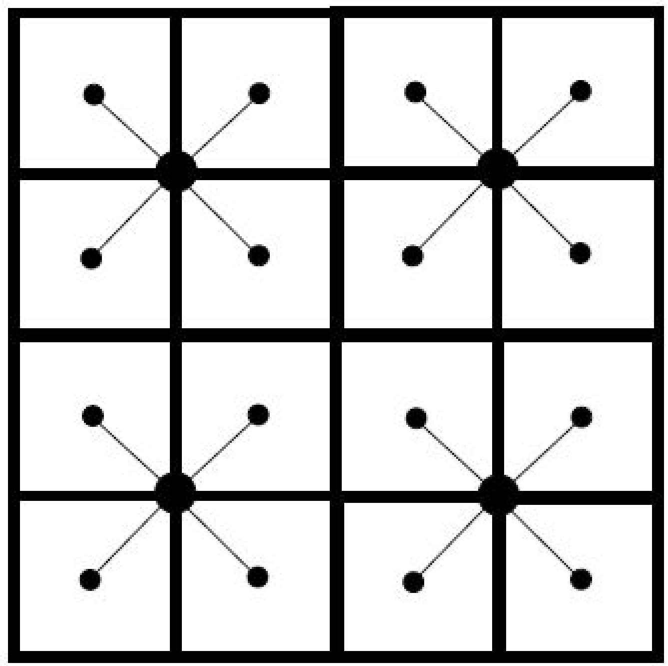


Fig. 6.1. To transport μ_i to μ_{i+1}, all the mass at the center of a cube with edge length $\frac{L}{2^{i-1}}$ is transported to the centers of its sub-cubes with edge length $\frac{L}{2^i}$.

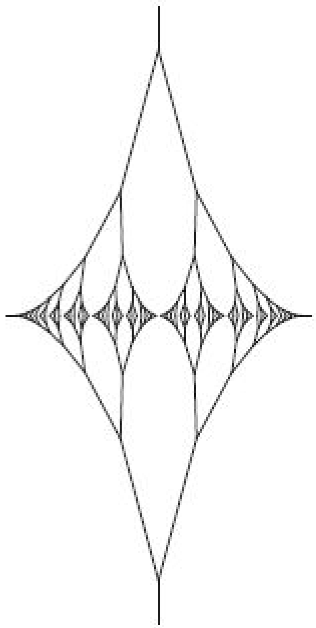
Traffic plans and distances between measures

for $\alpha > 1 - \frac{1}{N}$ where N is the dimension of the ambient space, the optimal cost to transport μ^+ to μ^- is finite. More precisely, if μ^+ and μ^- are two nonnegative measures on a domain X with the same total mass M and $\alpha > 1 - 1/N$, set

$$E^\alpha(\mu^+,\mu^-) := \min_{\chi \in \mathrm{TP}(\mu^+,\mu^-)} E^\alpha(\chi). \tag{6.1}$$

$$W_1(\mu^+,\mu^-) \leq E^\alpha(\mu^+,\mu^-) \leq c W_1(\mu^+,\mu^-)^\beta$$

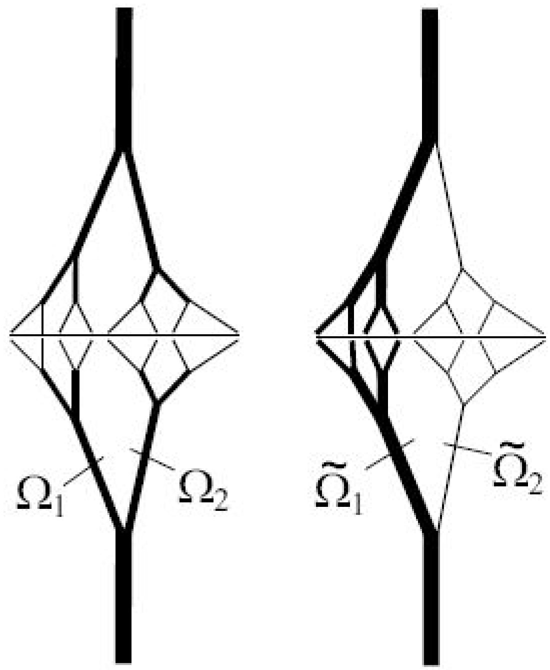
The tree-like structure: single path property



This traffic plan is obtained through the concatenation of a traffic plan transporting a Dirac mass to the Lebesgue measure on a segment and a traffic plan

such a structure is not optimal.

Technique to prove the « single path property »



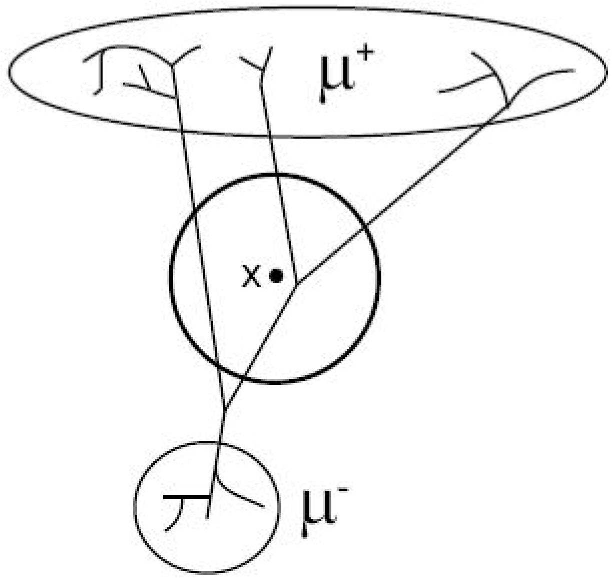
Suppose that in an optimal traffic plan χ two sets of fibers Ω_1 and Ω_2 go from x to y. Some part of the mass of Ω_2 can be conveyed through the fibers of Ω_1, or conversely, without changing the irrigated measures or the transference plan of χ . Thus the modified traffic plan displayed on the right has an energy larger or equal to the energy of χ.

Main structure result

We say that a traffic plan χ is normal *if it is loop-free, strictly single path and if for any fiber ω, $|\chi(\omega, t)|_\chi$ is bounded away from zero on any compact set contained in $]0, T(\omega)[$.*

One can transform a traffic plan into a normal traffic plan by modifying its domain on a set with zero measure.

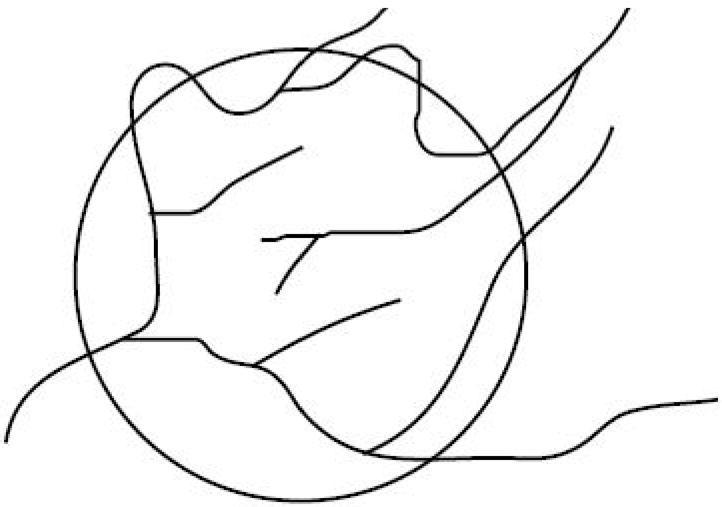
Main « interior regularity » result (works also if one of the irrigated or irrigating measure is finite atomic)



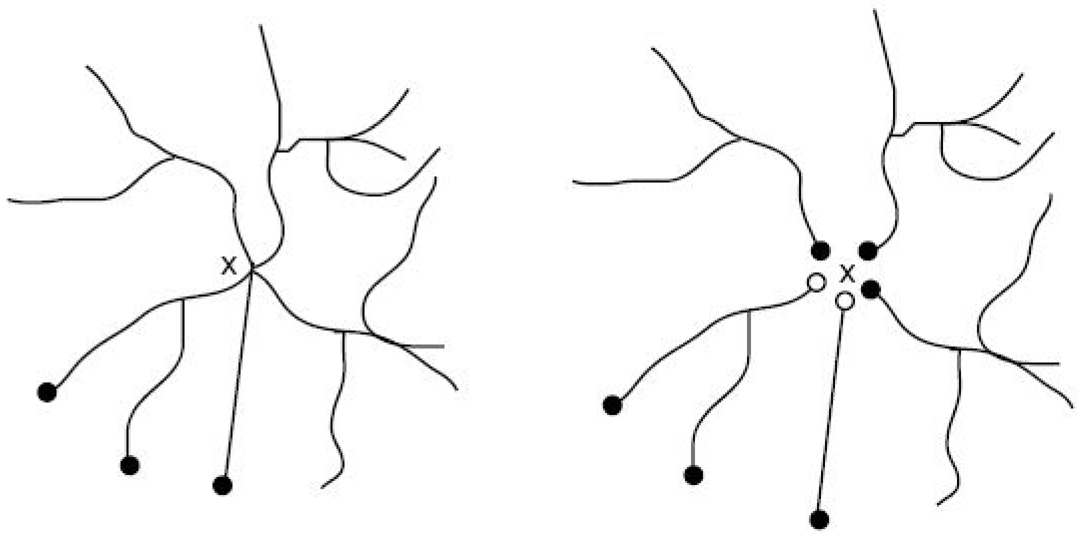
Theorem *Let* $\alpha \in (1 - \frac{1}{N}, 1)$ *and let* χ *be an optimal traffic plan in* $\mathrm{TP}(\mu^+, \mu^-)$. *Assume that the supports of* μ^+ *and* μ^- *are at positive distance. In any closed ball* $B(x, r)$ *not meeting the supports of* μ^+ *and* μ^-, *the traffic plan has the structure of a finite graph*

Excision technique

Interior and boundary regularity



Regularity tecnique : Cutting a traffic plan into optimal subtrees



Main regularity lemma

Lemma

Let $\alpha \in (1 - \frac{1}{N}, 1)$ *and* $(\chi_n)_{n=1}^{\infty}$ *a sequence of disjoint traffic plans such that* $\sum_n |\chi_n| < \infty$ *and*

$$\mathrm{dist}(\overline{\cup_n \mathrm{supp}(\mu^+(\chi_n))}, \overline{\cup_n \mathrm{supp}(\mu^-(\chi_n))}) > 0.$$

Then there is some n *or some pair* (n, m) *such that* χ_n *or* $\chi_n \cup \chi_m$ *is not optimal. Therefore,* $\cup_n \chi_n$ *is not optimal.*

Regularity technique: short cuts

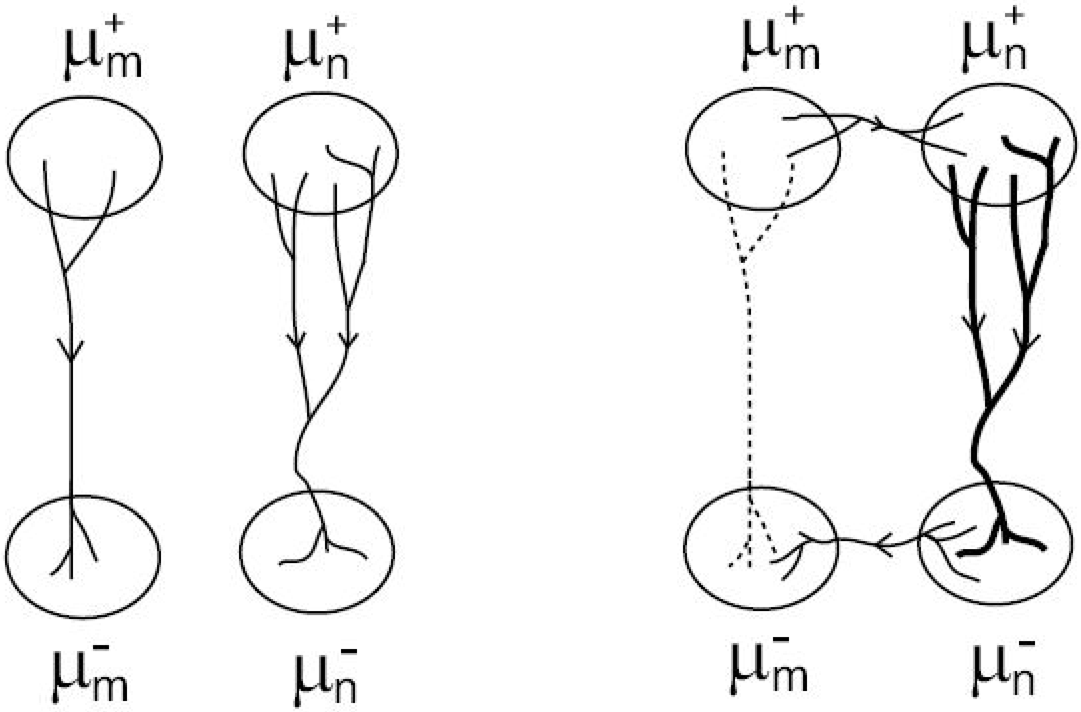


Fig. 8.3. Illustrates the proof of Lemma 8.9. Under the assumptions of Lemma 8.9, there is a shortcut χ_m through χ_n that has a better cost than the union of χ_m with χ_n that is represented on the left-hand side. The traffic plan represented on the right-hand side is the shortcut of χ_m through χ_n, i.e. all the mass that was transported by χ_m is transported through χ_n.

Boundary regularity

Theorem 8.20. **(bounded branching property)** *Let* $\alpha \in (0, 1]$. *At every point* x *of the support of an optimal traffic plan* χ *in* $\mathbb{R}^N$, *the number of branches at* x *is less than a constant* $\mathcal{N}(\alpha, N)$ *depending only on* N *and* α.

Other recent results for Lebesgue irrigated measures:

- One can associate a landscape function to any optimal traffic plan and the traffic plan is the steepest descent network of this landscape. The landscape is hölder (Santambrogio)
- More on regularity: curvature is a bounded measure on each path. Existence of a finite tangent cone at each point. (M. , Santambrogio)