Demagnetizing Factors of the General Ellipsoid

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Charts and tables of the demagnetizing factors of prolate and oblate spheroids are readily available; however, demagnetizing factors of ellipsoids of three different axes are incompletely tabulated and laborious to calculate. This article presents charts and tables which make possible easy determination of the demagnetizing factor for any principal axis of an ellipsoid of any shape. Formulas for the demagnetizing factors of the general ellipsoid are included together with supplementary formulas which cover a large number of special cases.

I. INTRODUCTION

If a ferromagnetic body of irregular shape is brought into a uniform applied field H_0, the magnetizing force H inside the material differs in magnitude from the applied field and varies in direction throughout the body in an unknown manner. This is usually a great disadvantage in research on magnetic materials, because such research usually requires studies of the relation between the magnetization J and the magnetizing force. However, for homogeneous bodies whose surface is of the second degree,[1] H and J (after suitable magnetic treatment) are uniform throughout, though they are not necessarily the same in direction as H_0. The ellipsoid has the only surface of the second degree that is finite, so materials in this form are frequently used in precise investigations of magnetic materials.

Inside any ellipsoid the component of H along any principal axis i is determined by the relation

$$H_i=(H_0)_i-N_iJ_i, \quad i=x,\, y,\, z, \tag{1.1}$$

where N_i is a constant called the demagnetizing factor; it is determined by the ratios of the axes. Hence, if the magnetization components and the lengths of the axes of the ellipsoid are known, H inside the body may be found.[2]

The determination of N is rather involved, except for the special cases of ellipsoids of revolution. Therefore, it is believed that calculation of N for various axial ratios of the *general* ellipsoid will serve a useful purpose. Naturally, it is impossible to include values of the demagnetizing factor for all axial ratios, so graphs of N have been made that will allow interpolation for those ratios not explicitly included. The following section gives the formulas from which the demagnetizing factors have been calculated, together with comments on the method by which they were obtained.

II. FORMULAS FOR THE DEMAGNETIZING FACTORS OF THE GENERAL ELLIPSOID

The equations given below are for the demagnetizing factors along the three axes of the general ellipsoid under the assumption that

$$a\geq b\geq c\geq 0,$$

where a, b, and c are the ellipsoid semi-axes. The demagnetizing factors (*corresponding to the semi-axes a, b, and c*) will be labeled L, M, and N. Then, as indicated in Eq. (1.1), to find the component of the magnetizing force along any principal axis, the component of the applied field $(H_0)_i$ along that axis must be considered together with the appropriate magnetization component J_i and demagnetizing factor. The formulas for L, M, and N are

$$L/4\pi=\frac{\cos\varphi\cos\vartheta}{\sin^3\vartheta\sin^2\alpha}[F(k,\vartheta)-E(k,\vartheta)], \tag{2.1}$$

$$M/4\pi=\frac{\cos\varphi\cos\vartheta}{\sin^3\vartheta\sin^2\alpha\cos^2\alpha}\left[E(k,\vartheta)-\cos^2\alpha F(k,\vartheta)-\frac{\sin^2\alpha\sin\vartheta\cos\vartheta}{\cos\varphi}\right], \tag{2.2}$$

[1] J. C. Maxwell, *Electricity and Magnetism* (The Clarendon Press, Oxford, 1904), third edition, Vol. 2, pp. 66–70.

[2] It is possible to find H at the center of a cylindrical bar if a kind of demagnetizing factor different from that for the ellipsoid is defined. The use of these so called *ballistic* demagnetizing factors is discussed by R. M. Bozorth and D. M. Chapin, J. App. Phys. **12**, 320 (1942).

$$N/4\pi=\frac{\cos\varphi\cos\vartheta}{\sin^3\vartheta\cos^2\alpha}\left[\frac{\sin\vartheta\cos\varphi}{\cos\vartheta}-E(k,\vartheta)\right], \quad (2.3)$$

where

$$\cos\vartheta=c/a, \quad (0\leq\vartheta\leq\pi/2), \quad (2.4)$$

$$\cos\varphi=b/a, \quad (0\leq\varphi\leq\pi/2), \quad (2.5)$$

$$\sin\alpha=\left[\frac{1-(b/a)^2}{1-(c/a)^2}\right]^{\frac{1}{2}}=\frac{\sin\varphi}{\sin\vartheta}=k, \quad (0\leq\alpha\leq\pi/2), \quad (2.6)$$

and $F(k,\vartheta)$ and $E(k,\vartheta)$ are elliptic integrals of the first and second kinds:[3] k is the modulus, and ϑ is the amplitude of these integrals. Equations (2.1)–(2.3) are given in forms that are convenient for calculation. Methods of obtaining general expressions for L, M, and N of the general ellipsoid are available in several places.[1,4] Maxwell has shown how the method of Poisson[5] for getting the potential of a uniformly magnetized body yields the demagnetizing factors for the general ellipsoid. The equations for the factors are not given in directly usable form however. Kellogg[4] briefly indicates the method of obtaining L, M, and N and gives expressions similar to Eqs. (2.1)–(2.3). The most complete discussion of the solution for the demagnetizing factors of the general ellipsoid by use of the Poisson method is given by Poritsky.[4] Stratton[4] obtains expressions for L, M, and N in the same form as found in Maxwell.

A useful relationship that exists between the demagnetizing factors of an ellipsoid is

$$L+M+N=4\pi. \quad (2.7)$$

This may be proved by using the following expression for the gravitational potential of an ellipsoid of uniform density ρ:

$$V=(1/2)\rho(c-Lx^2-My^2-Nz^2), \quad (2.8)$$

where the coefficients L, M, and N represent the usual demagnetizing factors as in the Poisson method. Poisson's operation

$$\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=-4\pi\rho, \quad (2.9)$$

gives Eq. (2.7).

If the ellipsoid is an ellipsoid of rotation, i.e., a prolate or oblate spheroid, Eqs. (2.1)–(2.3) reduce to more simple forms; the formulas for these and other limiting cases are given below:

(*a*) If $b=c$ (prolate spheroid),

$$L/4\pi=\frac{1}{m^2-1}\left[\frac{m}{2(m^2-1)^{\frac{1}{2}}}\times\ln\left(\frac{m+(m^2-1)^{\frac{1}{2}}}{m-(m^2-1)^{\frac{1}{2}}}\right)-1\right], \quad (2.10)$$

where $m=a/c$.

$$M/4\pi=N/4\pi=\frac{m}{2(m^2-1)}\left[m-\frac{1}{2(m^2-1)^{\frac{1}{2}}}\times\ln\left(\frac{m+(m^2-1)^{\frac{1}{2}}}{m-(m^2-1)^{\frac{1}{2}}}\right)\right]. \quad (2.11)$$

If $m\gg 1$ (very slender prolate spheroid),

$$L/4\pi\simeq(1/m^2)(\ln 2m-1), \quad (2.12)$$

$$M/4\pi=N/4\pi\simeq\tfrac{1}{2}[1-(\ln 2m-1)/m^2]. \quad (2.13)$$

(*b*) If $a\gg b\geq c$ (very slender ellipsoid),

$$L/4\pi=(bc/a^2)(\ln 4a/(b+c)-1), \quad (2.14)$$

$$M/4\pi=c/(b+c)-\tfrac{1}{2}(bc/a^2)\ln 4a/(b+c)+bc(3b+c)/4a^2(b+c), \quad (2.15)$$

$$N/4\pi=b/(b+c)-\tfrac{1}{2}(bc/a^2)\ln[4a/(b+c)]+bc(b+3c)/4a^2(b+c). \quad (2.16)$$

(*c*) If $a=\infty$, $b\geq c$ (elliptic cylinder),

$$M/4\pi=c/(b+c), \quad (2.17)$$

$$N/4\pi=b/(b+c). \quad (2.18)$$

(*d*) If $a=b$ (oblate spheroid),

$$L/4\pi=M/4\pi=\frac{1}{2(m^2-1)}\{m^2(m^2-1)^{-\frac{1}{2}}\times\arcsin[(m^2-1)^{\frac{1}{2}}/m]-1\}, \quad (2.19)$$

[3] See B. O. Pierce, *A Short Table of Integrals*, third edition (Ginn and Company, New York, 1929), p. 66 *et seq.*

[4] O. Kellogg, *Foundations of Potential Theory* (Verlagsbuchhandlung Julius Springer, Berlin, 1929), pp. 192–194, also p. 197, Exercise 7. H. Poritsky, *Magnetization of an Ellipsoid* (General Electric Company, Schenectady, New York). J. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 211–213.

[5] Poisson showed that if V is the gravitational potential of a body of uniform density ρ, the magnetic potential Ω_x of the same body is $-\partial V/\partial x$, if it is uniformly magnetized in the x-direction so that $J=\rho$.

TABLE I. Demagnetizing factors of the general ellipsoid calculated from formulas of section II.

c/a	b/a	$L/4\pi$	$M/4\pi$	$N/4\pi$
017452	.99620		.013471	
	.98164		.013719	
	.95632		.014165	
	.91357		.014989	
	.40709		.037634	
	.29285		.053078	
	.19158		.080810	
	.071895		.19374	
	.055162		.23901	
	.039018		.30809	
.034900	.89892			.94467
	.79891			.94104
	.69511			.93617
	.60246			.93029
	.50091			.92126
	.40798		.071847	.90905
	.29427		.099697	.88398
	.19386		.14720	.83986
	.093838		.26746	.72453
	.077961		.30596	
	.062880		.35411	
.087156	1.00000	.06154	.06154	.87692
	.98492	.06108	.06281	.87611
	.94013	.06051	.06620	.87329
	.86727	.05908	.07222	.86870
	.76809	.05683	.08307	.86010
	.64625	.05347	.10028	.84626
	.50565	.04846	.12912	.82242
	.35171	.04087	.18185	.77728
	.19376	.02898	.29661	.67441
	.08716	.01641	.49180	.49180
.17365	1.00000	.11138	.11138	.77724
	.98527	.11085	.11327	.77588
	.94157	.10936	.11903	.77161
	.87037	.10665	.12955	.76380
	.77413	.10245	.14651	.75104
	.65641	.09627	.17296	.73077
	.52213	.08734	.21496	.69770
	.37895	.07455	.28191	.64354
	.24371	.05756	.38888	.55356
	.17365	.04582	.47709	.47709
.25882	1.00000	.15207	.15207	.69587
	.98583	.15136	.15440	.69424
	.94386	.14927	.16154	.68919
	.87562	.14553	.17452	.67995
	.78392	.13985	.19464	.66551
	.67269	.13167	.22498	.64335
	.54793	.12029	.26973	.60998
	.41970	.10524	.33356	.56120
	.30844	.08817	.41326	.49857
	.25882	.07892	.46054	.46054
.34202	1.00000	.18555	.18555	.62889
	.98660	.18477	.18805	.62718
	.94695	.18221	.19597	.62181
	.88274	.17755	.21045	.61200
	.79697	.17108	.23128	.59764
	.69414	.16176	.26200	.57624
	.58115	.14940	.30440	.54620
	.46933	.13428	.35885	.50687
	.37895	.11924	.41586	.46490
	.34202	.11218	.44391	.44391
.50000	1.00000	.23640	.23640	.52720
	.98863	.23555	.23885	.52560
	.95513	.23256	.24672	.52072
	.90139	.22760	.26012	.51228
	.83073	.22043	.27966	.49991
	.74825	.21105	.30560	.48335
	.66144	.19970	.33745	.46285
	.58115	.18772	.37190	.44037
	.52213	.17762	.40115	.42123
	.50000	.17356	.41322	.41322
.64279	1.00000	.27187	.27187	.45627
	.99111	.27097	.27403	.45499
	.96507	.26824	.28058	.45118
	.92374	.26370	.29152	.44478
	.87037	.25760	.30650	.43590
	.80971	.24999	.32522	.42479
	.74825	.24148	.34617	.41235
	.69414	.23330	.36647	.40023
	.65641	.22717	.38169	.39114
	.64279	.22486	.38757	.38757
.76604	1.00000	.29670	.29670	.40660
	.99375	.29598	.29830	.40572
	.97553	.29388	.30303	.40309
	.94696	.29048	.31062	.39890
	.91065	.28595	.32086	.39319
	.87037	.28064	.33285	.38651
	.83073	.27513	.34529	.37958
	.79697	.27018	.35644	.37338
	.77413	.26668	.36464	.36868
	.76604	.26542	.36729	.36729
.86603	1.00000	.31380	.31380	.37240
	.99622	.31352	.31462	.37186
	.98527	.31225	.31740	.37035
	.96824	.30983	.32224	.36789
	.94696	.30707	.32803	.36490
	.92374	.30396	.33473	.36131
	.90139	.30093	.34121	.35786
	.88274	.29825	.34692	.35484
	.87037	.29651	.35059	.35290
	.86603	.29584	.35208	.35208
.93969	1.00000	.32497	.32497	.35006
	.99823	.32474	.32548	.34978
	.99314	.32409	.32673	.34918
	.98527	.32306	.32889	.34805
	.97553	.32177	.33153	.34669
	.96507	.32038	.33435	.34527
	.95513	.31902	.33707	.34391
	.94695	.31790	.33947	.34260
	.94157	.31716	.34105	.34183
	.93969	.31690	.34155	.34155
.98481	1.00000	.33119	.33119	.33762
	.99955	.33120	.33143	.33737
	.99824	.33100	.33180	.33720
	.99622	.33078	.33230	.33692
	.99375	.33044	.33295	.33661
	.99111	.33010	.33364	.33628
	.98863	.32977	.33431	.33590
	.98660	.32950	.33487	.33563
	.98527	.32931	.33512	.33558
	.98481	.32925	.33537	.33537

TABLE II. Demagnetizing factors of the general ellipsoid for b/a equal to integral multiples of 0.1.

b/a	c/a	$L/4\pi$	$M/4\pi$	$N/4\pi$
.1000	.0872	.0181	.4624	.5194
	.0349		.2545	.7359
	.0175		.1530	
.2000	.1736	.0505	.4400	.5082
	.0872	.0294	.2890	.6828
	.0349		.1431	.8439
	.0175		.0785	
.3000	.2588	.0866	.4204	.4927
	.1736	.0653	.3369	.5990
	.0872	.0375	.2095	.7580
	.0349		.0977	.8854
	.0175		.0518	
.4000	.3420	.1230	.4014	.4760
	.2588	.1031	.3459	.5522
	.1736	.0768	.2692	.6532
	.0872	.0435	.1613	.7940
	.0349		.0727	.9083
	.0175		.0377	
.5000	.5000	.1735	.4132	.4132
	.3420	.1387	.3428	.5186
	.2588	.1149	.2920	.5930
	.1736	.0855	.2235	.6909
	.0872	.0481	.1308	.8217
	.0349			.9202
.6000	.5000	.1906	.3638	.4457
	.3420	.1516	.2962	.5521
	.2588	.1253	.2501	.6253
	.1736	.0926	.1895	.7173
	.0872	.0520	.1080	.8398
	.0349			.9301
.7000	.6428	.2341	.3641	.4018
	.5000	.2050	.3226	.4724
	.3420	.1625	.2600	.5776
	.2588	.1338	.2168	.6494
	.1736	.0987	.1622	.7385
	.0872	.0551	.0925	.8528
	.0349			.9364
.8000	.7660	.2706	.3555	.3739
	.6428	.2487	.3284	.4229
	.5000	.2172	.2887	.4941
	.3420	.1713	.2303	.5984
	.2588	.1408	.1908	.6684
	.1736	.1037	.1420	.7546
	.0872	.0576	.0798	.8630
	.0349			.9412
.9000	.8660	.3008	.3415	.3577
	.7660	.2845	.3240	.3915
	.6428	.2611	.2981	.4408
	.5000	.2275	.2605	.5121
	.3420	.1788	.2063	.6148
	.2588	.1470	.1697	.6832
	.1736	.1077	.1250	.7670
	.0872	.0597	.0691	.8710
	.0349			.9445
1.0000	1.0000	.3333	.3333	.3333
	.9848	.3314	.3312	.3374
	.9397	.3249	.3250	.3500
	.8660	.3138	.3138	.3724
	.7660	.2965	.2967	.4066
	.6428	.2718	.2718	.4561
	.5000	.2364	.2364	.5270
	.3420	.1856	.1858	.6288
	.2588	.1520	.1520	.6960
	.1736	.1114	.1117	.7775
	.0872	.0615	.0615	.8769

$$N/4\pi = m^2/(m^2-1)\{1-1/(m^2-1)^{\frac{1}{2}} \times \arc\sin[(m^2-1)^{\frac{1}{2}}/m]\}. \quad (2.20)$$

If $m \gg 1$ (very flat oblate spheroid),

$$L/4\pi = M/4\pi \simeq (\pi/4m)(1-4/\pi m), \quad (2.21)$$

$$N/4\pi \simeq 1-\pi/2m+2/m^2. \quad (2.22)$$

(*e*) If $a \geq b \gg c$ (very flat ellipsoid),

$$L/4\pi = \frac{c}{a}(1-e^2)^{\frac{1}{2}}\frac{K-E}{e^2}, \quad (2.23)$$

$$M/4\pi = \frac{c}{a}\frac{E-(1-e^2)K}{e^2(1-e^2)^{\frac{1}{2}}}, \quad (2.24)$$

$$N/4\pi = 1-\frac{cE}{a(1-e^2)^{\frac{1}{2}}}, \quad (2.25)$$

where K and E are complete elliptic integrals[3] whose argument is

$$e = (1-b^2/a^2)^{\frac{1}{2}}. \quad (2.26)$$

III. CONSTRUCTION AND USE OF TABLES AND GRAPHS

Construction

The values of the demagnetizing factors for the general ellipsoid given in Table I are determined by the quantities ϑ and k ($=\sin\alpha$) used in Eqs. (2.1)–(2.3). These were chosen so that ϑ and α covered the region from 0° to 70° in 10° steps, and from 70° to 90°, ϑ varied in 5° and α in 10° steps. Some gaps in the values of the demagnetizing factors were revealed when the final graphs were plotted, as explained below; hence, additional values were subsequently computed to make the graphs complete. The axial ratios of the ellipsoids were then determined by Eqs. (2.4)–(2.6). $L/4\pi$ and $N/4\pi$ were determined directly from Eqs.

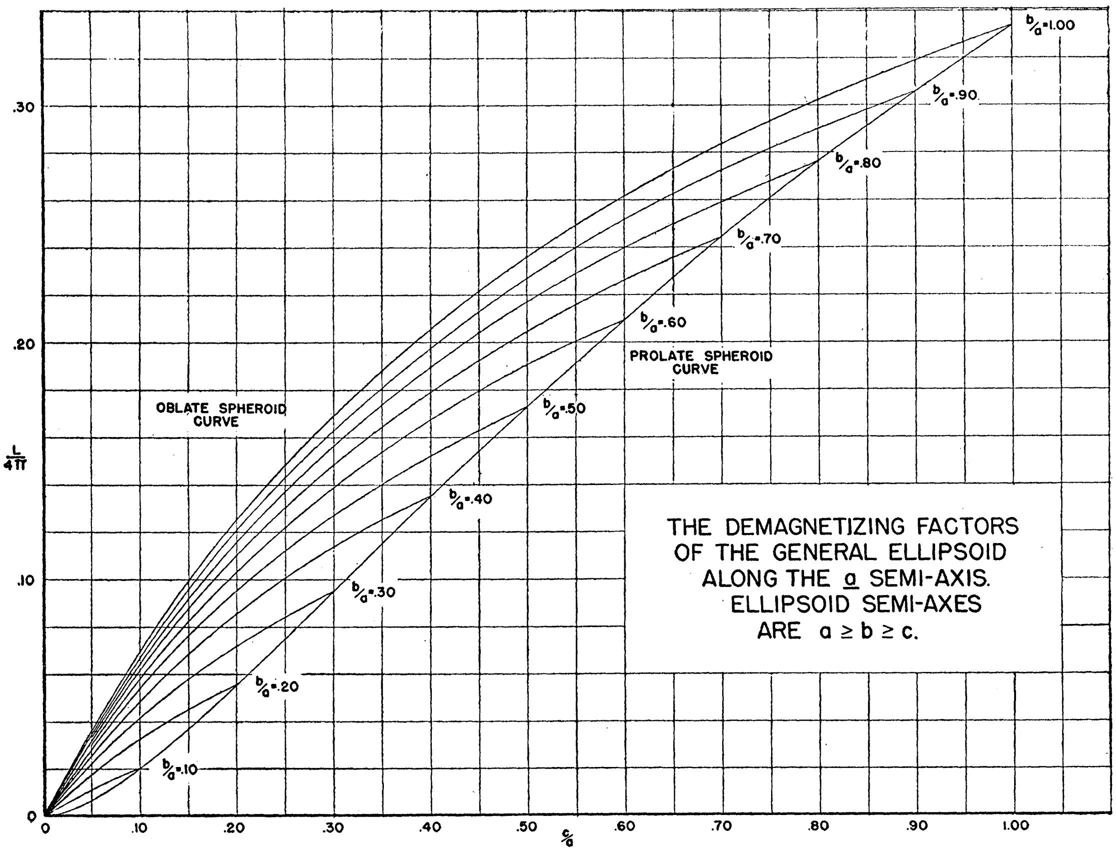

FIG. 1.

(2.1) and (2.3), and $M/4\pi$ was then determined in most cases by means of Eq. (2.7).

$L/4\pi$ and $N/4\pi$ in Table I are accurate to at least the fourth decimal place. $M/4\pi$ is somewhat less accurate than the other factors because it involves a difference.

The values of c/a and b/a obtained from ϑ and α are such that interpolation for the demagnetizing factors of other ellipsoids is not easy. To reduce this difficulty, each factor was separately plotted on a large scale against b/a, with c/a as a parameter. This gave a family of curves for each factor, L, M, and N. Each family of curves was then considered by itself. A value of b/a equal to an integral multiple of 0.1 was selected, and a value of the demagnetizing factor in question read from each curve. The constant value of b/a was now taken as a parameter and a new curve plotted with the factors (read from the above family of curves) as ordinates and the values of c/a (the parameter of the above family of curves) as abscissas. This process was repeated for different b/a values until a new family of curves was obtained with the parameter b/a equal to integral multiples of 0.1 from 0.1 to 1. These curves (shown in Figs. 1–3) warrant graphical interpolation to three decimal places. Table II contains the values of $L/4\pi$, $M/4\pi$, and $N/4\pi$ for various c/a (with b/a as a parameter) from which Figs. 1–3 were plotted. These values are accurate to three decimal places and are probably in error several units in the fourth place.

Use of Graphs

To use the graphs, any value of c/a may be chosen within the limitations

$$a \geq b \geq c \geq 0$$

and b/a may be found by interpolation. It is most important to remember that: *L corresponds*

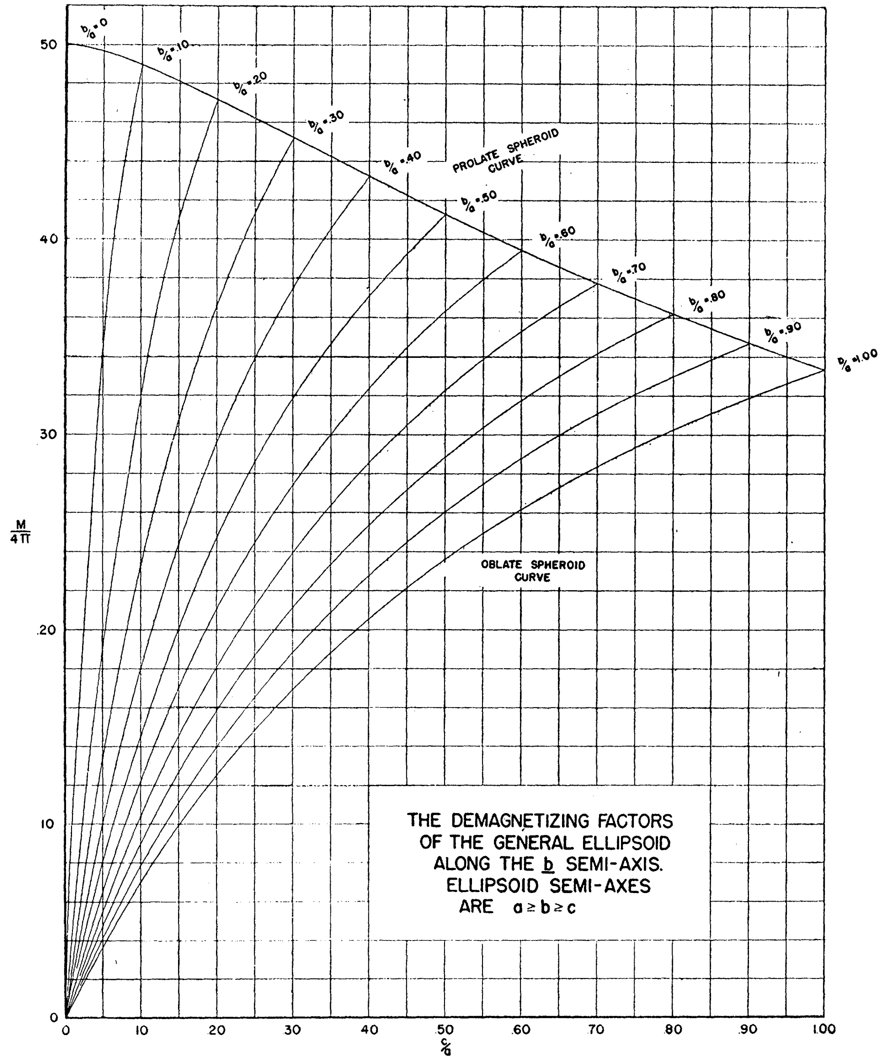

FIG. 2.

to a, the longest semi-axis; M corresponds to b, the intermediate semi-axis; and N corresponds to c, the shortest semi-axis. Two examples are given below illustrating the procedure for use of the graphs. Examples: (1) Suppose the semi-axes of an ellipsoid are 4, 3, and 2 and it is desired to find the demagnetizing factor along the axis of length 3. Then

$$a=4;\quad b=3;\quad \text{and}\quad c=2.$$

The factor is then read from the curves for $M/4\pi$ with axial ratios $c/a=0.500$ and $b/a=0.750$. $M/4\pi=0.306$ when read from the curve. (2) Suppose the axis lengths are 3, 2, and 1 and the factor along length 3 is wanted. Here

$$a=3;\quad b=2;\quad \text{and}\quad c=1.$$

In this case the factor is read from the curves for $L/4\pi$ with axial ratios $c/a=0.333$ and $b/a=0.667$. $L/4\pi=0.156$ when read from the curve.

IV. APPLICATION TO SPECIAL MATERIALS

If the magnetization curve is linear, and if the material either is isotropic or has its principal axes of magnetic anisotropy along the principal axes of the ellipsoid, then

$$J_i=\chi_i H_i=(\mu_i-1)H_i/4\pi, \tag{4.1}$$

where the susceptibility χ_i and the permeability

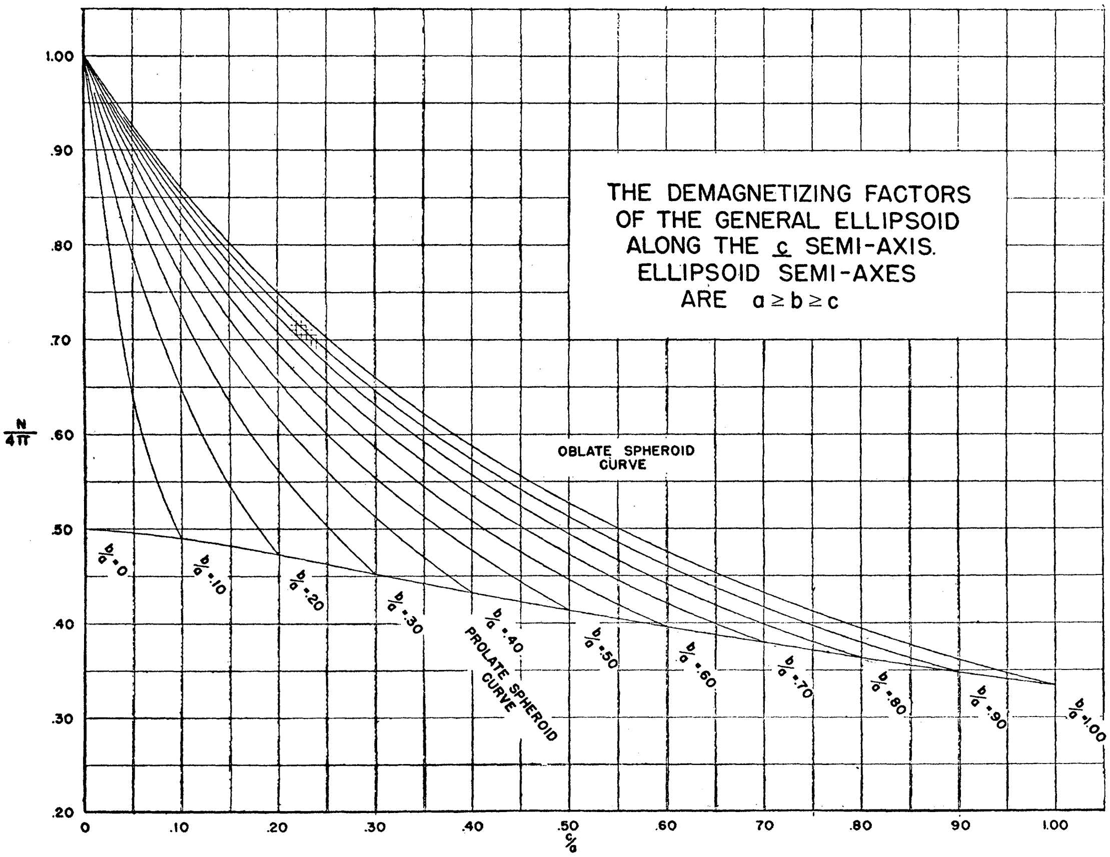

FIG. 3.

μ_i, corresponding to axis i, are constant. (For isotropic materials the three χ_i's are equal.) Therefore

$$H_i=(H_0)_i-N_iJ_i=(H_0)_i-N_i\chi_iH_i, \quad (4.2)$$

so that

$$H_i=(H_0)_i/(1+N_i\chi_i), \quad (4.3)$$

and

$$J_i=\chi_iH_i=\chi_i(H_0)_i/(1+N_i\chi_i) = (H_0)_i/(N_i+1/\chi_i). \quad (4.4)$$

By use of these formulas, the magnitude and direction of the magnetizing force and magnetization may be found if the magnitude and direction of the applied field are given.

V. ACKNOWLEDGMENT

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