Homogenization of ferromagnetic materials

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Abstract

In these notes we describe the homogenzation theory for ferromagnetic materials. A new phenomenon arises, in particular to take into account the geometric constraint that the magnetization is locally saturated.

1 Introduction

Ferromagnetic materials present a spontaneous magnetization. They are usually the main components of permanent magnets and are of everyday use in devices such as hard disks, magnetic tapes, cellular phones, etc. Nowadays, nonhomogeneous ferromagnetic materials are the subject of a growing interest. Actually, such configurations often combine the attributes of the constituent materials, while sometimes, their properties can be strikingly different from the properties the different constituents. To predict the magnetic behaviour of these composite materials is of prime importance for applications [9].

The main objective of this paper is to perform, in the framework of DE GIORGI's notion of $\Gamma-$convergence [4] and ALLAIRE's notion of two-scale convergence [2] (see also the paper by NGUETSENG [10]), a mathematical homogenization study of the GIBBS-LANDAU free energy functional associated to a composite periodic ferromagnetic material, i.e. a ferromagnetic material in which the heterogeneities are periodically distributed inside the ferromagnetic media.

2 The Gibbs-Landau functional

A ferromagnetic material, occupying the domain $\Omega \subset \mathbb{R}^3$ presents a spontaneous magnetization, which is a vectorfield $\boldsymbol{M}: \Omega \to \mathbb{R}^3$. The magnetization is allowed to vary over Ω, and represents a continuum of magnetized pointers of as many compass distributed along the material (see Fig. 1)

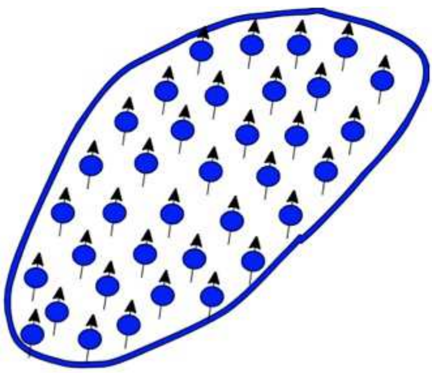

Figure 1. Schematization of the magnetization inside a ferromagnetic material. Here the magnetization vector is constant throughout the sample.

According to LANDAU and LIFSCHITZ micromagnetic theory of ferromagnetic materials (see [3], [7], [8]), the states of a rigid *single-crystal* ferromagnet subject to a given external magnetic field $\boldsymbol{h}_a$, are described by a vector field, the magnetization $\boldsymbol{M}$, verifying the so-called *fundamental constraint*: A ferromagnetic body is always locally saturated, i.e. there exists a positive constant M_S (the *saturation magnetization*) such that $|\boldsymbol{M}(x)| = M_S$ for a.e. $x \in \Omega$. We usually write $\boldsymbol{M} = M_S \boldsymbol{m}$ with

$$\boldsymbol{m}: \Omega \to \mathbb{S}^2$$

where $\mathbb{S}^2$ denotes the unit sphere of $\mathbb{R}^3$.

Eventhough the magnitude of the magnetization vector is constant in space for a given material, in general it is not the case for its direction, and the observable states can be mathematically characterized as local minimizers of the GIBBS-LANDAU free energy functional associated to the single-crystal ferromagnetic particle:

$$\mathcal{G}_{\mathcal{L}}(\boldsymbol{m}) := \underbrace{\int_\Omega a_{\text{ex}} |\nabla \boldsymbol{m}|^2 \,\mathrm{d}\mu}_{=:\mathcal{E}(\boldsymbol{m})} + \underbrace{\int_\Omega \varphi_{\text{an}}(\boldsymbol{m}) \,\mathrm{d}\mu}_{=:\mathcal{A}(\boldsymbol{m})} \underbrace{- \frac{\mu_0}{2} \int_\Omega \boldsymbol{h}_{\text{d}}[M_s \boldsymbol{m}] \cdot M_s \boldsymbol{m} \,\mathrm{d}\mu}_{=:\mathcal{W}(\boldsymbol{m})} \underbrace{- \mu_0 \int_\Omega \boldsymbol{h}_a \cdot M_s \boldsymbol{m} \,\mathrm{d}\mu}_{=:\mathcal{Z}(\boldsymbol{m})} .$$

The first term, $\mathcal{E}(\boldsymbol{m})$, is called *exchange energy*, and penalizes spatial variations of $\boldsymbol{m}$. The factor a_{ex} in the term is a phenomenological positive material constant which summarizes the effect of (usually very) short-range exchange interactions.

The second term, $\mathcal{A}(\boldsymbol{m})$, or the *anisotropy energy*, models the existence of preferred directions for the magnetization (the so-called *easy axes*). The anisotropy energy density φ_{an}: $S^2 \to \mathbb{R}^+$ is assumed to be a globally non-negative even and globally lipschitz continuous function, that vanishes only on a finite set of unit vectors, the easy axes, and is a function that depends on the crystallographic symmetry of the sample.

The third term, $\mathcal{W}(\boldsymbol{m})$, is called the *magnetostatic self-energy*, and is the energy due to the (dipolar) magnetic field, also known in literature as the stray field, $\boldsymbol{h}_{\text{d}}[\boldsymbol{m}]$ generated by $\boldsymbol{m}$. From the mathematical point of view, assuming Ω to be open, bounded and with a Lipschitz boundary, a given magnetization $\boldsymbol{m} \in L^2(\Omega, \mathbb{R}^3)$ generates the stray field $\boldsymbol{h}_{\text{d}}[\boldsymbol{m}] = \nabla u_{\boldsymbol{m}}$ where the potential $u_{\boldsymbol{m}}$ solves:

$$\Delta u_{\boldsymbol{m}} = -\mathrm{div}\,(\overline{\boldsymbol{m}}) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3). \tag{1}$$

In (1) we have indicated with $\overline{\boldsymbol{m}}$ the extension of $\boldsymbol{m}$ to $\mathbb{R}^3$ that vanishes outside Ω. LAX-MILGRAM theorem guarantees that Eq.(1) possesses a unique solution in the BEPPO-LEVI space:

$$BL(\mathbb{R}^3) = \left\{ u \in \mathcal{D}'(\mathbb{R}^3) : \frac{u(\cdot)}{1 + |\cdot|} \in L^2(\mathbb{R}^3) \;\text{ and }\; \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3) \right\}. \tag{2}$$

Eventually, the fourth term $\mathcal{Z}(\boldsymbol{m})$, is called the *interaction energy*, and models the tendency of a specimen to have its magnetization aligned with the external field $\boldsymbol{h}_a$, assumed to be unaffected by variations of $\boldsymbol{m}$.

The competition of those four terms explain most of the striking pictures of the magnetization that ones can see in most ferromagnetic material [6], in particular the so-called *domain structure*, that is large regions of uniform or slowly varying magnetization (the *magnetic domains*) separated by very thin transition layers (the *domain walls*). However, and in a sake of simplification, we will not consider the stray field energy nor the anisotropy term in what follows. Therefore we omit the corresponding contributions $\mathcal{A}(\boldsymbol{m})$ and $\mathcal{W}(\boldsymbol{m})$ in the sequel.

3 Homogenization

Physically speaking, when considering a ferromagnetic body composed of different magnetic materials (i.e. a non single-crystal ferromagnet), the material functions a_{ex}, M_s and φ_{an} are no longer constant on the region Ω occupied by the ferromagnet. Moreover one has to describe the local interactions of two grains with different magnetic properties at their touching interface [1].

We hereafter assume the *strong coupling* assumption for which the direction of the magnetization does not jump through an interface between two different materials. We insist on the fact that only the direction is continuous at an interface while the magnitude M_s is obviously not. The natural mathematical setting for the problem becomes to consider that the magnetization direction $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$, where the class of admissible maps we are interested in is defined as

$$H^1(\Omega, \mathbb{S}^2) := \{ \boldsymbol{m} \in H^1(\Omega, \mathbb{R}^3) : \boldsymbol{m}(x) \in \mathbb{S}^2 \text{ for a.e. } x \in \Omega \}.$$

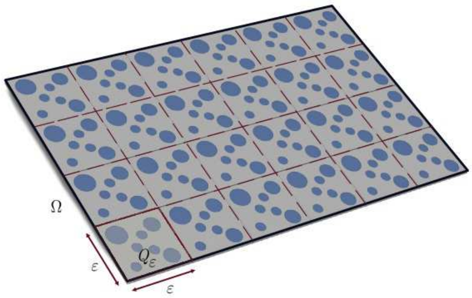

Figure 2. Schematization of a periodic ferromagnetic composite material. The periodization occurs at a scale ε.

3.1 The Gibbs-Landau energy functional associated to a composite ferromagnetic material

We start by recalling the basic idea of the mathematical theory of periodic homogenization. As illustrated in Fig. 2, this means that we can think of the material as being built up of small identical cubes, the side length of which being called ε. Let $Y = \mathbb{R}^3/\mathbb{Z}^3 \simeq [0,1)^3$ be the periodic unit cube of $\mathbb{R}^3$. We let for $y \in Y$, $a_{\text{ex}}(y), M_s(y)$ be the periodic repetitions of the functions that describe how the exchange constant a_{ex}, the saturation magnetization M_s and the anisotropy density energy $\varphi_{\text{an}}(y, \boldsymbol{m})$ vary over the representative cell Y (see Fig. 2). At every scale ε, the energy associated to the ε-heterogeneous ferromagnet, will be given by the following generalized Gibbs-Landau energy functional

$$\mathcal{G}_{\mathcal{L}}^{\varepsilon}(\boldsymbol{m}) := \underbrace{\int_\Omega a_{\text{ex}}(x/\varepsilon)|\nabla \boldsymbol{m}|^2 \,\mathrm{d}\mu}_{=:\mathcal{E}_\varepsilon(\boldsymbol{m})} \underbrace{- \mu_0 \int_\Omega \boldsymbol{h}_a \cdot M_s(x/\varepsilon)\boldsymbol{m} \,\mathrm{d}\mu}_{=:\mathcal{Z}_\varepsilon(\boldsymbol{m})} . \tag{3}$$

The asymptotic Γ-convergence analysis of the family of functionals $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$ as ε tends to 0, is the object of the present notes.

3.2 Setting of the problem

The main purpose of this paper is to analyze, by the means of both Γ-convergence and two-scale convergence techniques, the asymptotic behaviour, as $\varepsilon \to 0$, of the family of Gibbs-Landau free energy functionals $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$ expressed by (3). Let us make the statement more precise. We consider the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ and, for every $s \in \mathbb{S}^2$, the tangent space of $\mathbb{S}^2$ at a point s will be denoted by $T_s(\mathbb{S}^2)$.

For the energy densities appearing in the family $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$ we assume the following hypotheses:

- $\boldsymbol{H_1}$. The *exchange parameter* a_{ex} is supposed to be a Q-periodic measurable function belonging to $L^\infty(Q)$ which is bounded from below and above by two positive constants $c_{\text{ex}} > 0, C_{\text{ex}} > 0$, i.e. $0 < c_{\text{ex}} \leqslant a_{\text{ex}}(y) \leqslant C_{\text{ex}}$ for μ-a.e. $y \in Q$.
- $\boldsymbol{H_2}$. The saturation magnetization M_s is supposed to be a Q-periodic non-negative measurable function belonging to $L^\infty(Q)$.

4 Γ – Convergence

The notion of Γ-convergence has been introduced, mainly by De Giorgi in order to give a framework for the convergence of minimization problems. Roughly speaking, we consider a family of minimization problems (and associated functionals $\mathcal{F}_n$) defined on a metric space X (in our case, we have simply $X = H^1(\Omega)$) and try to find a limiting minimization problem (associated to the functional $\mathcal{F}_\infty$) such that

$$\text{“} \lim_{n \to +\infty} \min_{x \in X} \mathcal{F}_n(x) = \min_{x \in X} \mathcal{F}_\infty(x) \text{”}$$

in a suitable sense.

4.1 Γ-convergence of a family of functionals

We start by recalling DE GIORGI's notion of Γ-convergence and some of its basic properties (see [4, 5]). Although, this is much more general than what wa use in these notes, we restrict our attention to the case where the functionals are defined on $H^1(\Omega)$, and the notion of convergence is the weak one.

Definition 1. (Γ-convergence of a sequence of functionals) *Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of functionals defined on $H^1(\Omega)$ with values on $\overline{\mathbb{R}}$. The functional $\mathcal{F}\colon H^1(\Omega)\to\overline{\mathbb{R}}$ is said to be the* Γ-limit *of $(\mathcal{F}_n)_{n\in\mathbb{N}}$ with respect to the weak convergence, if for every $m\in H^1(\Omega)$ we have:*

$$\forall (m_n)\in (H^1(\Omega))^{\mathbb{N}}, m_n\rightharpoonup m \Rightarrow \liminf_{n\to\infty} \mathcal{F}_n(m_n)\geqslant \mathcal{F}(m) \tag{4}$$

and

$$\exists(\bar{m}_n)\in (H^1(\Omega))^{\mathbb{N}}, \bar{m}_n\rightharpoonup m \,\text{and}\, \mathcal{F}(m)=\lim_{n\to\infty}\mathcal{F}_n(\bar{m}_n). \tag{5}$$

In this case we write

$$\mathcal{F}=\Gamma\text{-}\lim_{n\to\infty}\mathcal{F}_n. \tag{6}$$

The condition (5) is sometimes referred in literature as the existence of a recovery sequence.

Definition 2. (Γ-convergence of a family of functionals) *Let $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ be a family of functionals defined on $H^1(\Omega)$ with values on $\overline{\mathbb{R}}$. The functional $\mathcal{F}\colon H^1(\Omega)\to\overline{\mathbb{R}}$ is said to be the* Γ-limit *of $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ with respect to the weak topology, as $\varepsilon\to 0$, if for every $\varepsilon_n\downarrow 0$*

$$\mathcal{F}=\Gamma\text{-}\lim_{n\to\infty}\mathcal{F}_{\varepsilon_n}. \tag{7}$$

In this case we write $\mathcal{F}=\Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon$.

One of the most important properties of Γ-convergence, and the reason why this kind of variational convergence is so important in the asymptotic analysis of variational problems, is that under appropriate compactness hypotheses it implies the convergence of (almost) minimizers of a family of functionals to the minimum of the limiting functional. More precisely, the following result holds.

Theorem 3. (Fundamental Theorem of Γ-convergence) *If $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is a family of functionals which Γ-converges on $H^1(\Omega)$ to the functional $\mathcal{F}$. Assume $(m_\varepsilon)_{\varepsilon>0}$ is a family of minimizers for $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ respectively. Assume that $m_\varepsilon\rightharpoonup m_0$ as $\varepsilon\rightharpoonup 0$, then m_0 is a minimizer of $\mathcal{F}$ and*

$$\mathcal{F}(m_0)=\min_{m\in H^1(\Omega)}\mathcal{F}(m)=\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(m_\varepsilon)=\lim_{\varepsilon\to 0}\min_{m\in H^1(\Omega)}\mathcal{F}_\varepsilon(m). \tag{8}$$

Proof. We already know (from the first statement of Γ-convergence) that

$$\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(m_\varepsilon)\geqslant\mathcal{F}(m_0)$$

Now, for any $n_0\in H^1(\Omega)$, we know that there exists a sequence $(n_\varepsilon)_{\varepsilon>0}$ such that $n_\varepsilon\rightharpoonup n_0$ and $\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(n_\varepsilon)=\mathcal{F}(n_0)$. Using the fact that m_ε is a minimizer of $\mathcal{F}_\varepsilon$, we get

$$\mathcal{F}(n_0)=\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(n_\varepsilon)\geqslant\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(m_\varepsilon)\geqslant \mathrm{iminf}_{\varepsilon\to 0}\mathcal{F}_\varepsilon(m_\varepsilon)\geqslant\mathcal{F}(m_0),$$

and therefore m_0 is a minimizer for $\mathcal{F}$. Eventually, taking $n_0=m_0$ leads to the fact that all inequalities are in fact equalities, and the result. □

4.2 Γ-convergence and homogenization

We describe here the Γ-convergence method applied to the model problem. We consider, as usual, a rigidity matrix $A(y)$ which is Y-periodic (we still denote by $Y=(0,1)^d$ the unit cell). For any $\varepsilon>0$, we call

$$\mathcal{F}_\varepsilon(u)=\frac{1}{2}\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\nabla u(x)\cdot\nabla u(x)\,dx-\int_\Omega f(x)u(x)\,dx\,,$$

and we work in $H_0^1(\Omega)$ (instead of $H^1(\Omega)$ as we have mentionned until now).

We also denote by

$$\mathcal{F}_0(u) = \frac{1}{2}\int_\Omega \bar{A}\,\nabla u(x)\cdot\nabla u(x)\,dx - \int_\Omega f(x)u(x)\,dx\,,$$

where $\bar{A}$ is he homogenized rigidity tensor. The aim of this part is to show the following theorem:

Theorem 4. *The family $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ Γ-converges to $\mathcal{F}_0$ in $H_0^1(\Omega)$ for the weak topology.*

Proof. We show the two properties of the Γ-convergence definition.

- Let $u_0 \in H_0^1(\Omega)$, and a family $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon \rightharpoonup u_0$ in $H_0^1(\Omega)$. Then we know that $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $H_0^1(\Omega)$, and we have

$$\begin{aligned} u_\varepsilon &\rightarrow u_0 \text{ strongly in } L^2(\Omega),\\ u_\varepsilon &\twoheadrightarrow u_0 \text{ two}-\text{scale in } L^2(\Omega\times Y),\\ \nabla u_\varepsilon &\twoheadrightarrow \nabla_x u_0(x) + \nabla_y u_1(x,y) \text{ two}-\text{scale in } L^2(\Omega\times Y). \end{aligned}$$

  This is sufficient to deduce that $\int_\Omega f(x)u_\varepsilon(x)\,dx \rightarrow \int_\Omega f(x)u_0(x)\,dx$ as $\varepsilon\to 0$. Now, expanding the inequality

$$\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\Big(\nabla u_\varepsilon(x) - \nabla u_0(x) - \nabla_y u_1\Big(x,\frac{x}{\varepsilon}\Big)\Big)\cdot\Big(\nabla u_\varepsilon(x) - \nabla u_0(x) - \nabla_y u_1\Big(x,\frac{x}{\varepsilon}\Big)\Big)\,dx \geq 0,$$

  and passing to the liminf as $\varepsilon\to 0$ leads to

$$\liminf_{\varepsilon\to 0}\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\nabla u_\varepsilon(x)\cdot\nabla u_\varepsilon(x) \geq \int_{\Omega\times Y} A(y)(\nabla_x u_0 + \nabla_y u_1)\cdot(\nabla_x u_0 + \nabla_y u_1)\,dx\,dy\,.$$

  Notice that strictly speaking this result is not true. We need indeed more regularity in u_1 to be able to pass to the limit. We omit the rigorous details here and comment a little bit afterwards.

  Nevertheless, minimizing this latter expression in u_1 leads to the fact that u_1 solves the cell problem (associated to u_0) and

$$\liminf_{\varepsilon\to 0}\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\nabla u_\varepsilon(x)\cdot\nabla u_\varepsilon(x) \geq \int_\Omega \bar{A}\,\nabla u_0(x)\cdot\nabla u_0(x)\,dx\,.$$

  Putting this together with the continuity of the linear term leads to

$$\liminf_{\varepsilon\to 0}\frac{1}{2}\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\nabla u_\varepsilon(x)\cdot\nabla u_\varepsilon(x) - \int_\Omega f(x)u_\varepsilon(x) \geq \frac{1}{2}\int_\Omega \bar{A}\,\nabla u_0(x)\cdot\nabla u_0(x) - \int_\Omega f(x)u_0(x)$$

  which is the desired result.

- In order to show the second statement of Γ-convergence, for any $u_0 \in H_0^2(\Omega)$, one constructs u_1 by solving the cell problem, that is to say $u_1(x,y) = \sum_{i=1}^d \omega_i(y)\frac{\partial u_0}{\partial x_i}(x)$ where ω_i are the correctors, and then we consider $u_\varepsilon(x) = u_0(x) + \varepsilon u_1\big(x,\frac{x}{\varepsilon}\big)$. It is easy to show (and left to the reader) that

$$\begin{aligned} &u_\varepsilon \rightarrow u_0 \text{ strongly (and thus weakly) in } H_0^1\,,\\ &\lim_{\varepsilon\to 0}\frac{1}{2}\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\nabla u_\varepsilon\cdot\nabla u_\varepsilon - \int_\Omega f u_\varepsilon = \frac{1}{2}\int_\Omega \bar{A}\,\nabla u_0\cdot\nabla u_0 - \int_\Omega f u_0\,. \end{aligned}$$

  This shows the second statement of the Γ-convergence, and finishes the proof.

- Notice that in order to prove both statements of $\Gamma-$convergence, we have been forced to assume more regularity on the limits than required. In the first step, the limit of the term which involves u_0 and u_1 requires a little bit of smoothness and in the second step, u_0 is assumed to have H^2 regularity. This latter assumption is not really problematic since redoing the proof of the convergence of minimizers in this context, we obtain

$$\mathcal{F}(n_0) \geqslant \mathcal{F}(m_0),$$

with n_0 in a dense subspace (here H^2) of H^1. however, since $\mathcal{F}$ is continuous in H^1, we have the desired result ($\mathcal{F}(n_0) \geqslant \mathcal{F}(m_0)$ $\forall n_0 \in H^1$) by density.

$\square$

Remark 5. Notice that in the preceding proof, we never assumed that u_0 (resp. u_ε) was a minimizer of $\mathcal{F}_0$ (resp. $\mathcal{F}_\varepsilon$).

5 Application to micromagnetics

The aim of this part is to apply the Γ-convergnce technique to the Gibbs-Landau functional. Before doing so, we see that the unit norm constraint leads to a new cell problem.

5.1 Preliminaries

The first question that we would like to answer is whether the constraint $|m| = 1$ a.e. leads to some new ingredients in the preceding proofs. Actually, we already know that if a family $(m_\varepsilon)_{\varepsilon>0}$ is bounded in H^1, then we can extract a subfamily $(m'_\varepsilon)_\varepsilon$ which converges to m_0 weakly in H^1 (and strongly in L^2 and a.e.). Expanding (formally) m'_ε in a multiscale expansion, we get

$$m'_\varepsilon(x) = m_0(x) + \varepsilon m_1\Big(x, \frac{x}{\varepsilon}\Big) + O(\varepsilon^2)$$

where m_0 is the weak H^1 limit, and m_1 is another map. Still formally, the second term εm_1 has to be understood as a perturbation to the first. Thus making an expansion (in ε) of $|m'_\varepsilon| = 1$ leads to

$$\begin{array}{rcl} |m_0(x)| & = & 1\,(\text{order}\,0)\\ 2m_0(x)\cdot m_1\Big(x, \dfrac{x}{\varepsilon}\Big) & = & 0\,(\text{order}\,1) \end{array}$$

which leads to the fact that

$$\forall x \in \Omega, \forall y \in Y, m_1(x, y) \cdot m_0(x) = 0.$$

Therefore, there is a hidden constraint that the corrector must be orthogonal to m_0. We now turn to the justification of this.

Proposition 6. *Let $(m_\varepsilon)_{\varepsilon>0}$ a family in $H^1(\Omega, \mathbb{S}^2)$ which is bounded in H^1. Then there exist $m_0 \in H^1(\Omega, \mathbb{S}^2)$, and $m_1 \in L^2(\Omega; H^1_\#(Y))$ such that $\forall x \in \Omega$, $m_1(x, y) \perp m_0(x)$ a.e. $y \in Y$, and up to the extraction of a subfamily, one has*

$$\begin{array}{rcl} m_\varepsilon(x) & \twoheadrightarrow & m_0(x)\,\text{two}-\text{scale},\\ \nabla m_\varepsilon(x) & \twoheadrightarrow & \nabla_x m_0(x) + \nabla_y m_1(x, y)\,\text{two}-\text{scale}. \end{array}$$

Proof. We already know by the application of the classical two-scale result that up to the extraction of a subfamily, there exist m_0 and m_1 such that

$$\begin{array}{rcl} m_\varepsilon(x) & \twoheadrightarrow & m_0(x)\,\text{two}-\text{scale},\\ \nabla m_\varepsilon(x) & \twoheadrightarrow & \nabla_x m_0(x) + \nabla_y m_1(x, y)\,\text{two}-\text{scale}. \end{array}$$

We just need to show that $|m_0(x)| = 1$ a.e. $x \in \Omega$ and $\forall x \in \Omega$, $m_1(x, y) \perp m_0(x)$ a.e. $y \in Y$.

For the former, we also know that $m_\varepsilon \rightharpoonup m_0$ in H^1 weakly, and strongly in L^2. Up to another extraction of a subfamily, we can thus further assume that $m_\varepsilon(x) \to m_0(x)$ for a.e. $x \in \Omega$. Since $|m_\varepsilon(x)| = 1$ a.e., we deduce that $|m_0(x)| = 1$ a.e. $x \in \Omega$. Now, we consider for $\phi \in \mathcal{C}^\infty(\Omega \times Y)$ the quantity

$$I_\varepsilon = \int_\Omega \phi\Big(x, \frac{x}{\varepsilon}\Big)(m_\varepsilon(x) - m_0(x)) \cdot \frac{\partial m_\varepsilon}{\partial x_i}(x)\,dx.$$

Since $m_\varepsilon - m_0 \to 0$ strongly in L^2 and $\frac{\partial m_\varepsilon}{\partial x_i}(x)$ is bounded in L^2, we know that

$$\lim_{\varepsilon \to 0} I_\varepsilon = 0.$$

On the other hand, since $|m_\varepsilon(x)| = 1$ a.e., we also know that $m_\varepsilon(x) \cdot \frac{\partial m_\varepsilon}{\partial x_i}(x) = 0$ a.e. $x \in \Omega$. Thus, expanding I_ε and passing to the 2-scale convergence, we get

$$\int_{\Omega \times Y} \left(\frac{\partial m_0}{\partial x_i}(x) + \frac{\partial m_1}{\partial y_i}(x, y) \right) \cdot m_0(x) \phi(x, y)\, dx\, dy = 0.$$

Now, using that $m_0(x) \cdot \frac{\partial m_0}{\partial x_i}(x) = 0$ a.e. $x \in \Omega$, we obtain

$$\int_{\Omega \times Y} \frac{\partial (m_1(x, y) \cdot m_0(x))}{\partial y_i} \phi(x, y)\, dx\, dy = 0,$$

which implies that $m_1(x, y) \cdot m_0(x)$ does not depend on y. Since moreover $\int_Y m_1(x, y)\, dy = 0$ a.e. $x \in \Omega$, we obtain that $m_1(x, y) \cdot m_0(x) = 0$ for a.e. $y \in Y$, and furthermore a.e. $x \in \Omega$. □

5.2 $\Gamma -$ convergence of the micromagnetic functional

The main result of these notes is the following.

Theorem 7. *Let $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$ be a family of Gibbs-Landau free energy functionals satisfying ($\boldsymbol{H_1}$) and ($\boldsymbol{H_2}$). The family $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$ Γ-converges in $(H^1(\Omega, S^2))$ for the weak convergence to the functional $\mathcal{G}_{\mathrm{hom}}: H^1(\Omega, S^2) \to \mathbb{R}^+$ defined by*

$$\mathcal{G}_{\mathrm{hom}}(\boldsymbol{m}) := \mathcal{E}_{\mathrm{hom}}(\boldsymbol{m}) - \mu_0 \mathcal{Z}_{\mathrm{hom}}(\boldsymbol{m}) \tag{9}$$

where

$$\mathcal{E}_{\mathrm{hom}}(\boldsymbol{m}) := \int_\Omega \bar{a}\, |\nabla \boldsymbol{m}|^2,$$

(here $\bar{a}\,\mathrm{Id}$ is the classical homogenized tensor) and where the homogenized interaction energy is given by

$$\mathcal{Z}_{\mathrm{hom}}(\boldsymbol{m}) = \langle M_s \rangle_Y \int_\Omega \boldsymbol{h}_a \cdot \boldsymbol{m}\, \mathrm{d}\mu.$$

Proof. We have to prove both statements of Γ- convergence. Namely we consider $\boldsymbol{m}_0 \in H^1(\Omega)$, and a family $\boldsymbol{m}_\varepsilon \in H^1(\Omega, \mathbb{S}^2)$ which is such that $\boldsymbol{m}_\varepsilon \rightharpoonup \boldsymbol{m}_0$ in $H^1(\Omega)$. As before, we know from this that up to the extraction of a subfamily, one can assume that

$$\begin{aligned} \boldsymbol{m}_\varepsilon &\to \boldsymbol{m}_0 \ \text{strongly in}\, L^2(\Omega), \\ \boldsymbol{m}_\varepsilon &\to \boldsymbol{m}_0 \ a.e. \,\text{in}\, \Omega, \\ \boldsymbol{m}_\varepsilon &\twoheadrightarrow \boldsymbol{m}_0 \ \text{two}-\text{scale}\, \text{in}\, L^2(\Omega \times Y), \\ \nabla \boldsymbol{m}_\varepsilon &\twoheadrightarrow \nabla_x \boldsymbol{m}_0(x) + \nabla_y \boldsymbol{m}_1(x, y) \ \text{two}-\text{scale}\, \text{in}\, L^2(\Omega \times Y). \end{aligned}$$

We also know that $\boldsymbol{m}_0$ only depends on x while from the a.e. convergence, we easily deduce that $\boldsymbol{m}_0 \in H^1(\Omega, \mathbb{S}^2)$.

Now, let us consider the second term $\mathcal{Z}_\varepsilon(\boldsymbol{m})$. We have

$$\begin{aligned} \mathcal{Z}_\varepsilon(\boldsymbol{m}) &= \int_\Omega \boldsymbol{h}_a \cdot M_s(x/\varepsilon) \boldsymbol{m}_\varepsilon(x) \mathrm{d}x \\ &= \boldsymbol{h}_a \cdot \int_\Omega M_s(x/\varepsilon) \boldsymbol{m}_\varepsilon(x)\, \mathrm{d}x \\ &\to \boldsymbol{h}_a \cdot \int_\Omega \int_Y M_s(y) \boldsymbol{m}_0(x)\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

as $\varepsilon \to 0$ from the 2-scale convergence. Then we remark that

$$\boldsymbol{h}_a \cdot \int_\Omega \int_Y M_s(y) \boldsymbol{m}_0(x)\, \mathrm{d}x\, \mathrm{d}y = \langle M_s \rangle_Y \int_\Omega \boldsymbol{h}_a \cdot \boldsymbol{m}_0\, \mathrm{d}x = \mathcal{Z}_{\mathrm{hom}}(\boldsymbol{m}_0)\,.$$

Therefore the second term is continuous with respect to weak H^1 convergence. As far as the first term is concerned, we have the following lemma whose proof is postponed.

Lemma 8. *Let $\mathcal{F}_\varepsilon(\boldsymbol{m}) = \int_\Omega A\big(\frac{x}{\varepsilon}\big)\nabla\boldsymbol{m}(x)\cdot\nabla\boldsymbol{m}(x)\mathrm{d}x$. Then $(\mathcal{F}_\varepsilon)_\varepsilon$ Γ-converges in H^1 for the weak H^1 convergence to $\mathcal{F}_0(\boldsymbol{m}) = \int_\Omega A_{\mathrm{hom}}(\boldsymbol{m})\nabla\boldsymbol{m}(x)\cdot\nabla\boldsymbol{m}(x)\mathrm{d}x$ where $A_{\mathrm{hom}}(\boldsymbol{m})$ is given by*

$$A_{\mathrm{hom}}(\boldsymbol{m})\xi\cdot\xi = \min_{\psi\in H^1_\#(Y,\boldsymbol{m}^\perp)}\int_Y A(y)(\xi+\nabla\psi(y))\cdot(\xi+\nabla\psi(y))\mathrm{d}y.$$

Assuming the lemma for the time being, we get from a direct application that $(\mathcal{E}_\varepsilon)_\varepsilon$ Γ-converges in H^1 for the weak H^1 convergence to $\mathcal{E}_0$ defined by

$$\mathcal{E}_0(\boldsymbol{m}) = \int_\Omega a_{\mathrm{hom}}(\boldsymbol{m})|\nabla\boldsymbol{m}(x)|^2\mathrm{d}x,$$

where

$$a_{\mathrm{hom}}(\boldsymbol{m})|\xi|^2 = \min_{\psi\in H^1_\#(Y,\boldsymbol{m}^\perp)}\int_Y a(y)|\xi+\nabla\psi(y)|^2\mathrm{d}y.$$

In order to get the theorem, we just need to show that actually this homogenized coefficient does not depend on $\boldsymbol{m}$ and is equal to the classical homogenized coefficient. This is simply seen by expanding the preceding formula in terms of coefficients. Indeed, it is not restrictive to assume that $\boldsymbol{m} = e_z = (0,0,1)$ is the vertical unit vector (by applying a global rotation to the formula), and we get

$$\begin{aligned}
a_{\mathrm{hom}}(\boldsymbol{m})|\xi|^2 &= a_{\mathrm{hom}}(e_3)|\xi'|^2\\
&= \min_{\psi\in H^1_\#(Y,e_z^\perp)}\int_Y a(y)|\xi'+\nabla\psi(y)|^2\mathrm{d}y\\
&= \min_{\psi_1,\psi_2\in H^1_\#(Y)}\int_Y a(y)(|\xi'_1+\nabla\psi_1(y)|^2+|\xi'_2+\nabla\psi_2(y)|^2)\mathrm{d}y\\
&= \bar{a}\,(|\xi'_1|^2+|\xi'_2|^2)\\
&= \bar{a}\,|\xi'|^2\\
&= \bar{a}\,|\xi|^2.
\end{aligned}$$

Therefore $\mathcal{E}_0(\boldsymbol{m}) = \int_\Omega \bar{a}\,|\nabla\boldsymbol{m}(x)|^2\mathrm{d}x$, which is the desired result. □

In order to finish the proof, we need to prove Lemma 8.

Proof. (Lemma 8). Let $\boldsymbol{m}_0\in H^1(\Omega)$, and a family $\boldsymbol{m}_\varepsilon\in H^1(\Omega,\mathbb{S}^2)$ which is such that $\boldsymbol{m}_\varepsilon\rightharpoonup\boldsymbol{m}_0$ in $H^1(\Omega)$. As before, we know from this that up to the extraction of a subfamily, one can assume that

$$\begin{aligned}
\boldsymbol{m}_\varepsilon &\to \boldsymbol{m}_0 \text{ strongly in } L^2(\Omega),\\
\boldsymbol{m}_\varepsilon &\to \boldsymbol{m}_0 \;a.e.\text{ in }\Omega,\\
\boldsymbol{m}_\varepsilon &\twoheadrightarrow \boldsymbol{m}_0 \text{ two}-\text{scale in } L^2(\Omega\times Y),\\
\nabla\boldsymbol{m}_\varepsilon &\twoheadrightarrow \nabla_x\boldsymbol{m}_0(x)+\nabla_y\boldsymbol{m}_1(x,y) \text{ two}-\text{scale in } L^2(\Omega\times Y).
\end{aligned}$$

We also know that $\boldsymbol{m}_0$ only depends on x while from the a.e. convergence, we easily deduce that $\boldsymbol{m}_0\in H^1(\Omega,\mathbb{S}^2)$. Moreover, an application of proposition 6 also gives that $\boldsymbol{m}_1(x,y)\cdot\boldsymbol{m}_0(x) = 0$ for all $x\in\Omega$ and all $y\in Y$. Now, applying the technique of proof of the preceding case (without the constraint), we get

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\nabla\boldsymbol{m}_\varepsilon(x)\cdot\nabla\boldsymbol{m}_\varepsilon(x) &\geq \int_{\Omega\times Y} A(y)(\nabla_x\boldsymbol{m}_0+\nabla_y\boldsymbol{m}_1)\cdot(\nabla_x\boldsymbol{m}_0+\nabla_y\boldsymbol{m}_1)\,dx\,dy\\
&\geq \min_{\phi\cdot\boldsymbol{m}_0=0}\int_{\Omega\times Y} A(y)(\nabla_x\boldsymbol{m}_0+\nabla_y\phi)\cdot(\nabla_x\boldsymbol{m}_0+\nabla_y\phi)\,dx\,dy\\
&\geq \int_\Omega A_{\mathrm{hom}}(\boldsymbol{m}_0)\nabla_x\boldsymbol{m}_0\cdot\nabla_x\boldsymbol{m}_0\,dx
\end{aligned}$$

by the definition of $A_{\mathrm{hom}}(\boldsymbol{m})$. This proves the first statement of Γ-convergence. For the second statement, we have to build a special family $(\boldsymbol{m}_\varepsilon)_\varepsilon$ that converges weakly to $\boldsymbol{m}_0$ and such that

$$\lim_{\varepsilon\to 0}\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\nabla\boldsymbol{m}_\varepsilon(x)\cdot\nabla\boldsymbol{m}_\varepsilon(x) = \int_\Omega A_{\mathrm{hom}}(\boldsymbol{m}_0)\nabla_x\boldsymbol{m}_0\cdot\nabla_x\boldsymbol{m}_0\,dx.$$

As before, we consider $\boldsymbol{m}_1$ solution to the cell problem, namely

$$\boldsymbol{m}_1(x,\cdot) = \operatorname{argmin}_{\phi\cdot\boldsymbol{m}_0=0}\int_Y A(y)(\nabla_x\boldsymbol{m}_0(x)+\nabla_y\phi(x,y))\cdot(\nabla_x\boldsymbol{m}_0(x)+\nabla_y\phi(x,y))\,dy.$$

We then build the family

$$\boldsymbol{m}_\varepsilon(x) = \frac{\boldsymbol{m}_0(x)+\varepsilon\boldsymbol{m}_1\big(x,\frac{x}{\varepsilon}\big)}{\big|\boldsymbol{m}_0(x)+\varepsilon\boldsymbol{m}_1\big(x,\frac{x}{\varepsilon}\big)\big|}.$$

If $\boldsymbol{m}_0\in H^2$, it is easily seen that $\boldsymbol{m}_\varepsilon\in H^1(\Omega,\mathbb{S}^2)$, and $\boldsymbol{m}_\varepsilon(x)=\boldsymbol{m}_0(x)+\varepsilon\boldsymbol{m}_1\big(x,\frac{x}{\varepsilon}\big)+O(\varepsilon^2)$ in H^1. Therefore,

$$\begin{aligned}\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\nabla\boldsymbol{m}_\varepsilon(x)\cdot\nabla\boldsymbol{m}_\varepsilon(x) &= \int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\Big(\nabla_x\boldsymbol{m}_0+\nabla_y\boldsymbol{m}_1\Big(x,\frac{x}{\varepsilon}\Big)\Big)\cdot\Big(\nabla_x\boldsymbol{m}_0+\nabla_y\boldsymbol{m}_1\Big(x,\\&\quad\frac{x}{\varepsilon}\Big)\Big)\,dx+O(\varepsilon)\\&\to \int_{\Omega\times Y} A(y)(\nabla_x\boldsymbol{m}_0+\nabla_y\boldsymbol{m}_1)\cdot(\nabla_x\boldsymbol{m}_0+\nabla_y\boldsymbol{m}_1)\,dx\,dy\end{aligned}$$

as $\varepsilon\to 0$ by two-scale convergence. A direct computation shows that this latter expression is equal to

$$\int_\Omega A_{\mathrm{hom}}(\boldsymbol{m}_0)\nabla_x\boldsymbol{m}_0\cdot\nabla_x\boldsymbol{m}_0\,dx$$

which concludes the proof. □

6 Conclusion

In these notes, we have described an application of Γ-convergence applied to the periodic homogenization of the GIBBS-LANDAU functional which modelizes the behavior of ferromagnetic materials. Compared to the classical theory, we notice that two major difficulties have to be overcome:

- The geometric constraint the the magnetization is of constant magnitude almost everywhere throughout the sample leads to a cell problem which now depends also on the value of m_0 the two-scale limit, and not only on its gradient. As we have already remarked, this is a general fact when such constraints need to be handled on the unknown (typically, to belong to a manifold);
- The convergence is understood in terms of convergence of variational problems. The good framework for this kind of convergence is the so-called Γ-convergence theory of DE GIORGI.

Although some difficulties appear, the classical homogenization program can be performed and a homogenized functional (when the scale of the periodic structure tends to 0) can be shown to be the Γ-limit of the GIBBS-LANDAU functional.

Bibliography

[1] E. Acerbi, I. Fonseca and G. Mingione. *Proc. R. Soc.*, 462:0, 2006.

[2] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):0, 1992.

[3] W. F. Brown. The fundamental theorem of the theory of fine ferromagnetic particles. *J. Appl. Phys.*, 39:0, 1968.

[4] E. De Giorgi and T. Franzoni. Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* , 58(8):0, 1975.

[5] G. Dal Maso. *An introduction to Γ-convergence*. Progress in Nonlinear Differential Equations and their Applications. Birkhauser Boston, 1993.

[6] A. Hubert and R. Schaefer. *Magnetic Domains: The Analysis of Magnetic Microstructures*. Springer, 1998.

[7] L. Landau and E. Lifshitz. *Phys. Z. Sowjetunion*, 169(14):0, 1935.

[8] Serpico C. Mayergoyz I.D., Bertotti G. *Nonlinear magnetization dynamics in nanosystems*. Elsevier, 2008.

[9] G. W. Milton. *The Theory of Composites*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, 2002.

[10] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):0, 1989.