The effect of competition on the height and length of the forest of genealogical trees of a large population

M. Ba É. Pardoux

Rencontre ANR, Marseille

January 31, 2012

Introduction

- Consider a Galton–Watson branching process X^m in continuous time with m ancestors at time $t = 0$: for $k \geq 1$, the process

$$X^m \text{ jumps } \begin{cases} k \to k+1, & \text{at rate } \mu k; \\ k \to k-1, & \text{at rate } \lambda k. \end{cases}$$

- In order to model competition within the population described by X^m, we superimpose to each individual a death rate due to competition, equal to γ times the number of presently alive individuals in the population, which amounts to add a global death rate equal to $\gamma(X_t^m)^2$ at time t.
- Set $m = [Nx]$, $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma/N$, and $Z^{N,x} = \frac{X^{[Nx]}}{N}$. The "total population mass process" Z^N converges weakly to the solution of the Feller SDE with logistic drift

$$dZ_t^x = \left[\theta Z_t^x - \gamma(Z_t^x)^2\right] dt + 2\sqrt{Z_t^x} dW_t, \; Z_0^x = x.$$

Introduction

- Consider a Galton–Watson branching process X^m in continuous time with m ancestors at time $t = 0$: for $k \geq 1$, the process

$$X^m \text{ jumps } \begin{cases} k \to k+1, & \text{at rate } \mu k; \\ k \to k-1, & \text{at rate } \lambda k. \end{cases}$$

- In order to model competition within the population described by X^m, we superimpose to each individual a death rate due to competition, equal to γ times the number of presently alive individuals in the population, which amounts to add a global death rate equal to $\gamma(X_t^m)^2$ at time t.
- Set $m = [Nx]$, $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma/N$, and $Z^{N,x} = \frac{X^{[Nx]}}{N}$. The "total population mass process" Z^N converges weakly to the solution of the Feller SDE with logistic drift

$$dZ_t^x = \left[\theta Z_t^x - \gamma(Z_t^x)^2\right] dt + 2\sqrt{Z_t^x} dW_t, \; Z_0^x = x.$$

Introduction

- Consider a Galton–Watson branching process X^m in continuous time with m ancestors at time $t = 0$: for $k \geq 1$, the process X^m jumps $\begin{cases} k \to k+1, & \text{at rate } \mu k; \\ k \to k-1, & \text{at rate } \lambda k. \end{cases}$
- In order to model competition within the population described by X^m, we superimpose to each individual a death rate due to competition, equal to γ times the number of presently alive individuals in the population, which amounts to add a global death rate equal to $\gamma (X_t^m)^2$ at time t.
- Set $m = [Nx]$, $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma / N$, and $Z^{N,x} = \frac{X^{[Nx]}}{N}$. The "total population mass process" Z^N converges weakly to the solution of the Feller SDE with logistic drift

$$dZ_t^x = \left[\theta Z_t^x - \gamma (Z_t^x)^2\right] dt + 2\sqrt{Z_t^x} dW_t, \; Z_0^x = x.$$

Introduction

- This equation has been studied by Lambert, Pardoux and Wakolbinger. Clearly the forest of those m trees is finite a.s.
- One can define the height and the length of the discrete forest of genealogical trees

$$H^m = \inf\{t > 0,\ X_t^m = 0\}, \quad L^m = \int_0^{H^m} X_t^m dt, \quad \text{for } m \geq 1,$$

as well as the height and the lenght of the continuous "forest of genealogical trees"

$$T^x = \inf\{t > 0,\ Z_t^x = 0\}, \quad S^x = \int_0^{T^x} Z_t^x dt$$

.

Introduction

- This equation has been studied by Lambert, Pardoux and Wakolbinger. Clearly the forest of those m trees is finite a.s.
- One can define the height and the length of the discrete forest of genealogical trees

$$H^m = \inf\{t > 0,\ X_t^m = 0\}, \quad L^m = \int_0^{H^m} X_t^m dt, \quad \text{for } m \geq 1,$$

as well as the height and the lenght of the continuous "forest of genealogical trees"

$$T^x = \inf\{t > 0,\ Z_t^x = 0\}, \quad S^x = \int_0^{T^x} Z_t^x dt$$

.

Branching process with competition

- We generalize the above models, replacing the death rate $\gamma(X_t^m)^2$ by $\gamma(X_t^m)^\alpha$.
- Set $m = [Nx]$, $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma / N^{\alpha-1}$, and $Z^{N,x} := \frac{X^{[Nx]}}{N}$. We show that the process Z^N converges weakly to to a Feller SDE with a negative polynomial drift.

$$dZ_t^x = [\theta Z_t^x - \gamma(Z_t^x)^\alpha]\, dt + 2\sqrt{Z_t^x} dW_t, \;\; Z_0^x = x. \tag{1}$$

- Consider the population generating a forest of genealogical trees in continuous time, we study the height and the length of the genealogical tree, as an (increasing) function of the initial population in the discrete and the continuous model.

Branching process with competition

- We generalize the above models, replacing the death rate $\gamma(X_t^m)^2$ by $\gamma(X_t^m)^\alpha$.
- Set $m = [Nx]$, $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma/N^{\alpha-1}$, and $Z^{N,x} := \frac{X^{[Nx]}}{N}$. We show that the process Z^N converges weakly to to a Feller SDE with a negative polynomial drift.

$$dZ_t^x = [\theta Z_t^x - \gamma(Z_t^x)^\alpha]\, dt + 2\sqrt{Z_t^x} dW_t, \ Z_0^x = x. \tag{1}$$

- Consider the population generating a forest of genealogical trees in continuous time, we study the height and the length of the genealogical tree, as an (increasing) function of the initial population in the discrete and the continuous model.

Results

- Both $\mathbb{E}[\sup_m H^m] < \infty$ and $\mathbb{E}[\sup_x T^x] < \infty$ if $\alpha > 1$, while $H^m \to \infty$ as $m \to \infty$ and $T^x \to \infty$ as $x \to \infty$ a. s. if $\alpha \le 1$.
- Both $\mathbb{E}[\sup_m L^m] < \infty$ and $\mathbb{E}[\sup_x S^x] < \infty$ if $\alpha > 2$, while $L^m \to \infty$ as $m \to \infty$ and $S^x \to \infty$ as $x \to \infty$ a. s. if $\alpha \le 2$.

This necessitates to define in a consistent way the population processes jointly for all initial population sizes, i. e. we will need to define the two–parameter processes $\{X_t^m,\ t \ge 0, m \ge 1\}$ and $\{Z_t^x,\ t \ge 0, x > 0\}$.

The model with asymetric effect of the competition

- The description of the process $(X_t^m, t \geq 0)$ is valid for one initial condition m, but it is not sufficiently precise to describe the joint evolution of $\{(X_t^m, X_t^n),\ t \geq 0\}$, $1 \leq m < n$.
- Modelize the effect of the competition in a asymmetric way. The idea is that the progeny X_t^m of the m "first" ancestors should not feel the competition due to the progeny $X_t^n - X_t^m$ of the $n - m$ "additional" ancestors which is present in the population X_t^n.
- We order the ancestors from left to right, this order being passed to their progeny.

The model with asymetric effect of the competition

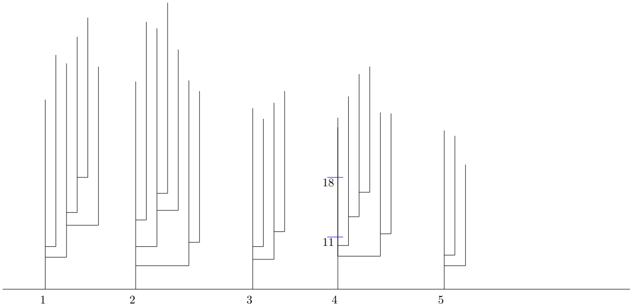
- The description of the process $(X_t^m, t \geq 0)$ is valid for one initial condition m, but it is not sufficiently precise to describe the joint evolution of $\{(X_t^m, X_t^n),\ t \geq 0\}$, $1 \leq m < n$.
- Modelize the effect of the competition in a asymmetric way. The idea is that the progeny X_t^m of the m "first" ancestors should not feel the competition due to the progeny $X_t^n - X_t^m$ of the $n - m$ "additional" ancestors which is present in the population X_t^n.
- We order the ancestors from left to right, this order being passed to their progeny.

The model with asymetric effect of the competition

- The description of the process $(X_t^m, t \geq 0)$ is valid for one initial condition m, but it is not sufficiently precise to describe the joint evolution of $\{(X_t^m, X_t^n),\ t \geq 0\}$, $1 \leq m < n$.
- Modelize the effect of the competition in a asymmetric way. The idea is that the progeny X_t^m of the m "first" ancestors should not feel the competition due to the progeny $X_t^n - X_t^m$ of the $n - m$ "additional" ancestors which is present in the population X_t^n.
- We order the ancestors from left to right, this order being passed to their progeny.

Order killing

- The individual placed at the position i at time t dies becuse of competition at rate $\gamma[\mathcal{L}_i(t)^\alpha - (\mathcal{L}_i(t)-1)^\alpha]$, $\mathcal{L}_i(t)$ is the number of alive individuals at time t, who are located at his left on the planar tree.



Discret model in the asymetric competition picture

- $\{X_t^m,\ t \geq 0\}$ is a continuous time $\mathbb{Z}_+$–valued Markov process, which evolves as follows. X_t^m jumps to $\begin{cases} k+1, & \text{at rate } \mu k; \\ k-1, & \text{at rate } \lambda k + \gamma(k-1)^\alpha. \end{cases}$
- The above description specifies well the evolution of the two$_p$*arametersprocess*$\{X_t^m,\ t \geq 0, m \geq 0\}$.
- If $\alpha \neq 1$, $\{X_t^m,\ m \geq 1\}$ is not a Markov chain for fixed t. The conditional law of X_t^{n+1} given X_t^n differs from its conditional law given $(X_t^1, X_t^2, \dots, X_t^n)$.
- However, $\{X_\cdot^m,\ m \geq 0\}$ is a Markov chain with values in the space $D([0,\infty); \mathbb{Z}_+)$, which starts from 0 at $m = 0$.

Discret model in the asymetric competition picture

- $\{X_t^m,\ t \geq 0\}$ is a continuous time $\mathbb{Z}_+$–valued Markov process, which evolves as follows. X_t^m jumps to $\begin{cases} k+1, & \text{at rate } \mu k; \\ k-1, & \text{at rate } \lambda k + \gamma(k-1)^\alpha. \end{cases}$
- The above description specifies well the evolution of the two$_parametersprocess\{X_t^m,\ t \geq 0, m \geq 0\}$.
- If $\alpha \neq 1$, $\{X_t^m,\ m \geq 1\}$ is not a Markov chain for fixed t. The conditional law of X_t^{n+1} given X_t^n differs from its conditional law given $(X_t^1, X_t^2, \ldots, X_t^n)$.
- However, $\{X_\cdot^m,\ m \geq 0\}$ is a Markov chain with values in the space $D([0,\infty); \mathbb{Z}_+)$, which starts from 0 at $m = 0$.

Discret model in the asymetric competition picture

- $\{X_t^m,\ t \geq 0\}$ is a continuous time $\mathbb{Z}_+$–valued Markov process, which evolves as follows. X_t^m jumps to $\begin{cases} k+1, & \text{at rate } \mu k; \\ k-1, & \text{at rate } \lambda k + \gamma(k-1)^\alpha. \end{cases}$
- The above description specifies well the evolution of the two$_parametersprocess\{X_t^m,\ t \geq 0, m \geq 0\}$.
- If $\alpha \neq 1$, $\{X_t^m,\ m \geq 1\}$ is not a Markov chain for fixed t. The conditional law of X_t^{n+1} given X_t^n differs from its conditional law given $(X_t^1, X_t^2, \ldots, X_t^n)$.
- However, $\{X_\cdot^m,\ m \geq 0\}$ is a Markov chain with values in the space $D([0,\infty);\mathbb{Z}_+)$, which starts from 0 at $m = 0$.

Discription of jointly laws

For arbitrary $0 \leq m < n$, let $V_t^{m,n} := X_t^n - X_t^m$, $t \geq 0$. Conditionally upon $\{X_\cdot^\ell,\ \ell \leq m\}$, and given that $X_t^m = x(t)$, $t \geq 0$, $\{V_t^{m,n},\ t \geq 0\}$ is a $\mathbb{Z}_+$–valued time inhomogeneous Markov process starting from $V_0^{m,n} = n - m$, whose time–dependent infinitesimal generator $\{Q_{k,\ell}(t),\ k, \ell \in \mathbb{Z}_+\}$ is such that its off–diagonal terms are given by

$$
\begin{aligned}
Q_{0,\ell}(t) &= 0, \quad \forall \ell \geq 1, \quad \text{and for any } k \geq 1, \\
Q_{k,k+1}(t) &= \mu k, \\
Q_{k,k-1}(t) &= \lambda k + \gamma(x(t) + k - 1)^\alpha, \\
Q_{k,\ell}(t) &= 0, \quad \forall \ell \notin \{k-1, k, k+1\}.
\end{aligned}
$$

The continuous Model

- $\{Z_t^x,\ t \geq 0, x \geq 0\}$ which such that for each fixed $x > 0$, $\{Z_t^x, t \geq 0\}$ is continuous process, solution of the SDE (1).
- For any $0 < x < y$, $\{V_t^{x,y} := Z_t^y - Z_t^x,\ t \geq 0\}$ solves the SDE

$$dV_t^{x,y} = \left[\theta V_t^{x,y} - \gamma \left\{(Z_t^x + V_t^{x,y})^\alpha - (Z_t^x)^\alpha\right\}\right] dt + 2\sqrt{V_t^{x,y}} dW_t',$$
$$V_0^{x,y} = y - x,$$

  $\{W_t',\ t \geq 0\}$ is independent from the Brownian motion W which drives the SDE (1) for Z_t^x.
- $\{Z_\cdot^x,\ x \geq 0\}$ is a Markov process with values in $C([0,\infty), \mathbb{R}_+)$, starting from 0 at $x = 0$.

Convergence result

Theorem

As $N \to \infty$,

$$\{\tilde{Z}_t^{N,x},\ t \geq 0, x \geq 0\} \Rightarrow \{Z_t^x,\ t \geq 0, x \geq 0\}$$

in $D([0,\infty); C([0,\infty); \mathbb{R}_+))$, equipped with the Skohorod topology of the space of càdlàg functions of x, with values in the space $C([0,\infty); \mathbb{R}_+)$ equipped with the topology of locally uniform convergence.

Height of the discrete tree

Theorem

- *If* $0 < \alpha \leq 1$, *then*

$$\sup_{m \geq 1} H^m = +\infty \qquad a.\ s.$$

- *If* $\alpha > 1$, *then*

$$\mathbb{E}\left[\sup_{m \geq 1} H^m\right] < \infty.$$

Proof of the Theorem[Height of the discrete tree] for $\alpha > 1$

Let $H_1^m = \inf\{s \geq 0; X_s^m = 1\}$.

Proposition

For $\alpha > 1$, $\lambda = 0$, $\forall\ m \geq 1$, $\mathbb{E}(H_1^m)$ is given by

$$\mathbb{E}(H_1^m) = \sum_{k=2}^{m} \frac{1}{\gamma(k-1)^\alpha} \sum_{n=0}^{\infty} \frac{\mu^n}{\gamma^n} \frac{1}{[k(k+1)\cdots(k+n-1)]^{\alpha-1}} < \infty.$$

Moreover we have

$$H^m \leq H_1^m + GH_1^2 + \sum_{i=1}^{G} T_i,$$

where G is geometric variable with parameter $\frac{\lambda}{\lambda+\mu}$ and T_i is exponential with mean $1/(\lambda+\mu)$.

Proof of the Theorem[Height of the discrete tree] for $\alpha \leq 1$

We have $X_t^{\alpha,m} \geq X_t^{1,m}$, for all $m \geq 1$, $t \geq 0$, a. s.. $\{X_t^m,\ t \geq 0\}$ is the sum of m mutually independent copies of $\{X_t^1,\ t \geq 0\}$. The result follows from the fact that $\mathbb{P}(H^1 > t) > 0$, for all $t > 0$.

Length of the discrete tree

- Time change of X^m:

$$A_t^m := \int_0^t X_r^m dr, \quad \eta_t^m = \inf\{s > 0;\ A_s^m > t\}.$$

$$U^m := X^m \circ \eta^m, \quad \text{and} \quad S^m = \inf\{r > 0; U_r^m = 0\}.$$

- We have $L^m = S^m$ since $S^m = \int_0^{H^m} X_r^m dr$.

Length of the discrete tree

$$X_t^m = m + P_1\left(\int_0^t \mu X_r^m dr\right) - P_2\left(\int_0^t [\lambda X_r^m + \gamma(X_r^m - 1)^\alpha]\, dr\right),$$

$$U_t^m = m + P_1(\mu t) - P_2\left(\int_0^t \left[\lambda + \gamma(U_r^m)^{-1}(U_r^m - 1)^\alpha\right] dr\right).$$

On the interval $[0, S^m)$, $U_t^m \geq 1$, and consequently we have

$$\begin{aligned} m - P_2\left(\int_0^t \left[\lambda U_r^m + \gamma(U_r^m - 1)^{\alpha-1}\right] dr\right) &\leq U_t^m \\ &\leq m + P_1\left(\int_0^t \mu U_r^m dr\right) - P_2\left(\int_0^t \left[\frac{\gamma}{2}(U_r^m - 1)^{\alpha-1}\right] dr\right). \end{aligned}$$

Length of the discrete tree

$$X_t^m = m + P_1\left(\int_0^t \mu X_r^m dr\right) - P_2\left(\int_0^t [\lambda X_r^m + \gamma(X_r^m - 1)^\alpha]\, dr\right),$$

$$U_t^m = m + P_1(\mu t) - P_2\left(\int_0^t \left[\lambda + \gamma(U_r^m)^{-1}(U_r^m - 1)^\alpha\right] dr\right).$$

On the interval $[0, S^m)$, $U_t^m \geq 1$, and consequently we have

$$\begin{aligned} m - P_2\left(\int_0^t \left[\lambda U_r^m + \gamma(U_r^m - 1)^{\alpha-1}\right] dr\right) &\leq U_t^m \\ &\leq m + P_1\left(\int_0^t \mu U_r^m dr\right) - P_2\left(\int_0^t \left[\frac{\gamma}{2}(U_r^m - 1)^{\alpha-1}\right] dr\right). \end{aligned}$$

Length of the discrete tree

Theorem

If $\alpha \leq 2$*, then*

$$\sup_{m \geq 0} L^m = \infty \quad a.\ s..$$

If $\alpha > 2$*, then*

$$\mathbb{E}\left[\sup_{m \geq 0} L^m\right] < \infty.$$

Height of the continuous tree

Consider again $\{Z_t^x,\ t \geq 0\}$ solution of (1). We have

Theorem

- *If* $0 < \alpha < 1$, $0 < \mathbb{P}(T^x = \infty) < 1$ *if* $\theta > 0$, *while* $T^x < \infty$ *a. s. if* $\theta = 0$.
- *If* $\alpha = 1$, $T^x < \infty$ *a. s. if* $\gamma \geq \theta$, *while* $0 < \mathbb{P}(T^x = \infty) < 1$ *if* $\gamma < \theta$.
- *If* $\alpha > 1$, $T^x < \infty$ *a. s.*

Height of the continuous tree

Theorem

- *It* $\alpha \leq 1$, *then* $T^x \to \infty$ *a. s., as* $x \to \infty$.
- *If* $\alpha > 1$, *then* $\mathbb{E}\left[\sup_{x>0} T^x\right] < \infty$.

Proof of of Theorem 5 for $\alpha > 1$

We first need to establish some preliminary results on SDEs with infinite initial condition. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz and such that

$$\lim_{x \to \infty} \frac{|f(x)|}{x^\alpha} = 0. \tag{2}$$

Theorem

Let $\alpha > 1$, $\gamma > 0$ and f satisfy the assumption (2). Then there exists a minimal $X \in C((0, +\infty); \mathbb{R})$ which solves

$$\begin{cases} dX_t = [f(X_t) - \gamma(X_t)^\alpha]\mathbf{1}_{\{X_t \geq 0\}}dt + dW_t; \\ X_t \to \infty, \text{ as } t \to 0. \end{cases} \tag{3}$$

Moreover, if $T_0 := \inf\{t > 0,\ X_t = 0\}$, then $\mathbb{E}[T_0] < \infty$.

Proof of the Theorem for $\alpha > 1$

The process $Y_t^x := \sqrt{Z_t^x}$ solves the SDE

$$dY_t^x = \left[\frac{\theta}{2} Y_t^x - \frac{\gamma}{2}(Y_t^x)^{2\alpha-1} - \frac{1}{8Y_t^x}\right] dt + dW_t, \;\; Y_0^x = \sqrt{x}.$$

By a well–known comparison theorem, $Y_t^x \leq U_t^x$, where U_t^x solves

$$dU_t^x = \left[\frac{\theta}{2} U_t^x - \frac{\gamma}{2}(U_t^x)^{2\alpha-1}\right] dt + dW_t, \;\; U_0^x = \sqrt{x}.$$

The result follows from the previous Theorem.

Proof of the Theorem for $\alpha < 1$

The result is equivalent to the fact that the time to reach 1, starting from x, goes to ∞ as $x \to \infty$. But when $Z_t^x \geq 1$, a comparison of SDEs for various values of α shows that it suffices to consider the case $\alpha = 1$. But in that case, T^n is the maximum of the extinction times of n mutually independent copies of Z_t^1, hence the result.

Length of the continuous tree

Theorem

If $\alpha \leq 2$, *then* $S^x \to \infty$ *a. s. as* $x \to \infty$.
If $\alpha > 2$, *then* $\mathbb{E}\left[\sup_{x>0} S^x\right] < \infty$.

Proof of the Theorem for $\alpha > 2$

- Time change of Z^x:

$$A_t = \int_0^t Z_s^x ds,\ \ \eta(t) = \inf\{s > 0,\ A_s > t\}.\ t \geq 0,\ \text{and}\ U_t^x = Z^x \circ \eta(t)$$

- The process U^x solves the SDE

$$dU_t^x = \left[\theta - \gamma(U_t^x)^{\alpha-1}\right] dt + 2dW_t,\ \ U_0^x = x. \tag{4}$$

- Let $\tau^x := \inf\{t > 0,\ \ U_t^x = 0\}$. It follows from the above that $\eta(\tau^x) = T^x$, hence $S^x = \tau^x$.

The result follows for $\alpha > 2$.

Proof of the Theorem for $\alpha \leq 2$

It suffice to consider the case $\alpha = 2$. In that case, we have

$$U_t^x = e^{-\gamma t}x + \int_0^t e^{-\gamma(t-s)}[\theta ds + 2dW_s],$$

hence

$$S^x = \inf\left\{t > 0,\ \int_0^t e^{\gamma s}(\theta ds + 2dW_s) \leq -x\right\},$$

which clearly goes to infinity, as $x \to \infty$.

Merci pour votre attention