Dynamics of Genealogical Trees for Autocatalytic Branching Processes

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Construct nice process which models dynamics of genealogy and total mass for finite populations and large population approximations . . .

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Replace individual branching rate

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where

$$\begin{aligned}\gamma \in \mathcal{G}^{\alpha} := \{g \in C(\mathbb{R}_+, \mathbb{R}_+) : {} & g|_{(0,\infty)} > 0\,,\\ & x \mapsto xg(x) \text{ locally Lipschitz},\\ & g(x) = \mathcal{O}(x^{\alpha}) \text{ as } x \uparrow \infty\,,\\ & g(x) \sim x^{\alpha} \text{ as } x \downarrow 0\}\,.\end{aligned}$$

$$:\Longleftrightarrow \lim_{x\downarrow 0} \frac{g(x)}{x^{\alpha}} = c > 0$$

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- At each time $t \in [0, \tau_{\mathrm{ex}})$, population represented by a non-empty set:

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- Weight of each individual in $\mathcal{I}_t$ given by $\mu_t = \sum_{i \in \mathcal{I}_t} w_t(i)\delta_i$:

$$\boxed{(\mathcal{I}_t, r_t, \mu_t)}$$

⇨ $$w_t(i) = \begin{cases} 1 & \text{if } t \neq \text{death time of } i \\ k - 1 & \text{if } t = \text{death time of } i \text{ and } k \text{ offspring} \end{cases}$$

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Ultrametric
measure space!

1. Intro
2. Metric Measure Spaces
3. Discrete Process
4. Tightness
5. Continuous Processes
6. Skew Product MGPs

Metric Measure Spaces

Metric measure spaces:

$\mathfrak{x} = [X, r, \mu] : (X, r)$ Polish metric space,

μ positive finite measure on $\mathcal{B}(X)$

“sampling measure”

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- Distance matrix map:

$$R^{(X,r),n} : \begin{cases} X^n & \to \mathbb{D}^n \,, \\ (x_1, \ldots, x_n) & \mapsto (r(x_i, x_j))_{1 \leq i < j \leq n} \end{cases}$$

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- Characterisation on $\mathbb{M}_1$ *[Gromov/Vershik 99]* :

$$\mathfrak{x} = \mathfrak{x}' \iff \nu^{n,\mathfrak{x}} = \nu^{n,\mathfrak{x}'} \ \forall n \geq 2$$

Polynomials, Topology and Metrisation on $\mathbb{M}_1$

- Φ-polynomials $\Phi : \mathbb{M}_1 \to \mathbb{R}$,

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Polish! *[Greven/Pfaffelhuber/Winter 09]*

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$$d_{\mathrm{GP}}(\mathfrak{x}, \mathfrak{x}') := \inf_{(\varphi_{\boldsymbol{X}}, \varphi_{\boldsymbol{X}'}, Z)} d_{\mathrm{P}}\left((\varphi_X)_*\mu, (\varphi_{X'})_*\mu'\right),$$

complete!

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over all isometric embeddings
$\varphi_X : X \hookrightarrow Z$, $\varphi_{X'} : X' \hookrightarrow Z$
into common metric space (Z, r_Z)

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$$\pi : \begin{cases} \mathbb{M}_{>0} & \to \ \mathbb{R}_{>0} \times \mathbb{M}_1\,, \\ \mathfrak{x} & \mapsto \ (m^{\mathfrak{x}}, \hat{\mathfrak{x}})\,. \end{cases}$$

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- Inverse = skew product:

$$\pi^{-1}(x, \mathfrak{x}) = x \otimes \mathfrak{x} = [X, r, x\mu]\,.$$

The Space $\mathbb{M}$

- Polar Gromov-Prohorov metric on $\mathbb{M}_{>0}$

$$d_{\mathrm{pGP}}(\mathfrak{x},\mathfrak{x}') := d_{\mathrm{eucl}} \otimes d_{\mathrm{GP}}(\pi(\mathfrak{x}),\pi(\mathfrak{x}')) = |m^{\mathfrak{x}} - m^{\mathfrak{x}'}| + d_{\mathrm{GP}}(\hat{\mathfrak{x}},\hat{\mathfrak{x}}')$$

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- Completion of $(\mathbb{M}_{>0}, d_{\mathrm{pGP}})$ ⟹ complete separable metric space:

$$(\mathbb{M}, d_{\mathrm{pGP}}) \cong (\mathbb{R}_+ \times \mathbb{M}_1, d_{\mathrm{eucl}} \otimes d_{\mathrm{GP}}) \,.$$

[Extend π to completion in obvious way.]

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- If
  - $\mathcal{C}_1 \subseteq C_b(\mathbb{R}_+)$ separates points on $\mathbb{R}_+$
  - $\mathcal{C}_2 \subseteq C_b(\mathbb{D}^n)$ separating on $\mathcal{M}_1(\mathbb{D}^n)$
  - $\mathcal{C}_1$ or $\mathcal{C}_2$ contains $\mathbb{1}$

Then ⟹ $\Pi_\Psi(\mathcal{C}_1,\mathcal{C}_2)$ separates points on $\mathbb{M}$.

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The Discrete Tree-Valued ACBP

Explicit construction from lexicographic representative process

⇨ $(\mathfrak{U}^{(\alpha,1)}(t))_{t\geq 0}$

Tree-valued α-autocatalytic branching process

❈ piecewise deterministic Markov process taking values in $\mathbb{U}$ ❈

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Rescale:

- time $t \rightsquigarrow Nt$
- mass $\mu \rightsquigarrow \frac{1}{N}\mu$
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Effective state space:

$$\mathbb{U}^{(N)} = \left\{ [U, r, \mu] \in \mathfrak{U} : \mu = \frac{1}{N} \sum_{i=1}^{n} w_i \delta_{u_i} \right\} \cup \{\mathfrak{n}\}$$

Properties

Generator:

$$\Omega^{(\alpha,N)}\ \Psi^N\ = \Omega^{(N)}_{\text{grow}}\Psi^N + \Omega^{(\alpha,N)}_{\text{bran}}\Psi^N$$

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$\Psi^N : \mathbb{U}^{(N)} \to \mathbb{R}$
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- $\mathfrak{U}^{(\alpha,N)}$ is a Borel process.
- $\mathfrak{U}^{(\alpha,N)}$ is a strong Markov process.

Tightness

From now on assume:

- $\limsup_{N\to\infty} \mathbb{E}\left[m^{\mathfrak{U}^{(\alpha,\boldsymbol{N})}(0)}\right] < \infty$
- $\exists \nu \in \mathcal{M}_1(\mathbb{U}):\ \mathcal{L}\left[\mathfrak{U}^{(\alpha,N)}(0)\right] \overset{N\to\infty}{\Longrightarrow} \nu$
- $\nu(\mathbb{U}_{>0}) = 1$

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Proposition

(1) For each $\alpha \in [0,1]$, $R \in \mathbb{N}$, $\{ \mathfrak{U}^{(\alpha,N,R)} \ : N \in \mathbb{N}\}$ is tight.

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(2) Any limit process will never hit zero ❏

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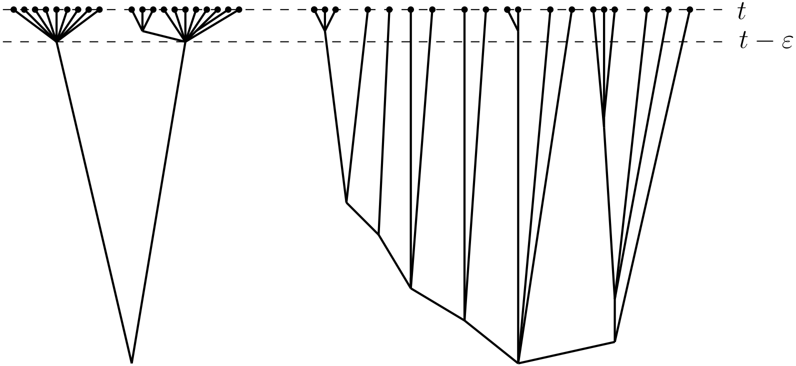
[Crit nter 12]:

$\#$ ancestors at time $t - \varepsilon$ of population at time t

- Relative proportion of subpopulations doesn't grow too much in $[0, T]$
- $\{ S_\varepsilon^{(\alpha,N)}(t) : N \in \mathbb{N}\}$ tight for every $0 < \varepsilon \leq t < T$ ❑

Good vs bad genealogy



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Proposition $(*)$

For all $0 \leq s < t < \infty$, $\varepsilon := t - s$,

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Let

- $X^{(1,\infty)} :=$ geometric BM with diffusion coefficient σ^2

variance of offspring distribution

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Define *tree-valued* 1*-autocatalytic branching diffusion* by

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is well-posed and $\mathfrak{U}^{(1,\infty)}$ is the solution.

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Assume $\mathfrak{U}^{(1,N)}(0) \Longrightarrow \nu$ such that $\nu \in \mathcal{M}_1(\mathbb{U})$. Then, on $D_{\mathbb{U}}[0,\infty)$,

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Corollary

Polar coordinates converge.

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- Results for α-autocatalytic true for $\gamma \in \mathcal{G}^\alpha$, in particular, for $\alpha = 1$.

The γ-Autocatalytic Case

Let $\gamma \in \mathcal{G}^\alpha$. Replace individual branching rate

$$(\text{total mass})^\alpha \rightsquigarrow \gamma(\text{total mass})$$

- Results for α-autocatalytic true for $\gamma \in \mathcal{G}^\alpha$, in particular, for $\alpha = 1$.
- Generator of tree-valued γ-autocatalytic branching diffusion $\mathfrak{U}^{(\gamma,\infty)}$

$$\Omega^{(\gamma,\infty)}\Psi^{\psi;n,\phi}(\mathfrak{u}) = \left(A^{(\gamma,\infty)}\psi(m^{\mathfrak{u}}) \right) \Phi^{n,\phi}(\hat{\mathfrak{u}}) + \psi(m^{\mathfrak{u}}) \left(\Omega^{(\infty)}_{\text{grow}}\Phi^{n,\phi}(\hat{\mathfrak{u}}) + \frac{\gamma(m^{\mathfrak{u}})}{m^{\mathfrak{u}}} \Omega^{(\infty)}_{\text{res}}\Phi^{n,\phi}(\hat{\mathfrak{u}}) \right)$$

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⇨ For all $f \in \mathcal{D}(A)$, $g \in \mathcal{D}(B)$, the following is a martingale:

Skew Product MGP

$$f(X(t))g(Y(t)) \\ - \int_0^t \big[(Af)(X(s))\big]g(Y(s)) + f(X(s))\big[\gamma(X(s))(Bg)(Y(s))\big]\,\mathrm{d}s.$$

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+ some minor technical assumptions

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Uniqueness

Q (X, Y) unique solution of MGP for $(L, \mathcal{D}(L) = \mathcal{D}(A)\mathcal{D}(B))$?

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Theorem

Consider setting above. Let $(x, y) \in E_1 \times E_2$.

Then The $D_{E_1 \times E_2}[0,\infty)$ MGP for $(L, \mathcal{D}(L), \delta_{(x,y)})$ has unique solution.

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Proof: Key Idea ⟹ If (X, Y) solves the skew product MGP, then

$$g(Y(t)) - \int_0^t \gamma(X(s))(Bg)(Y(s))\,\mathrm{d}s$$

is a martingale conditional on X . . . ❑

Proof (1) — The Conditional MGP

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Define:

- $\mathbb{P}_X := \mathbb{P} \circ X^{-1}$.
- $\mathbb{Q}(\,\cdot \mid \underline{x}) :=$ regular version of $\mathbb{P}\{\,\cdot \mid X = \underline{x}\}$.

Then

- Y solves MGP for $((\gamma \circ \underline{x}(t)B)_{t \geq 0}, \mathcal{D}(B), \delta_y)$ under $\mathbb{Q}(\,\cdot\,|\underline{x})$ for $\mathbb{P}_X$-a.a. $\underline{x}$.
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- Y solves MGP for $((\gamma \circ \underline{x}(t)B)_{t \geq 0}, \mathcal{D}(B), \delta_y)$ under $\mathbb{Q}(\,\cdot\,|\underline{x})$ for $\mathbb{P}_X$-a.a. $\underline{x}$.
- Fdd of Y under $\mathbb{Q}(\,\cdot\,|\underline{x})$ uniquely determined by $Y(0) = y$.

Proof (1) — The Conditional MGP

Q Is $g(Y(t)) - \int_0^t \gamma(X(s))(Bg)(Y(s))\,\mathrm{d}s$ martingale conditional on X?

A Yes!

Proposition

Assume setting from above and …

- $(x, y) \in E_1 \times E_2$.
- (X, Y) slv. $D_{E_1 \times E_2}[0, \infty)$ MGP for $(L, \mathcal{D}(L), \delta_{(x,y)})$ on $(\Omega, \mathcal{A}, \mathbb{P})$.

Define:

- $\mathbb{P}_X := \mathbb{P} \circ X^{-1}$.
- $\mathbb{Q}(\,\cdot\,|\,\underline{x}) :=$ regular version of $\mathbb{P}\{\,\cdot\,|\,X = \underline{x}\}$.

Then

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Proof (2) — Preservation of Martingale Property

Proposition

Assume:

- $(\Omega, \mathcal{A}, \mathcal{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a filtered probability space.
- $N = (N(t))_{t \geq 0}$ a locally square integrable martingale.
- $\mathcal{M}$ a family of locally square integrable martingales.
- $\mathcal{M} \perp N$, that is, $\langle M, N \rangle(t) = 0$, a.s., for all $t \geq 0$ and $M \in \mathcal{M}$.
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