On the measure division construction of Λ n-coalescents.

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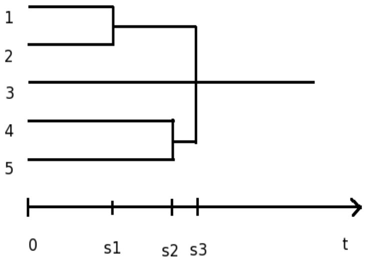
Aussois, 4 April 2012

Λ n-coalescent

1. $\Pi^{(n)} = (\Pi^{(n)}_t, t \geq 0)$ is a continuous time càdlàg Markov process taking values in $\mathcal{P}_n$, the set of partitions of $\{1, 2, \cdots, n\}$.
2. $\Pi^{(n)}_0 = \{\{1\}, \cdots, \{n\}\}$.
3. The process evolues by merging the blocks. The mechanism is determined by a measure Λ (to precise in the next slide).



$n = 5, \Pi^{(5)}_0 = \{\{1\}, \cdots, \{5\}\}$, $\Pi^{(5)}_{s1} = \{\{1,2\}, \{3\}, \{4\}, \{5\}\}$, $\Pi^{(5)}_{s2} = \{\{1,2\}, \{3\}, \{4,5\}\}$, $\Pi^{(5)}_{s3} = \{\{1,2,3,4,5\}\}$.

The measure Λ

If the processus $\Pi^{(n)}$ has b blocks at some time, then each k-tuple $(2 \le k \le b)$ of blocks merge independently into a big block at rate :

$$\lambda_{b,k} = \int_0^1 x^k (1-x)^{b-k} x^{-2} \Lambda(dx),$$

where Λ is a finite measure on $[0,1]$. Throughout this talk, Λ is assumed to be a probability measure.

In other words, one needs to wait an exponential time with parameter $g_b := \sum_{k=2}^{b} \binom{b}{k} \lambda_{b,k}$, and then each k-tuple of blocks merge together with probability

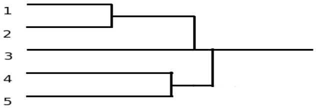
$$\frac{\lambda_{b,k}}{g_b}.$$

So the measure Λ describes completely the behavior.

Arising of Λ n-coalescent from biology

Given a large number population, we pick randomly a sample of n individuals and look at the genealogical tree of this sample. The larger the total population number, the more generations to backward to have coalescences for this sample. If the population number is very large and the time between two successive generations is well scaled, and also the variance of the number of descendants of one individual is controlled, the genealogical tree will tend to Λ n-coalescent.

$\Lambda = \delta_0$: **Kingman n-coalescent**. $\lambda_{b,2} = 1, \lambda_{b,k} = 0$ for $k \neq 2$. Only binary coalescences can happen.



This coalescent is the one mostly used by biologists.

Anton's comment :

Kingman n-coalescent is to model the genealogical tree of an n-sample of a large population by scaling many generations. In particular, the variance of the number of descendants of one individual should be small.

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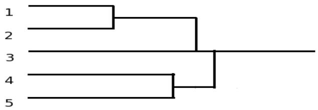

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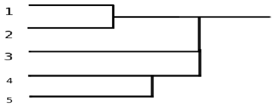

This process is related to Neveu CSBP (Bertoin and Le Gall, 1999), to random recursive trees (Goldschmidt and Martin, 2005), to spin glass theory in physics (Bolthausen and Sznitman, 1998), etc.

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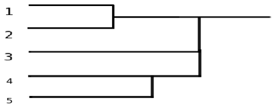

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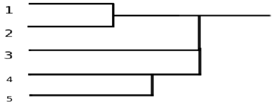
Beta$(2-\alpha,\alpha)$ n-coalescent with $1<\alpha<2$

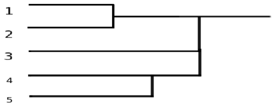
$\Lambda(dx) = \frac{x^{1-\alpha}(1-x)^{\alpha-1}dx}{\Gamma(\alpha)\Gamma(2-\alpha)} = Beta(2-\alpha,\alpha)$ measure with $1<\alpha<2$: **Beta$(2-\alpha,\alpha)$ n-coalescent**.



This coalescent is related to Alpha stable branching process (Birkner et al, 2005), to supercritical Galton-Waltson processes (Schweinsberg, 2003), etc.

coalescents with dust

$\int_0^1 x^{-1}\Lambda(dx) < +\infty$: **coalescents with dust**.



Anton's comment : Λ n-coalescent with no mass on 0, such as Bolthausen-Sznitman n-coalescent, Beta$(2-\alpha,\alpha)$ n-coalescent and also coalescents with dust, could be used to model the genealogical tree of a n-sample when the variance of the number of descendants of one individual is large.
Hence Λ coalescents with no mass on 0 is an analog of Lévy process which takes into account the variables with large variances.

Why call the measure Λ ?

Thanks to Prof Anton Wakolbinger for this anecdote !

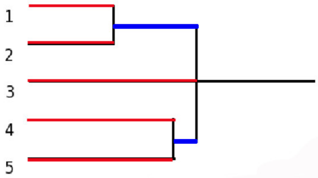
Simplified alphabet and pronunciation

Α α	(alpha) pronounced ‘*cu*p’ or ‘*ca*lm’
Β β	(beta) pronounced ‘b’ as in English
Γ γ	(gamma) a hard ‘g’, like ‘*g*ot’
Δ δ	(delta) a clean ‘d’, like ‘*d*ot’
Ε ϵ	(epsilon) short ‘e’ like ‘p*e*t’
Ζ ζ	(zeta) like ‘wi*sd*om’
Η η	(eta) pronounced as in ‘h*ai*r’
Θ θ	(theta) – blow a hard ‘t’ (‘*t*are’)
Ι ι	(iota) like ‘b*ea*d’ or like ‘b*i*n’
Κ κ	(kappa) a clean ‘k’ like ‘s*k*in’
Λ λ	(lambda) like ‘*l*ock’
Μ μ	(mu) like ‘*m*ock’
Ν ν	(nu) like ‘*n*et’
Ξ ξ	(xi) like ‘bo*x*’
Ο ο	(omicron) a short ‘o’, like ‘p*o*t’
Π π	(pi) a clean ‘p’, like ‘s*p*ot’
Ρ ρ	(rho) a rolled ‘r’, like ‘*rr*at’
Σ σ ς	(sigma) a soft ‘s’, like ‘*s*ing’
Τ τ	(tau) a clean ‘t’, like ‘s*t*ing’
Υ υ	(upsilon) French ‘l*u*ne’ or German ‘M*ü*ller’
Φ ϕ	(phi) – blow a hard ‘p’, like ‘*p*ool’
Χ χ	(khi) – blow a hard ‘c’, like ‘*c*ool’
Ψ ψ	(psi) as in ‘la*ps*e’
Ω ω	(omega) like ‘s*aw*’

Note ‘clean’ indicates no ‘h’ sound; ‘blow hard’ indicates plenty of ‘h’ aspiration (e.g. ϕ as in ‘top–hole’).

$$Lévy - - - - - - - - > L - - - - - - - - > \Lambda.$$

The red branches are external branches and the blue branches are internal branches.



The external branch length of individual i, denoted by $T_i^{(n)}$, is <u>one way</u> to measure the genetic diversity of the population(Rauch and Bar-Yam, 2004).

Question : What's the value of $T_1^{(n)}$ for any Λ ?

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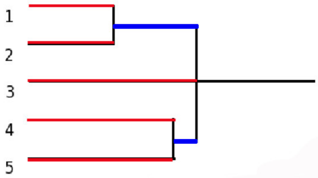

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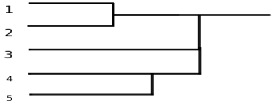


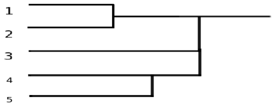
Freund and Möhle (2009) :

$$\ln nT_1^{(n)} \stackrel{(d)}{\to} Exp(1).$$

Remark that

$$\ln n = \int_{1/n}^{1} x^{-1}\Lambda(dx).$$

$\Lambda = Beta(2-\alpha,\alpha)$ measure with $1<\alpha<2$.



Dhersin, Freund, Siri-Jégousse, Y (2013) :

$$n^{\alpha-1}T_1^{(n)} \stackrel{(d)}{\to} T,$$

where T has density function $\frac{1}{(\alpha-1)\Gamma(\alpha)}(1+\frac{x}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}-1}\mathbf{1}_{x\geq 0}$.

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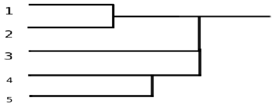

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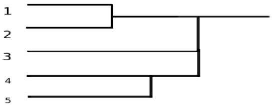
$\int_0^1 x^{-1}\Lambda(dx) < +\infty.$



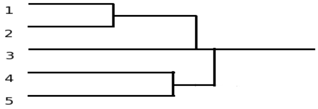
Möhle (2010) :

$$\int_0^1 x^{-1}\Lambda(dx)\, T_1^{(n)} \stackrel{(d)}{\rightarrow} Exp(1).$$

Remark that

$$\lim_{n\to+\infty} \frac{\int_{1/n}^1 x^{-1}\Lambda(dx)}{\int_0^1 x^{-1}\Lambda(dx)} = 1.$$

$\Lambda = \delta_0$. $\lambda_{b,2} = 1, \lambda_{b,k} = 0$ for $k \neq 2$. Only binary coalescences can happen.



Caliebe et al (2007) :

$$nT_1^{(n)} \overset{(d)}{\to} T,$$

where T has density function $\frac{8}{(2+x)^3}\mathbf{1}_{x\geq 0}$.

Remark that the Beta$(2-\alpha,\alpha)$ measure converges weakly to $\Lambda = \delta_0$ when $\alpha \to 2$. Since $n^{\alpha-1}$ is equivalent to $\int_{1/n}^{1} x^{-1}\frac{x^{1-\alpha}(1-x)^{\alpha-1}}{\Gamma(\alpha)\Gamma(2-\alpha)}dx$ and $n^{\alpha-1} \to n$, we can consider informally n as being equivalent to $\int_{1/n}^{1} x^{-1}\Lambda(dx)$ (not true, but I like it...)

External branch length in Kingman n-coalescent

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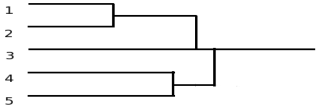

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Question : Is the normalization factor $\int_{1/n}^{1} x^{-1}\Lambda(dx)$ universal ?

Answer : at least for those satisfying condition (*). Define $\mu^{(\Lambda,n)} = \int_{1/n}^{1} x^{-1}\Lambda(dx)$, $\Pi^{(\Lambda,n)} = \Pi^{(n)}$, $T_i^{(\Lambda,n)} = T_i^{(n)}$, $g^{(\Lambda,n)} = g_n$.

Condition (*) : $\lim_{n\to+\infty} \frac{g^{(\Lambda,n)}}{n\mu^{(\Lambda,n)}} = 0.$

Theorem

(Y, 2013) If the condition (*) is satisfied, then

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Coalescents satisfying $(*)$

1. $\int_0^1 x^{-1}\Lambda(dx) < +\infty$.
2. Λ has a bounded density function $f(x)$ for $x \in [0, t]$ with $0 < t \leq 1$. This class includes the Bolthausen-Sznitman n-coalescent.
3. Λ has a density function $f(x) = p(-\ln x)^q$ for $x \in [0, t]$ with $0 < t \leq 1, q > 0, p > 0$.

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A characterization of condition (*)

Proposition

The following two assertions are equivalent :
(1) : Λ *satisfies condition (*) ;*
(2) : $\Lambda(\{0\}) = 0$ *and there exists a càglàd (limit from right, continuous from left) function* $g : [0,1] \to [0,1]$*, continuous on* 0 *with* $g(0) = 0$ *and a constant* $C > 0$*, such that*

$$\mu^{(\Lambda,n)} = C exp(\int_{1/n}^{1} \frac{g(x)}{x} dx)(1 - g(1/n)).$$

Remark that if $\lim_{x \to 0+} g(x) = \alpha - 1$ with $1 < \alpha < 2$, then it looks like a $Beta(2-\alpha, \alpha)$ coalescent. So this class of coalescents are "next to and below" the $Beta(2-\alpha,\alpha)$ coalescents.

The definition of $\mu^{(\Lambda,n)}$ concerns only the measure $\Lambda\mathbf{1}_{[1/n,1]}$. Does it mean that $\Lambda\mathbf{1}_{[0,1/n)}$ is negligible in the construction of $\Pi^{(n)}$ as $n \to +\infty$? How to evaluate the importance of each measure ?

Let $\xi_n = \{A_1, \cdots, A_{|\xi_n|}\}$, $\chi_n = \{B_1, \cdots, B_{|\chi_n|}\}$ be two partitions of $\{1, 2, \cdots, n\}$. We say that ξ_n **is finer than** χ_n, denoted by $\xi_n \preceq \chi_n$, if each B_i is a union of some blocks in ξ_n.

Examples :

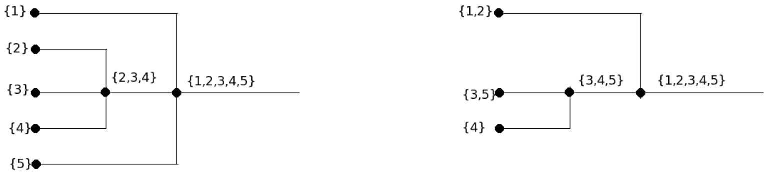
1 :

$$\begin{aligned} \xi_5 &= \{A_1 = \{1,2\}, A_2 = \{3\}, A_3 = \{4\}, A_4 = \{5\}\} \\ \preceq &\chi_5 = \{B_1 = \{1,2\}, B_2 = \{3\}, B_3 = \{4,5\}\}. \end{aligned}$$

2 :

$$\begin{aligned} \xi_5 &= \{A_1 = \{1,2\}, A_2 = \{3,4\}, A_3 = \{5\}\} \\ ! \preceq &\chi_5 = \{B_1 = \{1,2\}, B_2 = \{3\}, B_3 = \{4,5\}\}. \end{aligned}$$

Tool 2/2 : Restriction by the smallest element



(a) $\Pi^{(\Lambda,5)}$

(b) A restriction by the smallest element of $\Pi^{(\Lambda,5)}$ from $\xi_5 = \{\{1\},\cdots,\{5\}\}$ to $\chi_5 = \{\{1,2\},\{3,5\},\{4\}\}$

Let $\xi_n \preceq \chi_n$ and s_i^A (resp. s_i^B) be the smallest element in A_i (resp. B_i). We define the stochastic process $\tilde{\Pi}^{(\Lambda,\chi_n)}$, called the restriction by the smallest element of $\Pi^{(\Lambda,\xi_n)}$ from ξ_n to χ_n :

- $\tilde{\Pi}^{(\Lambda,\chi_n)}(0) = \chi_n$;
- For any $t \geq 0$, if $\Pi^{(\Lambda,\xi_n)}(t) = \{D_i\}_{1\leq i\leq |\Pi^{(\Lambda,\xi_n)}|(t)}$, where D_i denotes a block, then

$$\tilde{\Pi}^{(\Lambda,\chi_n)}(t) = \{ \bigcup_{s_j^B \in D_i} B_j\}_{1\leq i\leq |\Pi^{(\Lambda,\xi_n)}|(t)}.$$

(A block is represented by its smallest element.)

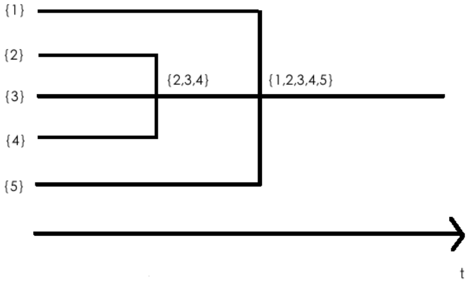
Equivalence in distribution

Lemma

$$\tilde{\Pi}^{(\Lambda,\chi_n)} \overset{(d)}{=} \Pi^{(\Lambda,\chi_n)}.$$

Measure division construction of $\Pi^{(\Lambda,n)}$

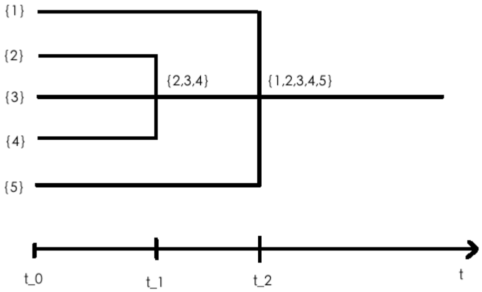
Let Λ_1, Λ_2 be two measures such that $\Lambda = \Lambda_1 + \Lambda_2$.
Step 0 : Get a realization or a path Π of $\Pi^{(\Lambda_1,n)}$:



Set a new process $\Pi_{1,2}^{(\Lambda,n)} = \Pi$.

Measure division construction of $\Pi^{(\Lambda,n)}$

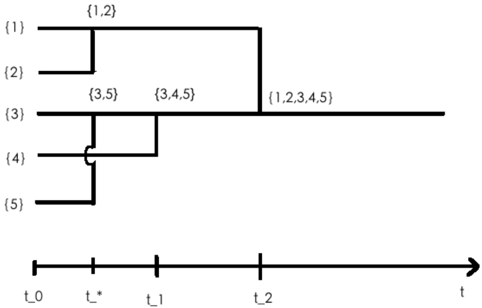
Step 1 : Let $t_1, t_2, \cdots$ be the coalescent times after t_0 of the given path of $\Pi_{1,2}^{(\Lambda,n)}$ (if there is no collision after t_0, we set $t_i = +\infty, i \geq 1$). Within $[t_0, t_1)$, $\Pi_{1,2}^{(\Lambda,n)}$ is constant. Then we run an independent Λ_2 coalescent with initial value $\Pi_{1,2}^{(\Lambda,n)}(t_0)$ from time t_0.



- If the Λ_2 coalescent has no collision on $[t_0, t_1)$, we pass to $[t_1, t_2)$. Similarly, we construct another independent Λ_2 coalescent with initial value $\Pi_{1,2}^{(\Lambda,n)}(t_1)$ from time t_1, and so on.
- Otherwise, we go to the next step.

Measure division construction of $\Pi^{(\Lambda,n)}$

Step 2 : If finally within $[t_{i-1}, t_i)$, the related independent Λ_2 coalescent has its first collision at time t_* and its value at t_* is ξ. We set the new $(\Pi_{1,2}^{(\Lambda,n)}(t))_{t\geq t_*}$ as the restriction by the smallest element of previous $(\Pi_{1,2}^{(\Lambda,n)}(t))_{t\geq t_*}$ from previous $\Pi_{1,2}^{(\Lambda,n)}(t_*)$ to ξ. Then we go to step 1 taking t_* as the new starting time.



In this case, the related Λ_2 coalescent with initial value $\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$ gets a coalescence at time t_* and $\xi = \{\{1,2\},\{3,5\},\{4\}\}$.

Theorem

$$\Pi_{1,2}^{(\Lambda,n)} \stackrel{(d)}{=} \Pi^{(\Lambda,n)}.$$

Advantages :

- One can take $\Lambda_1 = 0$ and $\Lambda_2 = \Lambda$. In this case, in step 0, we take a path of n parallel lineages.
- we call Λ_1 the noise measure, Λ_2 the main measure. If Λ_1 is "small", then $\Pi^{(\Lambda_1,n)}$ almost looks like n parallel lineages at small times. Then the behaviors of $\Pi^{(\Lambda_2,n)}$ is very close to that of $\Pi_{1,2}^{(\Lambda,n)}$. For $\Pi^{(\Lambda_2,n)}$, we often have many results known.

Control of the noise measure :

Lemma

Assume that Λ satisfies condition $(*)$ and $\Lambda_1 = \Lambda \mathbf{1}_{[0,1/n)}$. Then for any $t > 0, 0 < \epsilon \leq 1$, we have

$$\mathbb{P}(|\Pi^{(\Lambda_1,n)}|(\frac{t}{\mu^{(\Lambda,n)}}) \leq n - n\epsilon) = o(n^{-1}).$$

Notice that conditional on $\{|\Pi^{(\Lambda_1,n)}|(\frac{t}{\mu^{(\Lambda,n)}}) > n - n\epsilon\}$, we have lost at most $n\epsilon$ individuals. Then we have at most $2n\epsilon$ singletons and each of them is involved in a collision somewhere before $\frac{t}{\mu^{(\Lambda,n)}}$. Using the exchangeability of individuals, $\mathbb{P}(\{1\} \in \Pi^{(\Lambda_1,n)}(\frac{t}{\mu^{(\Lambda,n)}})) > 1 - 2\epsilon$.
So in this case, $(\Pi^{(\Lambda_1,n)}(s), 0 \leq s \leq \frac{t}{\mu^{(\Lambda_1,n)}})$ is very close to n parallele lineages.

Property of the main measure :

Lemma

Assume that Λ *satisfies condition* $(*)$ *and* $\Lambda_2 = \Lambda \mathbf{1}_{[1/n,1]}$. *Then for any* $t > 0$, *we have*

$$\lim_{n \to +\infty} \mathbb{P}(T_1^{(\Lambda_2,n)} \geq \frac{t}{\mu^{(\Lambda,n)}}) \to e^{-t}.$$

Assume that Λ satisfies condition $(*)$ and $\int_0^1 x^{-1}\Lambda(dx) = +\infty$. Define $L_{ext}^{(\Lambda,n)}$ as the total external branch length and $L^{(\Lambda,n)}$ the total branch length.
Then

Proposition

$$\frac{\mu^{(\Lambda,n)} L_{ext}^{(\Lambda,n)}}{n} \xrightarrow{\mathbb{P}} 1,$$

$$\frac{\mu^{(\Lambda,n)} L^{(\Lambda,n)}}{n} \xrightarrow{\mathbb{P}} 1,$$

Thank you for your attention !