On the extinction of CSBP with catastrophes

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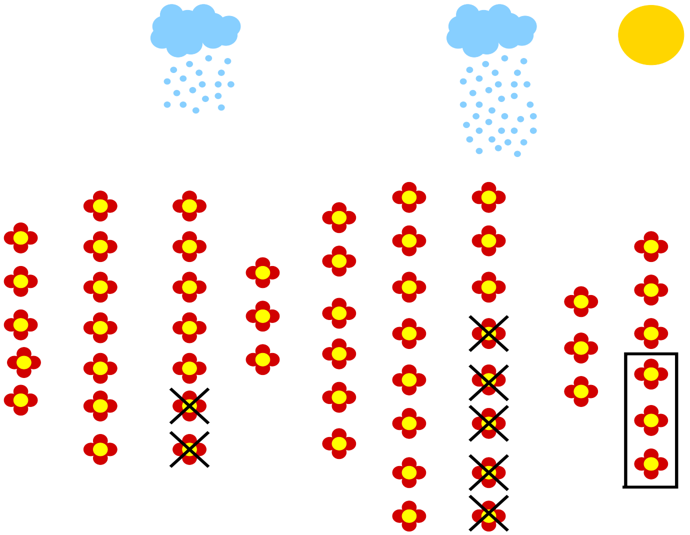

1. CSBP's with catastrophes
2. Existence and long time behavior
3. Speed of extinction
4. Application : parasite infection in dividing cells

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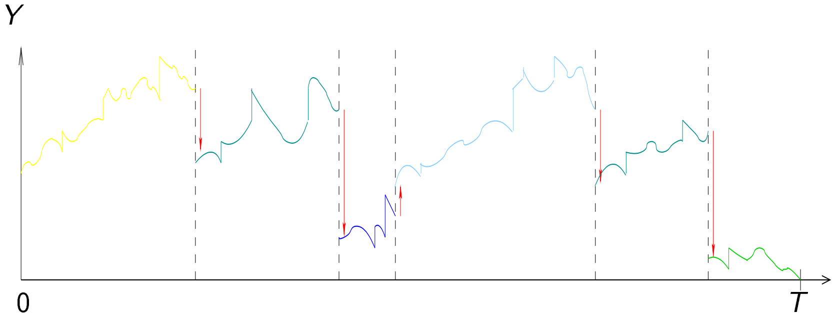
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CSBP $\Longleftrightarrow$ Scaling limit of GW (Lamperti 1967)

CSBP : Definition 2

Solution of an SDE (Fu and Li 2010)

$$Z_t = Z_0 + \int_0^t g Z_s ds + \int_0^t \sqrt{2\sigma^2 Z_s} dB_s + \int_0^t \int_0^\infty \int_0^{Z_{s-}} z \widetilde{N}_0(ds, dz, du)$$

- g malthusian parameter
- B brownian motion, N_0 Poisson random measure $(ds\mu(dz)du)$, independent of B
- $\tilde{N}_0$ compensated measure of N_0.

Stable CSBP's : a particular class (1)

Why ?

- Applications
- Simplest ones
- Exact speed of extinction
- Give bounds for speed of extinction of a wide class of CSBP's with catastrophes

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- Otherwise stable CSBP :

$$Y_t = Y_0 + \int_0^t gY_s ds + \int_0^t \int_0^\infty \int_0^{Y_{s^-}} z\tilde{N}_0(ds, dz, du)$$

where $N_0(ds, dz, du)$ is a Poisson random measure with intensity $ds\frac{Cdz}{z^{2+\beta}}du$

CSBP with catastrophes

- Catastrophes independent of the CSBP
- N_1 Poisson Point Process with intensity $dt\nu(dx)$
- Finally, process solution of

$$\begin{aligned} Y_t &= Y_0 + \int_0^t g Y_s ds + \int_0^t \sqrt{2\sigma^2 Y_s} dB_s \\ &+ \int_0^t \int_{[0,\infty)} \int_0^{Y_{s-}} z \widetilde{N}_0(ds, dz, du) \\ &+ \int_0^t \int_{[0,\infty)} (z-1) Y_{s-} N_1(ds, dz) \end{aligned}$$

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$$
\begin{aligned}
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&\quad + \int_0^t \int_{[0,\infty)} \int_0^{Y_{s-}} z \widetilde{N}_0(ds, dz, du) \\
&\quad + \int_0^t \int_{[0,\infty)} (z-1) Y_{s-} N_1(ds, dz)
\end{aligned} \tag{1}
$$

Proposition

If

$$\int_{(0,\infty)} 1 \wedge \left|x-1\right| \nu(dx) < \infty,$$

(1) has a unique strong solution (Fu and Li 2010)

Lévy process describing environment

$$\Delta_t = \int_0^t \int_{(0,\infty)} ln(x) N_1(ds, dx) = \sum_{s \leq t} ln(m_s)$$

Proposition

(i) (Subcritical) If $(\Delta_t + gt)_{t\geq 0}$ goes to $-\infty$, then $Y_t \to 0$ a.s.

(ii) (Critical) If $(\Delta_t + gt)_{t\geq 0}$ oscillates, then $\liminf_{t\to\infty} Y_t = 0$ a.s.

(iii) (Supercritical) If $(\Delta_t + gt)_{t\geq 0}$ goes to $+\infty$, then $\mathbb{P}(\forall t \geq 0, Y_t > 0) > 0$ and $\exists\ W \geq 0$ such that

$$e^{-gt-\Delta_t} Y_t \xrightarrow[t\to\infty]{} W \quad a.s., \qquad \{W = 0\} = \left\{ \lim_{t\to\infty} Y_t = 0 \right\}.$$

Proposition (stable case, $\beta \in (0,1]$)

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Proposition

Assume we are in the stable case ($\beta \in (0,1]$). For all $x_0 \geq 0$ and $t \geq 0$:

$$\mathbb{P}_{x_0}(Y_t > 0) = 1 - \mathbb{E}\left[\exp\left\{-x_0\left(c\int_0^t e^{-\beta(\Delta_s + gs)}ds\right)^{-1/\beta}\right\}\right].$$

Laplace exponent

- $\mathbb{E}[e^{\lambda\Delta_t}] = e^{t\phi(\lambda)} \quad for \lambda, t \geq 0$

The Laplace exponent ϕ is a convex function.

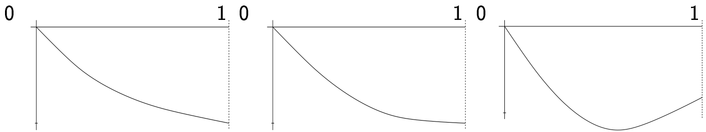
Theorem

We consider subcritical case ($\phi'(0) + g < 0$)

(a) (strongly) If $\phi'(1) + g < 0$: $\mathbb{P}_{x_0}(Y_t > 0) \underset{t \to \infty}{\simeq} x_0 e^{t(\phi(1)+g)}$

(b) (intermediate) If $\phi'(1) + g = 0$:
$\mathbb{P}_{x_0}(Y_t > 0) \underset{t \to \infty}{\simeq} x_0 t^{-1/2} e^{t(\phi(1)+g)}$

(c) (weakly) If $\phi'(1) + g > 0$: $\mathbb{P}_{x_0}(Y_t > 0) \underset{t \to \infty}{\simeq} c(x_0) t^{-3/2} e^{\tau t}$
($\tau = \min_{\lambda \in]0,1[} \{\phi(\lambda) + g\lambda\}$)



(a) $\phi'(1) + g < 0$ (b) $\phi'(1) + g = 0$ (c) $\phi'(1) + g > 0$

Figure: $t \mapsto \phi(t) + gt$

Heuristically

Equivalent linked to $\mathbb{E}[\exp(\inf_{s\in[0,t]}(\Delta_s + gs))]$

Strongly subcritical

Escher transform

Other cases

- Discretization of the Lévy process $\Delta_t + gt$ to obtain a random walk
- Limit theorems on functionals of arithmetico-geometric sequences $U_{n+1} = A_n U_n + B_n$ with (A_n, B_n) iid (Guivarch and Liu 2001)
- Continuous limit

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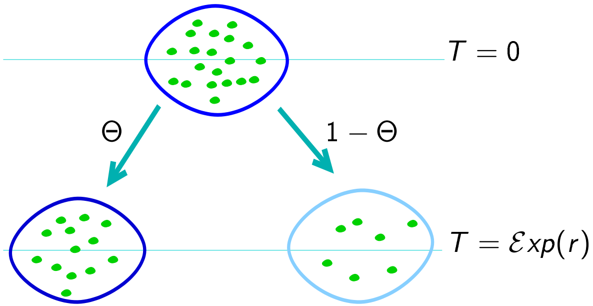
Model (Bansaye and Tran 2010)



Evolution of parasites in a cell

$$dX_s^i = gX_s^i ds + \sqrt{2\sigma^2 X_s^i} dB_s^i$$

Question

What is the quantity N_t^* of infected cells at time t ?

Result (Bansaye and Tran 2010)

$$\mathbb{E}(N_t^*) = e^{rt}\mathbb{P}(Y_t > 0) = \mathbb{E}(N_t)\mathbb{P}(Y_t > 0)$$

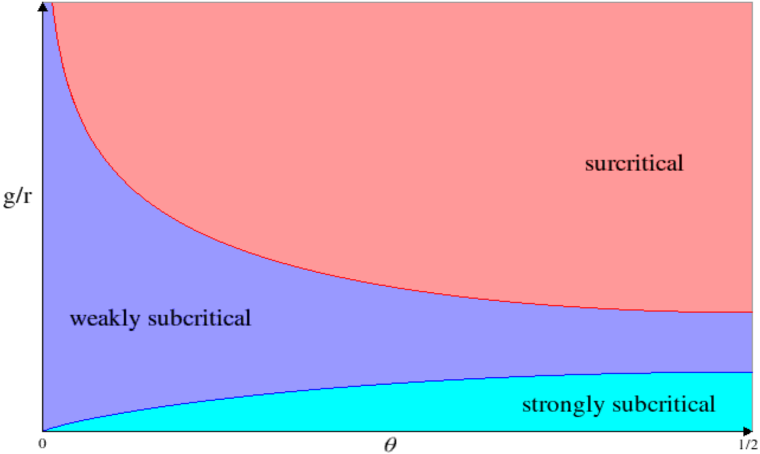
$$Y_t = x_0 + \int_0^t gY_s ds + \int_0^t \sqrt{2\sigma^2 Y_s} dB_s + \int_0^t \int_0^1 (\theta - 1) Y_{s_-} \rho(ds, d\theta)$$

$\rho(ds, d\theta)$ Poisson random measure with intensity $2rds\mathbb{P}(\Theta \in d\theta)$

Proposition

1. (Supercritical) If $g > 2r\mathbb{E}\left[\log(1/\Theta)\right]$, then $\mathbb{E}\left[N_t^*\right] \sim c_5 e^{rt}$,
2. (Critical) If $g = 2r\mathbb{E}\left[\log(1/\Theta)\right]$, then $\mathbb{E}\left[N_t^*\right] \sim c_4 t^{-1/2} e^{rt}$,
3. (Subcritical) If $g < 2r\mathbb{E}\left(\log(1/\Theta)\right)$:
   - (i) (Strongly) If $g < 2r\mathbb{E}\left(\Theta \log(1/\Theta)\right)$, $\mathbb{E}\left(N_t^*\right) \sim c_1 e^{gt}$,
   - (ii) (Intermediate) If $g = 2r\mathbb{E}\left(\Theta \log(1/\Theta)\right)$, $\mathbb{E}\left(N_t^*\right) \sim c_2 t^{-1/2} e^{gt}$,
   - (iii) (Weakly) If $g > 2r\mathbb{E}\left(\Theta \log(1/\Theta)\right)$, $\mathbb{E}\left(N_t^*\right) \sim c_3 t^{-3/2} e^{\alpha t}$, where $\alpha = \min_{\lambda \in]0,1[}\{(g\lambda + 2r(\mathbb{E}(\Theta^\lambda) - 1/2)\}$.

g =growth rate of parasites, r =division rate of cells,
$(\theta, 1-\theta)$ =sharing law of parasites



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Thanks for your attention