A time reversal duality for branching processes and applications

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Outline

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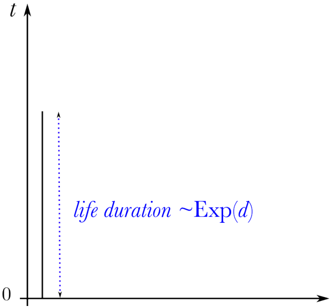
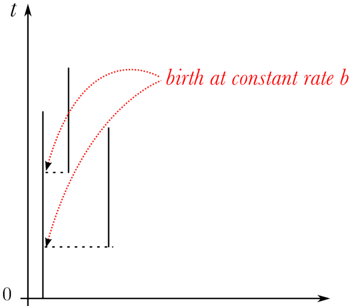
Birth-death (BD) process



Individuals

- have i.i.d. life durations $\sim \mathrm{Exp}(d)$
- reproduce at constant rate b during their life
- behave independently from one another

Birth-death (BD) process



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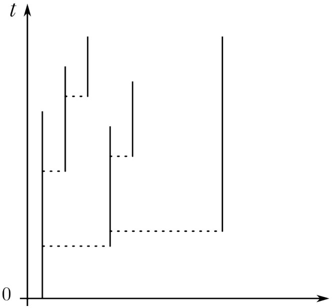

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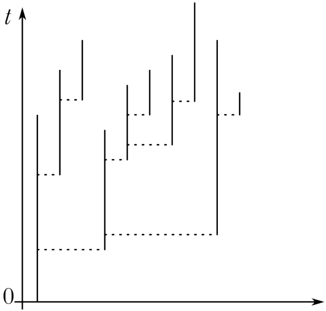

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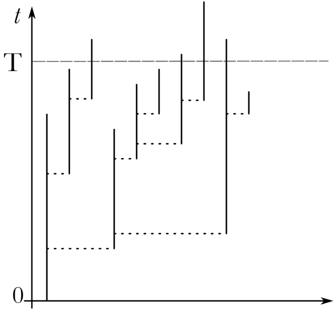

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$\mathcal{T}$: the BD tree starting from one ancestor

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$(\xi_t(\mathcal{T}), t \geq 0)$: the population size process

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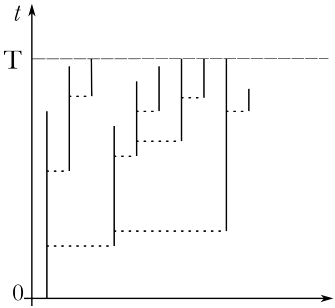

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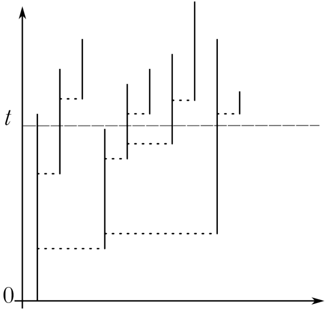

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Random forests

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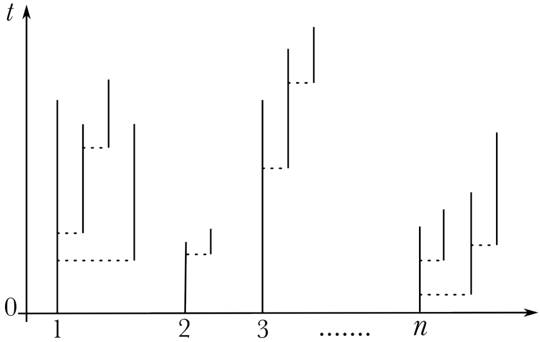

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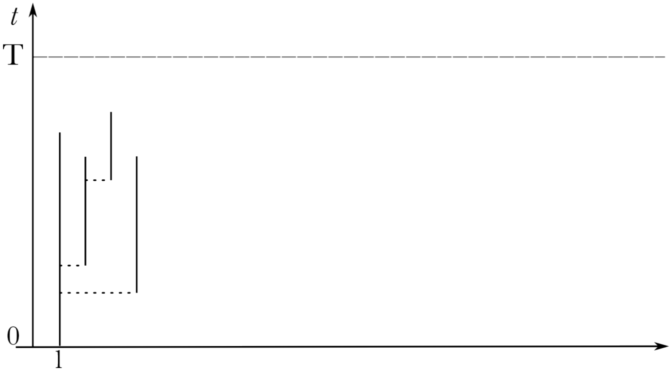

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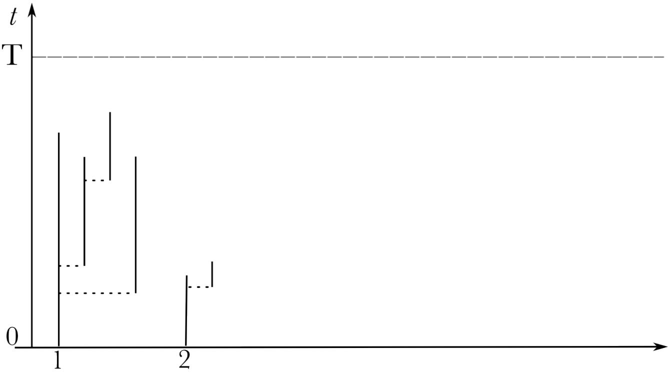

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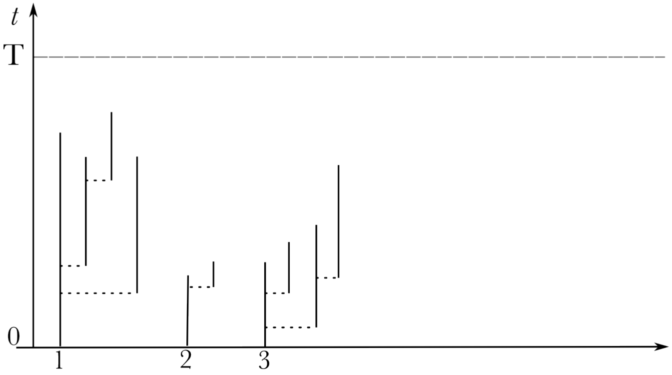

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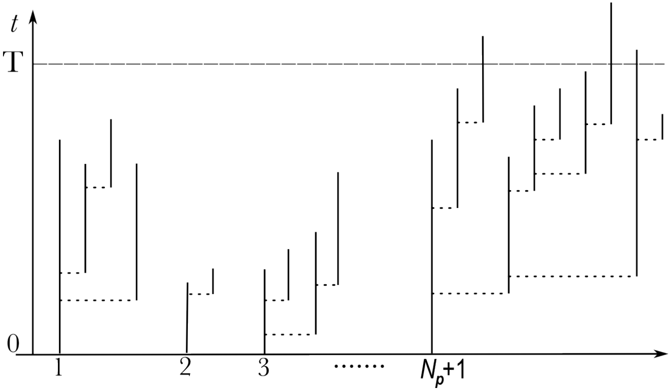

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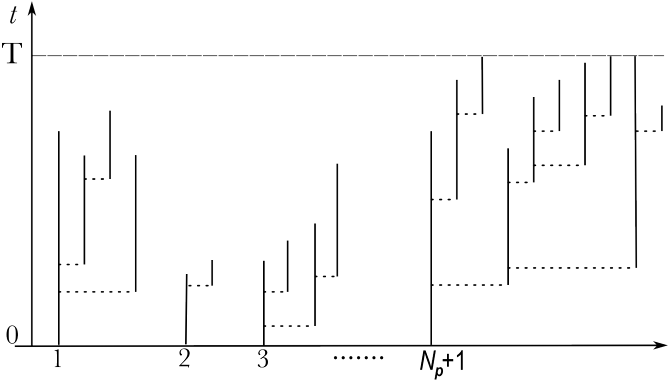

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For any forest $\mathcal{F}$, the population size process is denoted by,

$$(\xi_t(\mathcal{F}), t \geq 0)$$

Time-reversal duality

Fix $b \geq d$

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[Athreya and Ney 1972]

A supercritical BD process conditioned to die out is a subcritical BD process, obtained by swapping birth and death rates.

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[Athreya and Ney 1972]

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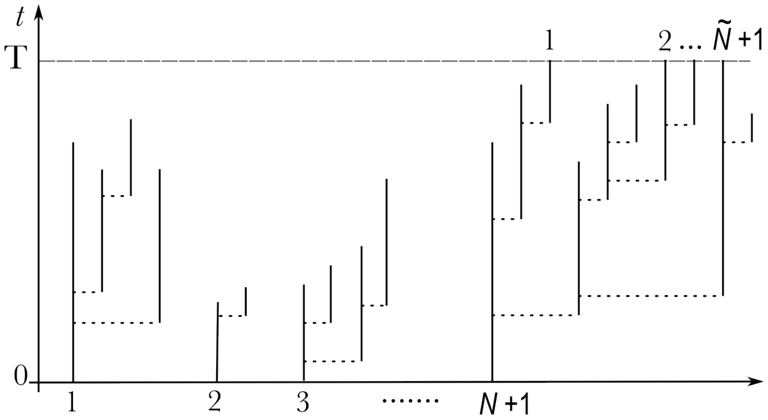
Theorem

We have the following identity in distribution,

$$(\xi_{T-t}(\mathcal{F}^*), 0 \leq t \leq T) \stackrel{d}{=} \left(\xi_t\left(\widetilde{\mathcal{F}}^*\right), 0 \leq t \leq T\right)$$

Time reversal duality



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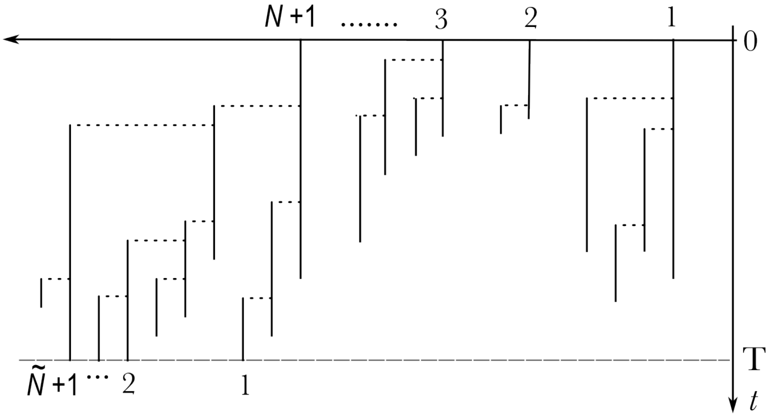

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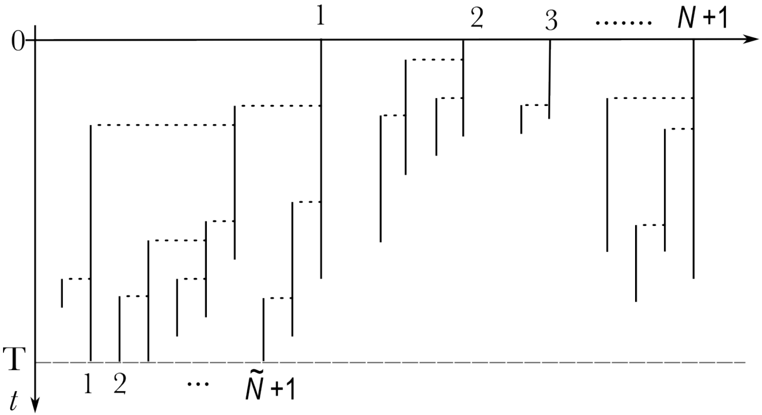

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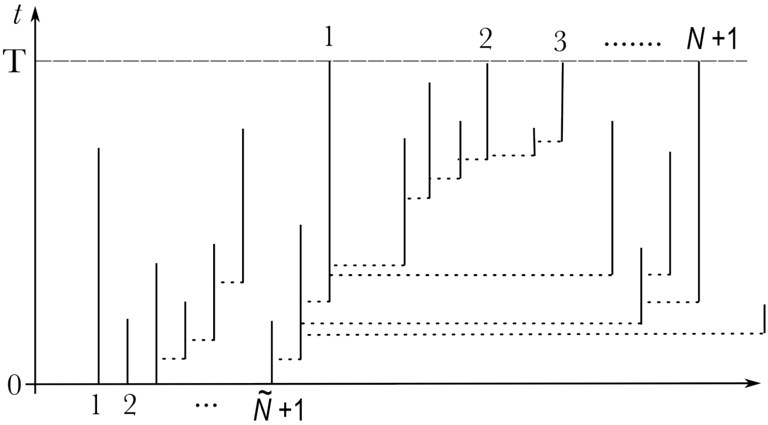

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Conditional on the reduced tree: applications in epidemiology

We want to characterize the population size process conditional on the coalescence times between individuals at present time T.

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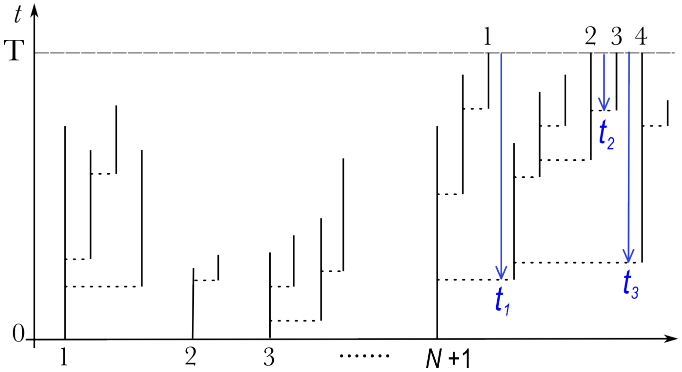

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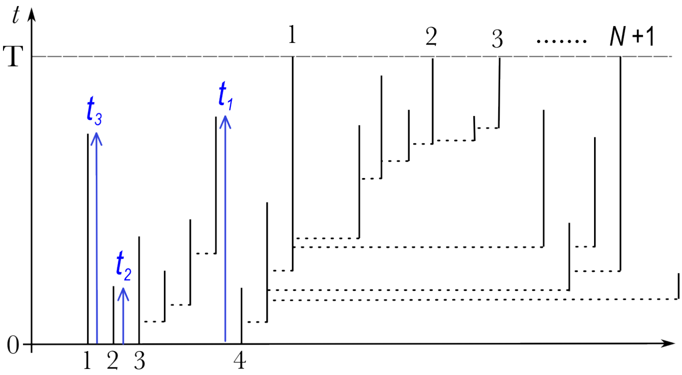
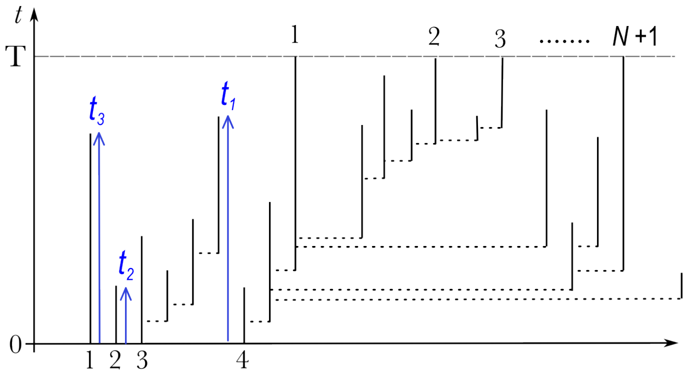
Conditional on the reduced tree: applications in epidemiology

When we return the time, thanks to the duality property, coalescence times become life durations



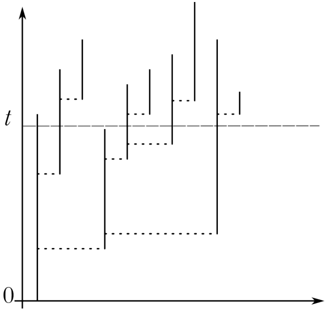
Conditional on the reduced tree: applications in epidemiology



Idea:

The population size process conditional on the coalescence times to be $t_1, \ldots, t_{\widetilde{N}+1}$, backward in time, is that of a sum of $\widetilde{N}$ BD trees, each conditioned on dying out before t_i for $1 \leq i \leq \widetilde{N}$, plus an additional tree conditioned on surviving up until time T.

Splitting trees



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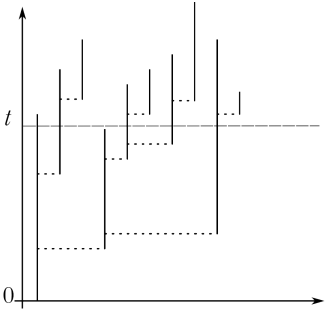

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A splitting tree is characterized by a σ-finite measure Π on $(0, \infty)$ satisfying $\int_{(0,\infty)}(1 \wedge r)\Pi(\mathrm{d}r) < \infty$ (the *lifespan measure*).

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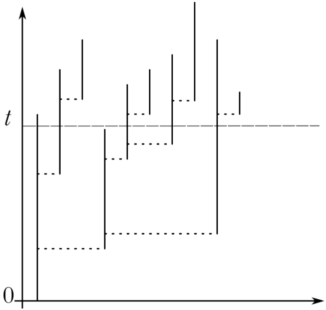

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We consider Π finite with mass b: individuals give birth at rate b and have life durations distributed as $\Pi(\cdot)/b$.

Time reversal duality for splitting trees

Define for Π:

- The Laplace exponent: $\psi(\lambda) := \lambda - \int_0^\infty \left(1 - \mathrm{e}^{-\lambda r}\right) \Pi(\mathrm{d}r),\ \lambda \geq 0$
- η the largest root of ψ
- A new measure $\widetilde{\Pi}(\mathrm{d}r) := \mathrm{e}^{-\eta r}\Pi(\mathrm{d}r)$

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The scale function W:

The unique continuous function $W : [0, +\infty) \to [0, +\infty)$, characterized by its Laplace transform,

$$\int_0^{+\infty} \mathrm{e}^{-\lambda x} W(x) = \frac{1}{\psi(\lambda)}, \qquad \lambda > \eta$$

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Define:

$$\gamma = \frac{1}{W(T)} \qquad \qquad \widetilde{\gamma} = \frac{1}{\widetilde{W}(T)}$$

Time reversal duality for splitting trees

Forest $\mathcal{F}^p$:

A sequence of i.i.d. splitting trees $(\mathcal{T}_1, \dots, \mathcal{T}_{N_p}, \mathcal{T}_{N_p+1}) \perp\!\!\!\perp N_p$, where,

- N_p: a geometric random variable with $\mathbb{P}(N_p = k) = (1-p)^k p$, $k \geq 0$
- $\mathcal{T}_1, \dots \mathcal{T}_{N_p}$: are conditioned on extinction before T
- $\mathcal{T}_{N_p+1}$: is conditionned on survival up until time T

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$\mathcal{F}^p_\top, \mathcal{F}^p_\perp$:

$\sim \mathcal{F}^p$, but lifetimes of the ancestors have a specific distribution $(\top, \perp)$, $\neq$ from $\Pi(\cdot)/b$

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$\mathcal{F}^p_\top, \mathcal{F}^p_\perp$:

$\sim \mathcal{F}^p$, but lifetimes of the ancestors have a specific distribution $(\top, \perp)$, $\neq$ from $\Pi(\cdot)/b$

Claim

$\mathcal{F}^{\tilde{\gamma}}_\perp$ = a sequence of i.i.d. splitting trees $(\perp, \Pi)$ stopped at the first tree having survived up to time T.

$\widetilde{\mathcal{F}}^{\gamma}_\top$ = a sequence of i.i.d. splitting trees $(\top, \widetilde{\Pi})$ stopped at the first tree having survived up to time T.

Time reversal duality for splitting trees

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$\mathcal{F}_{\perp}^{\widetilde{\gamma}}$ = a sequence of i.i.d. splitting trees $(\perp, \Pi)$ stopped at the first tree having survived up to time T.

$\widetilde{\mathcal{F}}_{\top}^{\gamma}$ = a sequence of i.i.d. splitting trees $(\top, \widetilde{\Pi})$ stopped at the first tree having survived up to time T.

Theorem

If the measure Π is supercritical (i.e. $m := \int_0^\infty r\Pi(\mathrm{d}r) > 1$) then,

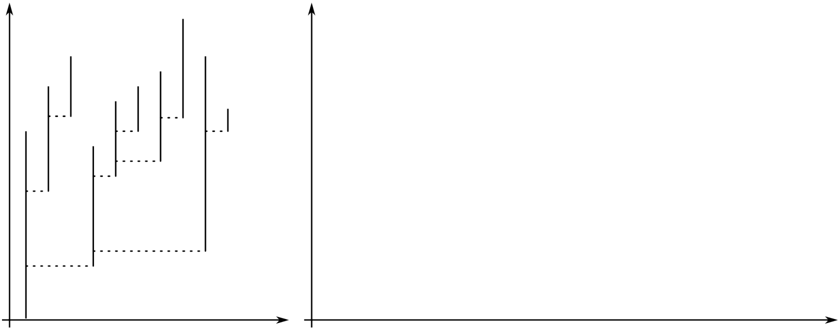
$$\left(\xi_{T-t}\left(\mathcal{F}_{\perp}^{\widetilde{\gamma}}\right), 0 \leq t \leq T\right) \overset{d}{=} \left(\xi_t\left(\widetilde{\mathcal{F}}_{\top}^{\gamma}\right), 0 \leq t \leq T\right)$$

In particular, if Π is subcritical, then,

$$\left(\xi_{T-t}\left(\mathcal{F}_{\perp}^{\gamma}\right), 0 \leq t \leq T\right) \overset{d}{=} \left(\xi_t\left(\mathcal{F}_{\top}^{\gamma}\right), 0 \leq t \leq T\right)$$

and actually in this case $\perp = \top$ since they have both density $\dfrac{\overline{\Pi}(r)}{m}\mathrm{d}r$.

The jumping chronological contour process



Example of a finite splitting tree and its contour process[1]

[1] Figure from C. Delaporte - Aussois 2013

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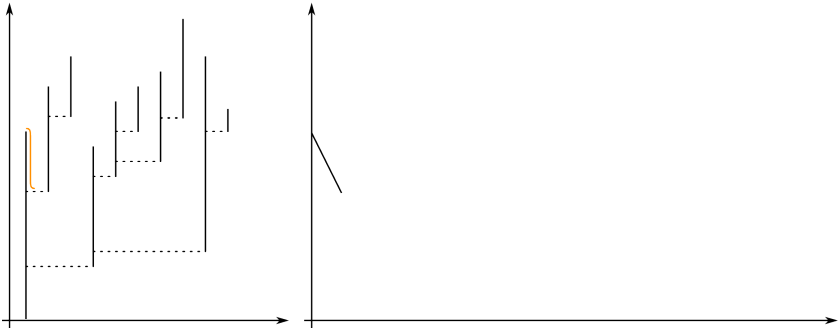

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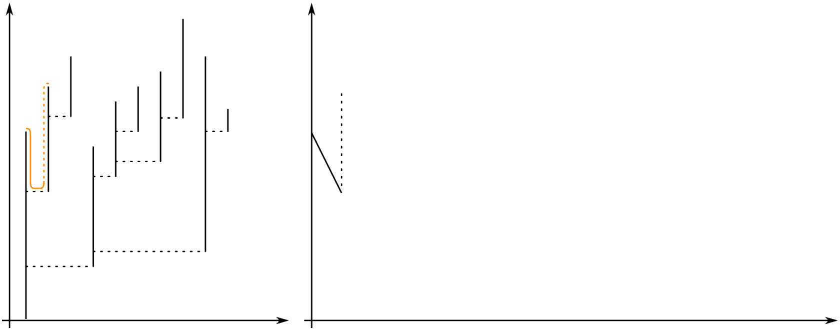

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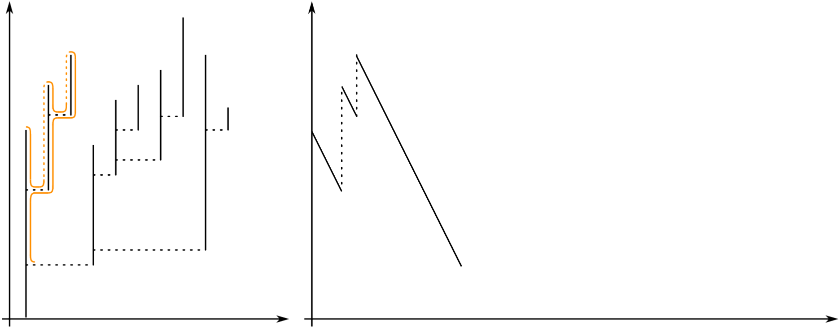

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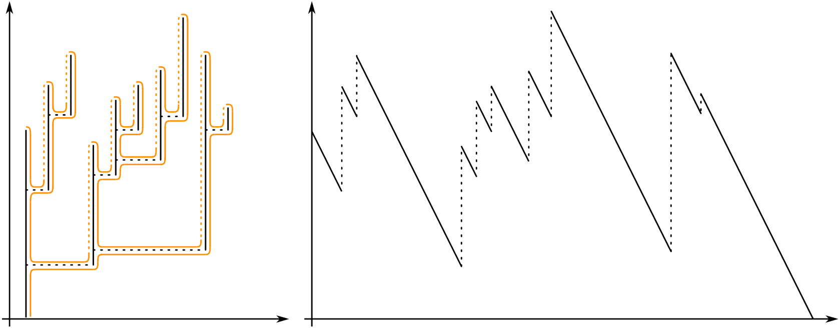

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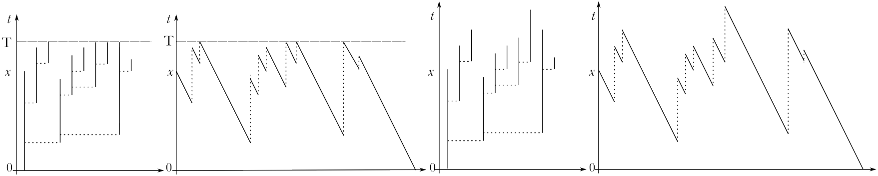
The contour of splitting trees is a Lévy process

Let Y be a a finite variation Lévy process with Lévy measure Π and drift -1.



Theorem [Lambert 2010]

Conditional on the lifespan of the ancestor to be x, the contour of $\mathcal{T}^{(T)}$, is distributed as Y, started at $x \wedge T$, reflected below T and killed upon hitting 0.

The contour of $\mathcal{T}$, conditional on extinction, has the law of Y started at x, conditioned on, and killed upon hitting 0.

Time reversal duality for spectrally positive Lévy processes

Theorem [Bertoin 1992]

The excursion measure has the following property of invariance under time reversal: under $\mathbb{P}_0\left(\cdot \middle| -Y_{(\tau_0^+)-} = u\right)$ the reverted excursion, $\left(-Y_{(\tau_0-t)-}, 0 \leq t < \tau_0\right)$ has the same distribution that $(Y_t, 0 \leq t < \tau_0)$ under $\mathbb{P}_u(\cdot|\tau_0 < +\infty)$.

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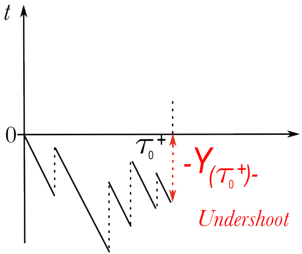

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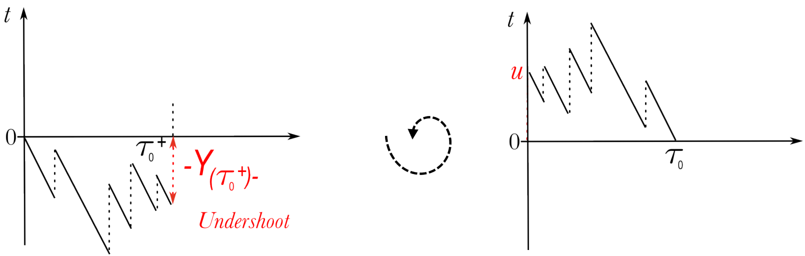

Time reversal duality for splitting trees

Define for a Lévy measure Π:

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- η the largest root of ψ
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Time reversal duality for splitting trees

Define for a Lévy measure Π:

- The Laplace exponent: $\psi(\lambda) := \lambda - \int_0^\infty \left(1 - \mathrm{e}^{-\lambda r}\right) \Pi(\mathrm{d}r),\ \lambda \geq 0$
- η the largest root of ψ
- A new measure $\widetilde{\Pi}(\mathrm{d}r) := \mathrm{e}^{-\eta r}\Pi(\mathrm{d}r)$
- $\tau_A = \in \{t \geq 0 : Y_t \in A\}$ the first hitting time of the real Borel set A

The scale function W:

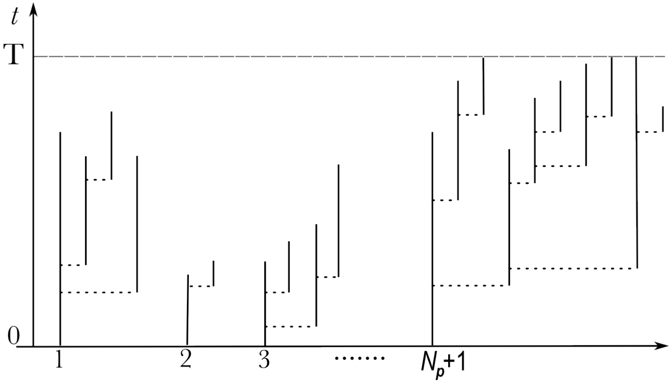
The unique continuous function $W : [0, +\infty) \to [0, +\infty)$, characterized by its Laplace transform,

$$\int_0^{+\infty} \mathrm{e}^{-\lambda x} W(x) = \frac{1}{\psi(\lambda)}, \qquad \lambda > \eta$$

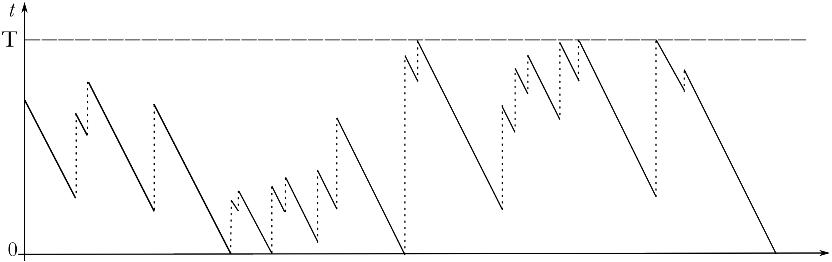
Define:

$$\gamma = \frac{1}{W(T)} = P_T\left(\tau_0 < \tau_T^+\right) \qquad \widetilde{\gamma} = \frac{1}{\widetilde{W}(T)} = \widetilde{P}_T\left(\tau_0 < \tau_T^+\right)$$

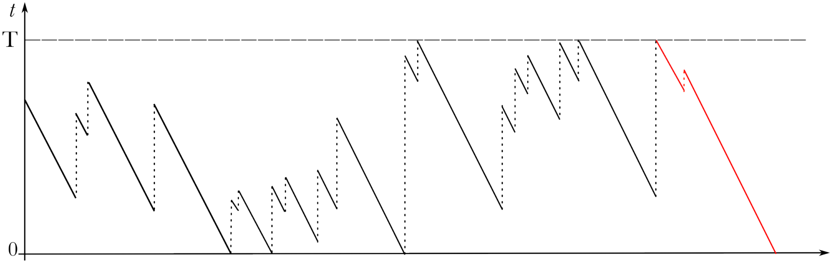
Contour of a forest



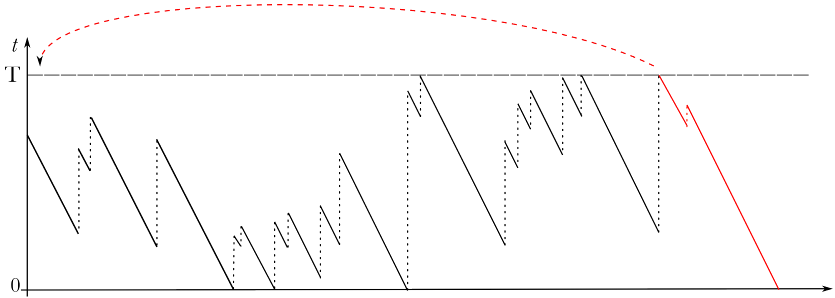
Contour of a forest



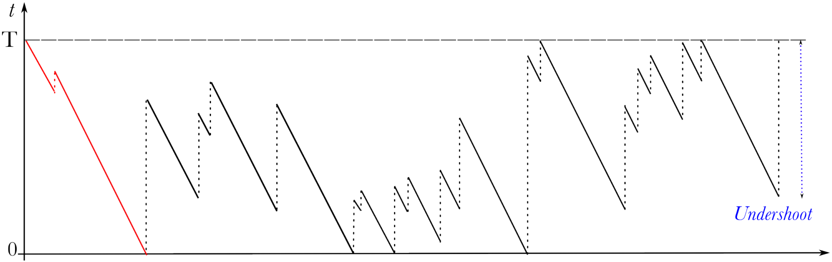
Contour of a forest



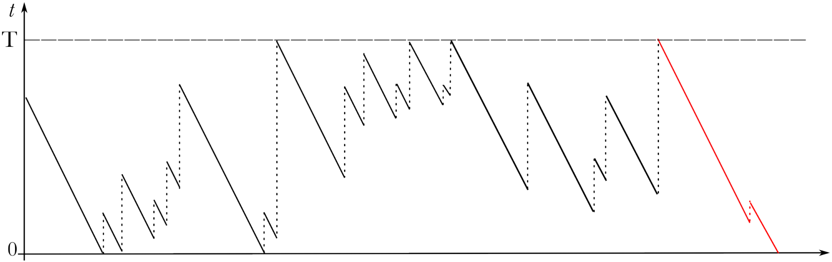
Contour of a forest



Contour of a forest



Contour of a forest



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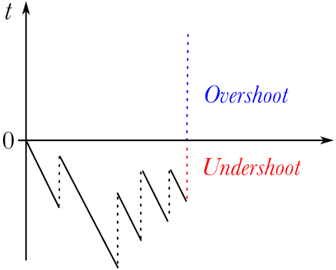
Thank You!

Undershoot and overshoot

$\mathcal{F}^p_\top, \mathcal{F}^p_\perp$:

Lifetimes of the ancestors have a specific distribution, different from $\Pi(\cdot)/b$:

The undershoot and overshoot at 0 of an excursion starting at 0 and conditional on $\tau_0^+ < +\infty$, are distributed as follows,



$$\text{Overshoot } (\perp): \sim \frac{\mathrm{e}^{\eta r}\overline{\widetilde{\overline{\Pi}}}(r)\mathrm{d}r}{m \wedge 1}$$

$$\text{Undershoot } (\top): \sim \frac{\mathrm{e}^{-\eta a}\overline{\Pi}(a)\mathrm{d}a}{m \wedge 1}$$