Branching processes evolving in asynchronous environment

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joint work with

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Galton-Watson processes in random environment

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- $\xi_j^{(n)} \stackrel{d}{=} \xi^{(n)}$ are i.i.d. **given** $f_0, f_1, \ldots$

Classification of Galton-Watson processes in **i.i.d.** random environment

$$\mathbf{E}\left[Z_n \middle| f_0, f_1, \dots\right] =: \mathbf{E}_f\left[Z_n\right]$$

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$$\mathbf{E}\left[Z_n \middle| f_0, f_1, \ldots\right] =: \mathbf{E}_f\left[Z_n\right] = \prod_{j=0}^{n-1} f_j'(1)$$

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$$= \quad e^{\sum_{j=0}^{n-1} \log f_j'(1)} = e^{S_n}$$

- $X_i := \log f_{i-1}'(1)$
- $S_0 = 0, \qquad S_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1.$

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Associated random walk

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- supercritical if $\lim_{n\to\infty} S_n = +\infty$ with probability 1;
- subcritical if $\lim_{n\to\infty} S_n = -\infty$ with probability 1;
- critical if $\limsup_{n\to\infty} S_n = +\infty$ and $\liminf_{n\to\infty} S_n = -\infty$ (both with probability 1);

Critical case includes

$$\mathbf{E} \log f'(1) = \mathbf{E} X = 0, \qquad 0 < \mathbf{E} X^2 < \infty$$

Quenched approach: Let

$$T := \min\{k : Z_k = 0\}.$$

We study

$$\mathbf{P}_f(Z_n > 0) = \mathbf{P}_f(T > n) = \mathbf{P}(T > n | f_0, f_1, ...),$$

as a random variable on the space of realizations of the environment $f_0, f_1, ...$

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as a random variable on the space of realizations of the environment $f_0, f_1, ...$ and

$$\mathbf{P}_f(Z_n \in dx \,|\, T > n)$$

as a random conditional law.

Basic assumptions

The distribution of $X = \log f'(1)$ is nonlattice and belongs (without centering) to an α-stable law with $\alpha \in (0, 2]$ and

$$\rho = \mathbf{P}(X > 0) \in (0, 1).$$

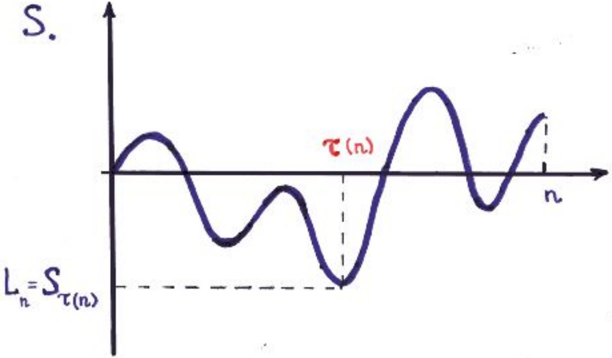
Set

$$M_n := \max\left(S_1, ..., S_n\right), \quad L_n := \min\left(S_1, ..., S_n\right)$$

and let

$$\mu(n) := \max\left\{1 \le i \le n : S_i = M_n\right\},$$

$$\tau(n) := \min\left\{0 \le i \le n : S_i = \min(0, L_n)\right\}.$$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It is known that if X belongs (without centering) to an α-stable law with $\rho \in (0,1)$ then for any $t \in (0,1]$, as $n \to \infty$

$$\frac{\tau(nt)}{n} \xrightarrow{d} \tau_t, \quad \frac{\mu(nt)}{n} \xrightarrow{d} \mu_t,$$

where $t^{-1}\tau_t$ and $t^{-1}\mu_t$ have the **generalized** arcsine distributions on $[0,1]$ with densities

$$\frac{C}{x^\rho(1-x)^{1-\rho}}, \qquad \frac{C}{x^{1-\rho}(1-x)^\rho},$$

respectively.

Let

$$\gamma_0 := 0, \quad \gamma_{j+1} := \min(n > \gamma_j : S_n < S_{\gamma_j})$$

and

$$\Gamma_0 := 0, \quad \Gamma_{j+1} := \min(n > \Gamma_j : S_n > S_{\Gamma_j}), \, j \geq 0,$$

be the strict descending and ascending ladder epochs of the random walk $\{S_n, n \geq 0\}$.

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be the strict descending and ascending ladder epochs of the random walk $\{S_n, n \geq 0\}$. Set

$$U(x) = 1 + \sum_{j=1}^{\infty} \mathbf{P}(S_{\Gamma_j} < x),\ x \geq 0, \quad U(x) = 0, x < 0,$$

$$V(x) = \sum_{j=0}^{\infty} \mathbf{P}(S_{\gamma_j} \geq -x),\ x \geq 0, V(x) = 0,\ x < 0.$$

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For any bounded, $\mathcal{F}_n$-measurable random variable R_n we construct probability measures $\mathbf{P}_x^-, x \leq 0,$ and $\mathbf{P}_y^+, y \geq 0,$ fulfilling for each n the equalities

$$\mathbf{E}_x^- \left[R_n\right] := \frac{1}{U(-x)} \mathbf{E}_x \left[R_n U(-S_n); M_n < 0\right]$$

and

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The measures specified in this way are consistent in n.

In particular, the measure $\mathbf{P}_0^-$ "corresponds"to the measure generated by the random walk conditioned to stay **negative**, while the measure $\mathbf{P}_0^+$ "corresponds"to the measure generated by the random walk conditioned to stay **nonnegative.**

Basic assumption:

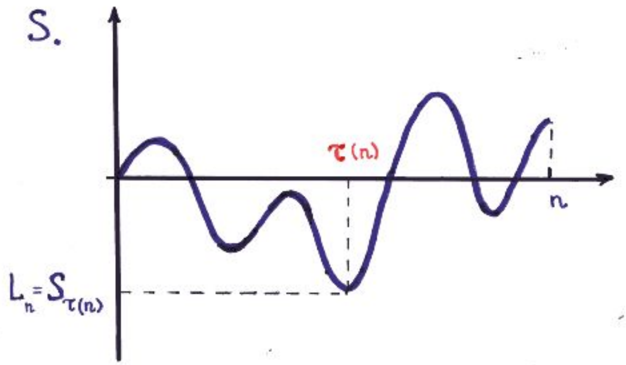
- **FROM NOW ON!!!**

$$f_n(s) = \frac{q_n}{1 - p_n s},\ n = 0, 1, 2, \ldots$$

(V. + Dyakonova)
In the critical case and **for the quenched approach** we have the following convergence in distribution

$$\frac{\mathbf{P}_f(Z_n > 0)}{e^{S_{\tau(n)}}} \xrightarrow{d} \zeta$$

where $\zeta \in (0, 1]$ with probability 1.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$\Rightarrow$ if $\sigma^2 = VarX < \infty$

$$\mathbf{P}_f(Z_n > 0) \approx \zeta e^{S_{\tau(n)}} = \zeta e^{\sigma\sqrt{n} \times \frac{1}{\sigma\sqrt{n}} S_{\tau(n)}} \approx \zeta e^{-|g|\sigma\sqrt{n}}$$

where g is a Gaussian random variable:

$$\mathbf{P}\left(g < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

Let now $S_0^{\pm} = 0$ and

$$\left(S_0^+, S_1^+, ..., S_n^+, ...\right)$$

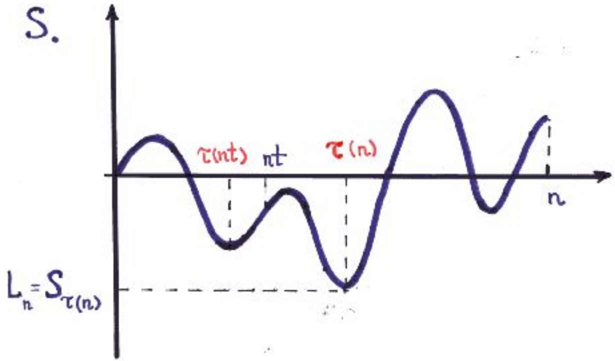
be a random walk conditioned to stay positive and

$$\left(S_1^1, ..., S_n^-, ...\right)$$

be a random walk conditioned to stay negative. Then

$$\frac{1}{\zeta} \stackrel{d}{=} \sum_{k=0}^{\infty} e^{-S_k^+} + \sum_{k=1}^{\infty} e^{S_k^-}.$$

THE NUMBER OF PARTICLES BEFORE THE POINT OF GLOBAL MINIMUM OF $S_k, 0 \leq k \leq n$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Observe that

$$\mathbf{E}_f\left[Z_{nt} | Z_{nt} > 0\right] = \frac{e^{S_{nt}}}{\mathbf{P}_f\left(Z_{nt} > 0\right)} \asymp e^{S_{nt} - S_{\tau(nt)}}$$

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Theorem 1. **(V. + Dyakonova)**

For any $t \in (0,1)$, any $x > 0$ as $n \to \infty$

$$\mathbf{P}_f \left(\frac{Z_{nt}}{\mathbf{E}_f \left[Z_{nt} | Z_{nt} > 0 \right]} \le x \middle| T > n;\ nt < \tau(n) \right) \xrightarrow{d} \int_0^x y e^{-y} dy.$$

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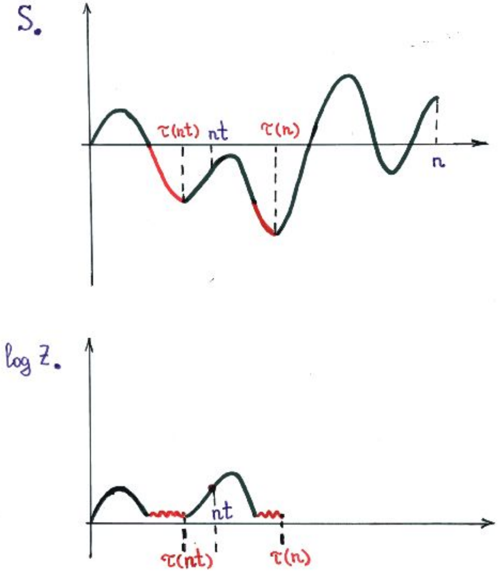
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BOTTLENECKS!!! earlier the time-point $\tau(n)$ of the global minimum of S_k within the interval $[0,n]$.

WHAT HAPPENS WITH THE PROCESS AFTER THE POINT OF GLOBAL MINIMUM??

THE NUMBER OF PARTICLES AFTER THE POINT OF GLOBAL MINIMUM OF $S_k, 0 \le k \le n$

Theorem 2. **(V. + Dyakonova)**

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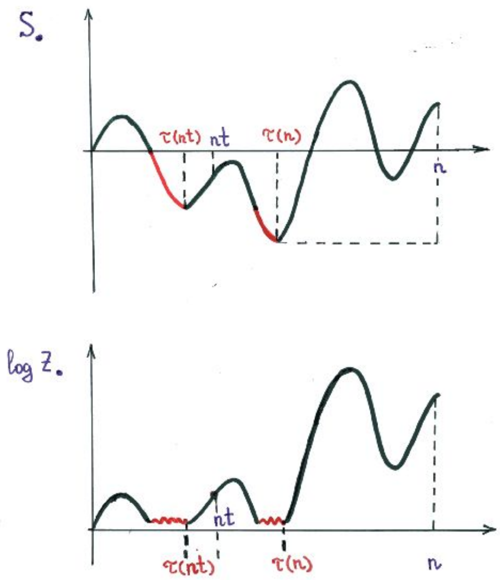

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Non rigorously: if $T > n$, $\tau(n) \leq nt$ $\Rightarrow$

$$\ln Z_{nt} \asymp S_{nt} - S_{\tau(n)}.$$

$\Rightarrow$ **NO tragic bottlenecks AFTER the moment $\tau(n)$.**

What happens at bottlenecks if $T > n$??

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If the probability generating functions are fractional linear then as $n \to \infty$

$$\mathbf{P}_f\left(Z_{\tau(nt)} = k \middle| \quad T>n; \quad nt<\tau(n) \quad \right) \xrightarrow{d} kq^2p^{k-1}, \quad k = 1, 2, ...$$

and

$$\mathbf{P}_f\left(Z_{\tau(n)} = k \middle| \quad T>n \quad \right) \xrightarrow{d} \tilde{q}\tilde{p}^{k-1}, \quad k = 1, 2, ...$$

where $q + p = 1, \tilde{q} + \tilde{p} = 1$ and $p, \tilde{p}$ are positive random variables.

BPRE with two types.

Let $\xi^{(1)}, \xi^{(2)}, \ldots;$ are i.i.d. random variables with (random) probability generating functions

$$f_n(s) \ := \ \mathbf{E}\left[s^{\xi^{(n)}} | f_n\right], \qquad 0 \leq s \leq 1$$

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Define a **pure decomposable two-types** BPRE (Z_n, Y_n) as

$$Z_n := \sum_{j=1}^{Z_{n-1}} \xi_j^{(n)}, \; Y_n := \sum_{j=1}^{Y_{n-1}} \eta_j^{(n)}$$

where, given the environment, Z_n and Y_n are ordinary inhomogeneous Galton-Watson branching processes.

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BASIC ASSUMPTION: Asynchronous environments (Predator (Z)-Pray (Y) competition). For any $k = 0, 1, 2, ...$

$$W_{k+1} = -X_k \quad \mathbf{P}-\text{a.s.}$$

Introduce two random walks

$$S_0 := 0, \quad S_n := \sum_{k=1}^{n} X_k, \quad R_0 := 0, \quad R_n := \sum_{k=2}^{n+1} W_k.$$

Clearly, $S_n = -R_n, n = 0, 1, \ldots$

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CRITICALITY.

The random walk $\{S_n, n \geq 0\}$ is oscillating.

$\Rightarrow$ The branching processes $Z(n), Y(n)$ are **critical**.

Denote

$$\zeta_n := \frac{\mathbf{P}_f\left(Z_n > 0\right)}{e^{S_{\tau(n)}}}, \quad \theta_n := \frac{\mathbf{P}_g\left(Y_n > 0\right)}{e^{R_{\tau(n)}}} = e^{S_{\mu(n)}}\mathbf{P}_g\left(Y_n > 0\right)$$

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Theorem

(V.+Liu). *As* $n \to \infty$

$$\left(\zeta_n, \theta_n\right) \xrightarrow{d} \left(\zeta, \theta\right),$$

where the random variables ζ *and* θ *are* ***independent*** *and take values in* $(0, 1]$ *with probability* 1*.*

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$\Rightarrow$

$$\mathbf{P}_f\left(Z_n > 0\right) \asymp e^{S_{\tau(n)}}, \qquad \mathbf{P}_g\left(Y_n > 0\right) \asymp e^{R_{\tau(n)}} = e^{-S_{\mu(n)}}$$

NOW

$$\mathbf{E}_f\left[Z_{nt} | Z_{nt} > 0\right] = \frac{e^{S_{nt}}}{\mathbf{P}_f\left(Z_{nt} > 0\right)} \asymp e^{S_{nt} - S_{\tau(nt)}}$$

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For $t \in (0, 1]$ consider

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For $t\in(0,1]$ consider

$$\mathcal{L}_{f,g}\left(\frac{Z_{nt}}{\mathbf{E}_f\left[Z_{nt}|Z_{nt}>0\right]}\leq z,\frac{Y_{nt}}{\mathbf{E}_g\left[Y_{nt}|Y_{nt}>0\right]}\leq y\middle|Z_n>0,Y_n>0\right)$$

$$=\mathcal{L}_f\left(\frac{Z_{nt}}{\mathbf{E}_f\left[Z_{nt}|Z_{nt}>0\right]}\leq z\middle|Z_n>0\right)\times\mathcal{L}_g\left(\frac{Y_{nt}}{\mathbf{E}_g\left[Y_{nt}|Y_{nt}>0\right]}\leq y\middle|Y_n>0\right)$$

For $t \in (0,1]$ we have **(V.+Liu)**

$$\mathcal{L}_{f,g}\left(\frac{Z_{nt}}{\mathbf{E}_f\left[Z_{nt}|Z_{nt}>0\right]} \le z, \frac{Y_{nt}}{\mathbf{E}_g\left[Y_{nt}|Y_{nt}>0\right]} \le y \middle| Z_n > 0, Y_n > 0\right)$$
$$\xrightarrow{p} \left(1-e^{-z}, 1-e^{-y}\right) I\left\{\max\left\{\tau_1, \mu_1\right\} < t\right\}$$

For $t \in (0,1]$ we have **(V.+Liu)**

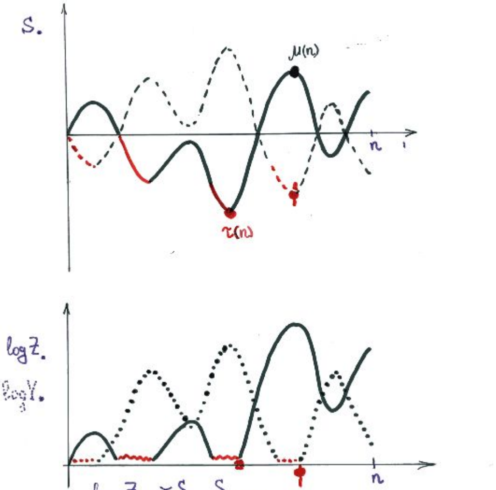
$$\mathcal{L}_{f,g}\Big(\frac{Z_{nt}}{\mathbf{E}_f\left[Z_{nt}|Z_{nt}>0\right]} \le z, \frac{Y_{nt}}{\mathbf{E}_g\left[Y_{nt}|Y_{nt}>0\right]} \le y \Big| Z_n > 0, Y_n > 0\Big)$$
$$\xrightarrow{p} \left(1-e^{-z}, 1-e^{-y}\right) I\left\{\max\left\{\tau_1, \mu_1\right\} < t\right\}$$
$$+ \left(1-e^{-z}, \int_0^y we^{-w} dw\right) I\left\{\tau_1 < t < \mu_1\right\}$$

For $t \in (0,1]$ we have **(V.+Liu)**

$$\mathcal{L}_{f,g}\left(\frac{Z_{nt}}{\mathbf{E}_f\left[Z_{nt}|Z_{nt}>0\right]} \leq z, \frac{Y_{nt}}{\mathbf{E}_g\left[Y_{nt}|Y_{nt}>0\right]} \leq y \middle| Z_n > 0, Y_n > 0\right)$$
$$\xrightarrow{p} \left(1-e^{-z}, 1-e^{-y}\right) I\left\{\max\left\{\tau_1, \mu_1\right\} < t\right\}$$
$$+\left(1-e^{-z}, \int_0^y we^{-w}dw\right) I\left\{\tau_1 < t < \mu_1\right\}$$
$$+\left(\int_0^z we^{-w}dw, 1-e^{-y}\right) I\left\{\mu_1 < t < \tau_1\right\}$$

For $t \in (0,1]$ we have **(V.+Liu)**

$$\begin{aligned}
\mathcal{L}_{f,g}\Bigg(\frac{Z_{nt}}{\mathbf{E}_f\left[Z_{nt}|Z_{nt}>0\right]} \le z, \frac{Y_{nt}}{\mathbf{E}_g\left[Y_{nt}|Y_{nt}>0\right]} \le y \Big| Z_n>0, Y_n>0\Bigg) \\
\xrightarrow{p} \left(1-e^{-z}, 1-e^{-y}\right) I\left\{\max\left\{\tau_1,\mu_1\right\}<t\right\} \\
+\left(1-e^{-z}, \int_0^y we^{-w}dw\right) I\left\{\tau_1<t<\mu_1\right\} \\
+\left(\int_0^z we^{-w}dw, 1-e^{-y}\right) I\left\{\mu_1<t<\tau_1\right\} \\
+\left(\int_0^z we^{-w}dw, \int_0^y we^{-w}dw\right) I\left\{\min\left\{\tau_1,\mu_1\right\}>t\right\}
\end{aligned}$$



$$\log Z_{nt} \asymp S_{nt} - S_{\tau(nt)}$$

$$\log Y_{nt} \asymp S_{\mu(nt)} - S_{nt}$$

WHAT HAPPENS AT THE POINTS OF LOCAL MINIMA OF THE ASSOCIATED RANDOM WALK ON THE INTERVAL

$$[0, n]$$

??

For $k = 1, 2, ...$ and $y > 0$ let

$$\mathcal{L}_{f,g}\left(Z_{\tau(nt)} = k, \frac{Y_{\tau(nt)}}{\mathbf{E}_g\left[Y_{\tau(nt)}|Y_{\tau(nt)} > 0\right]} \le y|Z_n > 0, Y_n > 0\right)$$

For $k = 1, 2, ...$ and $y > 0$ let

$$\mathcal{L}_{f,g}\left(Z_{\tau(nt)} = k, \frac{Y_{\tau(nt)}}{\mathbf{E}_g\left[Y_{\tau(nt)} | Y_{\tau(nt)} > 0\right]} \leq y | Z_n > 0, Y_n > 0\right)$$

$$= \mathcal{L}_f\left(Z_{\tau(nt)} = k | Z_n > 0\right) \times \mathcal{L}_g\left(\frac{Y_{\tau(nt)}}{\mathbf{E}_g\left[Y_{\tau(nt)} | Y_{\tau(nt)} > 0\right]} \leq y | Y_n > 0\right)$$

For $k = 1, 2, ...$ and $y > 0$ we have **(V.+Liu)**

$$\mathcal{L}_{f,g}\left(Z_{\tau(nt)} = k, \frac{Y_{\tau(nt)}}{\mathbf{E}_g\left[Y_{\tau(nt)}|Y_{\tau(nt)} > 0\right]} \leq y|Z_n > 0, Y_n > 0\right)$$
$$\xrightarrow{p} \left(\tilde{q}\tilde{p}^{k-1}, 1 - e^{-y}\right) I\left\{\mu_1 < \tau_t = \tau_1\right\}$$

For $k = 1, 2, \ldots$ and $y > 0$ we have **(V.+Liu)**

$$\mathcal{L}_{f,g}\left(Z_{\tau(nt)} = k, \frac{Y_{\tau(nt)}}{\mathbf{E}_g\left[Y_{\tau(nt)} | Y_{\tau(nt)} > 0\right]} \leq y | Z_n > 0, Y_n > 0\right)$$

$$\xrightarrow{p} \left(\tilde{q}\tilde{p}^{k-1}, 1 - e^{-y}\right) I\left\{\mu_1 < \tau_t = \tau_1\right\}$$

$$+ \left(\tilde{q}\tilde{p}^{k-1}, \int_0^y we^{-w} dw\right) I\left\{\mu_1 > \tau_t = \tau_1\right\}$$

For $k = 1, 2, \ldots$ and $y > 0$ we have **(V.+Liu)**

$$\mathcal{L}_{f,g}\left(Z_{\tau(nt)} = k, \frac{Y_{\tau(nt)}}{\mathbf{E}_g\left[Y_{\tau(nt)} | Y_{\tau(nt)} > 0\right]} \leq y | Z_n > 0, Y_n > 0\right)$$

$$\xrightarrow{p} \left(\tilde{q}\tilde{p}^{k-1}, 1 - e^{-y}\right) I\left\{\mu_1 < \tau_t = \tau_1\right\}$$

$$+ \left(\tilde{q}\tilde{p}^{k-1}, \int_0^y we^{-w}dw\right) I\left\{\mu_1 > \tau_t = \tau_1\right\}$$

$$+ \left(k\tilde{q}^2\tilde{p}^{k-1}, 1 - e^{-y}\right) I\left\{\mu_1 < \tau_t < \tau_1\right\}$$

For $k = 1, 2, ...$ and $y > 0$ we have **(V.+Liu)**

$$\mathcal{L}_{f,g}\left(Z_{\tau(nt)} = k, \frac{Y_{\tau(nt)}}{\mathbf{E}_g\left[Y_{\tau(nt)}|Y_{\tau(nt)} > 0\right]} \leq y|Z_n > 0, Y_n > 0\right)$$

$$\begin{aligned}
&\xrightarrow{p} \left(\tilde{q}\tilde{p}^{k-1}, 1 - e^{-y}\right) I\left\{\mu_1 < \tau_t = \tau_1\right\} \\
&+ \left(\tilde{q}\tilde{p}^{k-1}, \int_0^y we^{-w}dw\right) I\left\{\mu_1 > \tau_t = \tau_1\right\} \\
&+ \left(k\tilde{q}^2\tilde{p}^{k-1}, 1 - e^{-y}\right) I\left\{\mu_1 < \tau_t < \tau_1\right\} \\
&+ \left(k\tilde{q}^2\tilde{p}^{k-1}, \int_0^y we^{-w}dy\right) I\left\{\tau_t < \min\left\{\tau_1, \mu_1\right\}\right\}
\end{aligned}$$

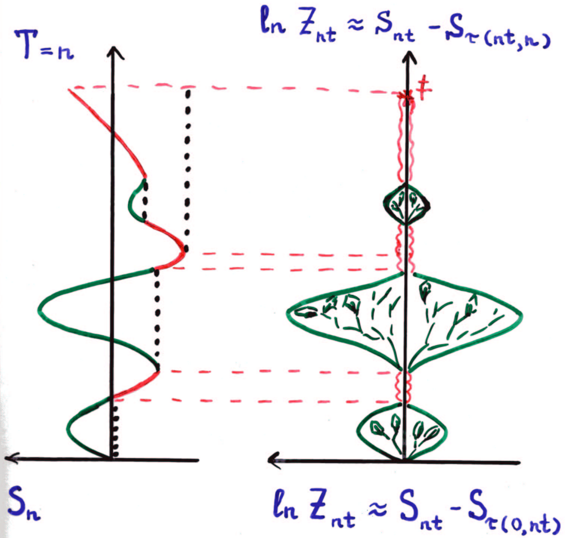
Variations:

1)

$$Y_n > 0, \qquad Z_{n-1} = 0, \quad Z_n = 0;$$

2) Correlation $\in (-1, 1)$;

3) Several types of individuals (> 2).



$\ln Z_{nt} \approx S_{nt} - S_{\tau(nt,n)}$
$T = n$
t
S_n
$\ln Z_{nt} \approx S_{nt} - S_{\tau(0,nt)}$

THANKS!