Law of Large numbers for the SIR epidemic on a random graph with given vertex degrees

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Soon to be on arXiv.

ANR Modèles aléatoires en écologie, génétique et évolution meeting, Paris 13, 30th January 2013

Overview

1. Define SIR epidemic model
2. Introduce family of random graphs
3. Subcritical regime
4. Supercritical regime; limiting evolution and final size
5. Sketch proof

SIR epidemic model on a graph

Idea: model an infectious disease (influenza, measles, ...) spreading through a population (e.g. of people, computers, ...).

Each individual is either **susceptible**, **infective** or **recovered**.

Classical formulations (Reed-Frost, ...) assume anyone can infect anyone.

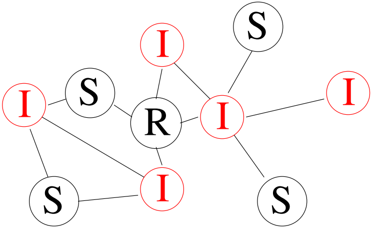
But populations usually have local structure limiting possible transmissions.

SIR epidemic model on a graph

Individuals = vertices in a graph G ('network').

Edge = potential transmissions.

Each vertex is either susceptible, infective or recovered.

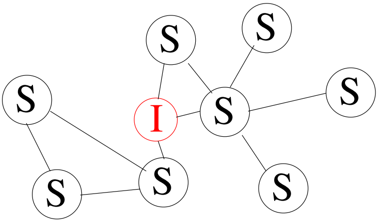


Continuous time Markovian dynamics

Epidemic evolves stochastically in time.

Infective vertices infect each susceptible neighbour at rate $\beta > 0$ and recover at rate $\rho \geq 0$. I.e. $S \to I \to R$.

No other transitions are possible.



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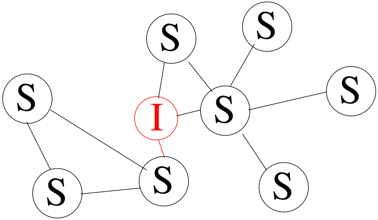

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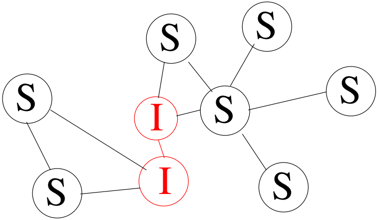

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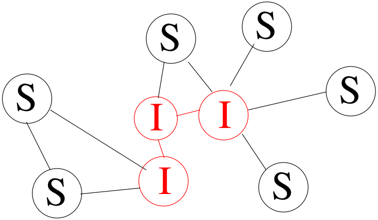

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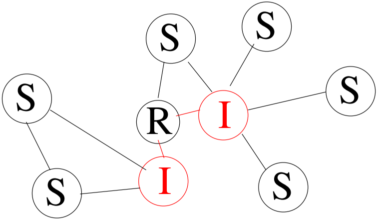

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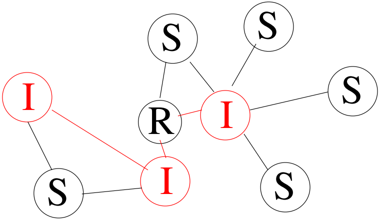

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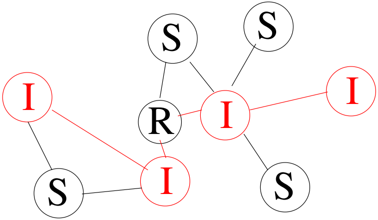


Take G = random graph

Behaviour and applicability of model depend on G.

Want $G \approx$ real world networks.

These are complicated and hard to capture.

But important features (small worlds, clustering, ...) are captured by *random graphs*.

Natural to study epidemics on random graphs. Thomas House has a recent survey (2012).

Random graphs are also interesting mathematically $\ddot\smile$

Random graph with given vertex degrees

Fix $n \in \mathbb{N}$. *Dependence on n will not be indicated explicitly.*

Let $(d_i)_{i=1}^n$ be a given sequence of positive integers.

$G \sim$ uniform over all graphs with n vertices s.t. vertex i has degree d_i

This is a flexible family of random graphs. Popular with theorists and practicioners alike.

Theory parallels that of Erdos-Renyi $G(n, p)$ graphs: giant component; k-core; chromatic number, matching number, ...

Previous studies of SIR epidemic on this graph: Newman '02, Volz '07, Miller '11, Decreusefond-Dhersin-Moyal-Tran '12, Bohman-Picollelli '12.

Notation

Let: $n_k = \#\{i : d_i = k\} = \#$ vertices of degree $k \geq 0$.

n_k^{S}, n_k^{I}, $n_k^{\mathrm{R}} = \#$ those that are initially susceptible, infective, recovered, resp.

$n^{\mathrm{S}} = \sum_k n_k^{\mathrm{S}} = \#$ initially susceptible vertices, ...

Assumptions

Assumptions on asymptotics of the degree sequence *(ALL limits are as $n \to \infty$)*:

D1) $n^{\mathrm{S}}/n \to \alpha_{\mathrm{S}} \in (0,1]$, $n^{\mathrm{I}}/n \to \alpha_{\mathrm{I}} \in [0,1]$, $n^{\mathrm{R}}/n \to \alpha_{\mathrm{R}} \in [0,1]$.

D2) $n_k^{\mathrm{S}}/n^{\mathrm{S}} \to p_k$, $k \geq 0$; and $\lambda := \sum_k kp_k < \infty$.

D3) $\sum_k kn_k^{\mathrm{S}}/n^{\mathrm{S}} \to \lambda$. [$\iff$ uniform integrability of degree of a randomly chosen susceptible].

D4) $\sum_k kn_k/n \to \mu$, $\sum_k kn_k^{\mathrm{I}}/n \to \mu_{\mathrm{I}}$, $\sum_k kn_R/n \to \mu_{\mathrm{R}}$.

D5) $\sum_i d_i^2 = O(n)$ [$\implies$ $\max_i d_i = O(n^{1/2})$. Also $\implies$ D3!]

Subcritical regime

Conditions to guarantee epidemic stays small:

Theorem

Suppose $\mu_{\mathrm{I}} = 0$ *and*

$$R_0 := \left(\frac{\beta}{\rho+\beta}\right)\left(\frac{\alpha_{\mathrm{S}}\lambda}{\mu}\right)\frac{\sum_k k(k-1)p_k}{\sum_k kp_k} \leq 1.$$

Then the number Z *of initially susceptible vertices that ever get infected is* $o_p(n)$, *i.e.* $\mathbb{P}(Z/n > \epsilon) \to 0$ *for any* $\epsilon > 0$.

Threshold identified heuristically by Newman '02 and Volz '07. Rigorous result by Bohman/Picollelli '12 for **bounded** degree sequences (and $\mu_{\mathrm{R}} = 0$).

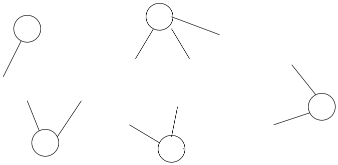
Proving results is easier on a multigraph

Working with the uniform simple graph is hard.

Instead we consider the following *multigraph* (loops and multiple edges are allowed):

Take n vertices and attach d_i half edges ('stubs') to vertex i. Pair the half edges uniformly at random to form complete edges.

E.g. $\mathbf{d} = (1, 2, 3, 2, 2)$



Called the *configuration model* [Canfield & Bender, Bollobas, ..] .

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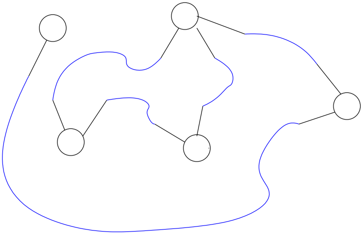

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Why is it sufficient to consider this multigraph?

Conditional on simplicity, it is UNIFORM.

Our results say $\mathbb{P}(A) \to 1$ as $n \to \infty$ for some event A

Thus,

$$\mathbb{P}(A, G \text{ is simple}) = \mathbb{P}(G \text{ is simple}) + o(1),$$

and

$$\mathbb{P}(A|G \text{ is simple}) = \frac{\mathbb{P}(G \text{ is simple}) + o(1)}{\mathbb{P}(G \text{ is simple})} = 1 + o(1)$$

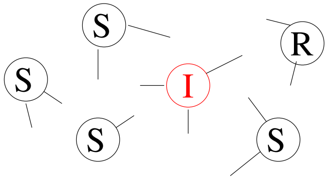
provided e.g. $\liminf_{n\to\infty} \mathbb{P}(G \text{ is simple}) > 0$.

Assumption (D5) guarantees this [Janson].

Results hold on multigraph if (D5) is replaced with $\max_i d_i = o(n)$

Evolving graph and epidemic process

Can reveal edges in G dynamically, as required by epidemic process:



Each infective half edge fires at rate β. It pairs up with a uniformly sampled half edge. If that half edge belongs to a susceptible vertex, then that vertex becomes infective.

Also each infective vertex recovers at rate ρ.

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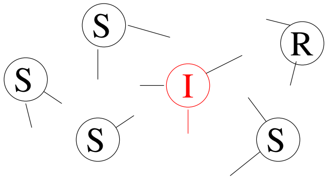

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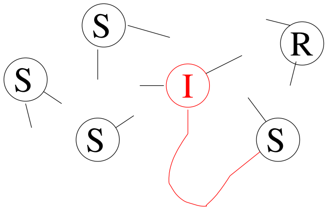

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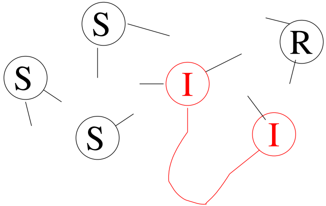

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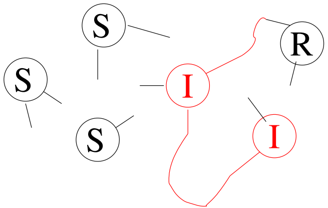

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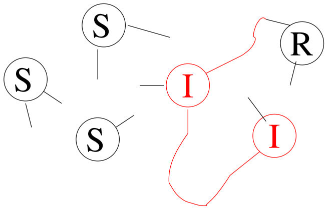

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Infective pressure = # infective half edges

Branching process approximation in early stages

Infective half edges 'reproduce' by pairing with a susceptible half edge.

Mean off spring =

$\mathbb{P}$(half edge fires before recovering) $\times$ $\mathbb{P}$(it hits a susceptible) $\times$ $\mathbb{E}$[other half edges attached to the susceptible]

$$\approx \left(\frac{\beta}{\rho+\beta}\right)\left(\frac{\alpha_{\mathrm{S}}\lambda}{\mu}\right)\left(\frac{\sum_k (k-1)kp_k}{\sum_k kp_k}\right) =: R_0.$$

If mean offspring < 1 then branching process dies out almost surely!

Supercritical regime

$\mu_{\mathrm{I}} > 0$ means many initially infective half edges.

Epidemic is guaranteed to take off.

Let X_t = total # of half edges at time $t \geq 0$.

$X_{\mathrm{S},t}$ = # susceptible half edges, $X_{\mathrm{I},t}$ = # infective half edges, etc.

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Limiting evolution is governed by

$$f_{\mathrm{S}}(\theta) := \alpha_{\mathrm{S}} \sum_k k\theta^k p_k, \quad f_{\mathrm{R}}(\theta) := \mu_{\mathrm{R}}\theta + \mu\frac{\rho}{\beta}\theta(1-\theta), \quad f_{\mathrm{X}}(\theta) := \mu\theta^2,$$

and $f_{\mathrm{I}}(\theta) := f_{\mathrm{X}}(\theta) - f_{\mathrm{S}}(\theta) - f_{\mathrm{R}}(\theta)$, $0 \leq \theta \leq 1$,

$\theta = \theta_t$ = suitable function of time.

Theorem

Suppose $\mu_{\mathrm{I}} > 0$.

(a) f_{I} *has a unique root* $\theta_\infty \in (0,1)$. *Further,* $f_{\mathrm{I}}(\theta) > 0$ *for* $\theta \in (\theta_\infty, 1]$.

(b) $\exists!\ \theta_t : [0,\infty) \to (\theta_\infty, 1]$ *s.t.* $\theta_0 = 1$ *and*

$$\frac{d}{dt}\theta_t = -\beta f_{\mathrm{I}}(\theta_t)(\theta_t / f_{\mathrm{X}}(\theta_t))$$

Interpretation: $\theta_t = \mathbb{P}$*(that a given susceptible half edge has not received infection).*

(c) *Let* $S_t = \#$ *susceptible vertices. Then uniformly in probability*[1]

$$S_t/n \overset{\mathrm{u.p.}}{\to} \alpha_{\mathrm{S}} \sum_k p_k \theta_t^k, \quad X_{\mathrm{S},t}/n \overset{\mathrm{u.p.}}{\to} \alpha_{\mathrm{S}} \sum_k k p_k \theta_t^k = f_{\mathrm{S}}(\theta_t).$$

[1] 'Uniformly in probability' $\overset{\mathrm{u.p.}}{\to}$ means $\sup_{t\geq 0} |X_{\mathrm{S},t}/n - f_{\mathrm{S}}(\theta_t)| \overset{\mathrm{p}}{\to} 0$ etc

(d) Further,

$$X_t\,/n \overset{\text{u.p.}}{\to} f_{\mathrm{X}}(\theta_t),\ X_{\mathrm{I},t}\,/n \overset{\text{u.p.}}{\to} f_{\mathrm{I}}(\theta_t),$$

$$X_{\mathrm{R},t}\,/n \overset{\text{u.p.}}{\to} f_{\mathrm{R}}(\theta_t).$$

If I_t, R_t denote the number of infective and recovered then

$$I_t/n \overset{\text{u.p.}}{\to} \alpha_{\mathrm{I}}(t),\ R_t/n \overset{\text{u.p.}}{\to} \alpha_{\mathrm{R}}(t),$$

where $\alpha_{\mathrm{I}}(0) = \alpha_{\mathrm{I}}$, $\alpha_{\mathrm{R}}(0) = \alpha_{\mathrm{R}}$, and

$$\alpha_{\mathrm{I}}'(t) = \beta f_{\mathrm{I}}(\theta_t)\frac{f_{\mathrm{S}}(\theta_t)}{f_{\mathrm{X}}(\theta_t)} - \rho\alpha_{\mathrm{I}}(t),\ \ \alpha_{\mathrm{R}}'(t) = \rho\alpha_{\mathrm{I}}(t).$$

(d) $Z = \#$ susceptible vertices that ever get infected satisfies

$$Z/n^{\mathrm{S}} \overset{\mathrm{p}}{\to} 1 - \sum_k p_k\theta_\infty^k.$$

Credits

Newman '02: identified the final size heuristically.

Volz '07: differential equations heuristic; let $g_{\mathrm{S}}(\theta) = \sum_k \theta^k p_k$, $\mathrm{p_I}(\theta) = f_{\mathrm{I}}(\theta)/f_{\mathrm{X}}(\theta)$, $\mathrm{p_S}(\theta) = f_{\mathrm{S}}(\theta)/f_{\mathrm{X}}(\theta)$

$$\frac{d\mathrm{p_I}(\theta_t)}{dt} = \mathrm{p_I}(\theta_t)\left(-(\rho+\beta) + \beta\mathrm{p_I}(\theta_t) + \beta\mathrm{p_S}(\theta_t)\theta_t\frac{g''_{\mathrm{S}}(\theta_t)}{g'_{\mathrm{S}}(\theta_t)}\right),$$

$$\frac{d\mathrm{p_S}(\theta_t)}{dt} = \beta\mathrm{p_I}(\theta_t)\mathrm{p_S}(\theta_t)\left(1 - \theta_t\frac{g''_{\mathrm{S}}(\theta_t)}{g'_{\mathrm{S}}(\theta_t)}\right).$$

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Decreusefond-Dhersin-Moyal-Tran '12: proved a related result involving measure valued processes

- assume **fifth** moment bound for degree distributions.

Super criticality with few initially infectives

Suppose $\mu_{\mathrm{I}} = 0$ and $R_0 > 1$.

Similar result holds. However, (a) epidemic may die before taking off (b) time to infect ϵn vertices may be random.

Theorem

Suppose $\mu_{\mathrm{I}} = 0$, $R_0 > 1$ *[and* $p_1 > 0$ *or* $\rho > 0$ *and* $\mu_{\mathrm{R}} > 0$*].*

Take $\epsilon, \delta > 0$. *Then with high probability, either:*

(a) *less than* ϵn $\#$ *vertices ever get infected OR*

(b) *After a random time*

$$T_{\uparrow} = \inf\{t \geq 0 : X_{\mathrm{I},t} > \epsilon n\},$$

the epidemic becomes macroscopic and time shifted versions of the concentration statements hold;
$\sup_{t \geq 0} |X_{T_{\uparrow}+t}/n - f_{\mathrm{X}}(\theta_t)| < \delta$, ... *where*
$\frac{d}{dt}\theta_t = -\beta\theta_t f_{\mathrm{I}}(\theta_t)/f_{\mathrm{X}}(\theta_t)$ *and* θ_0 *is s.t.* $f_{\mathrm{I}}(\theta_0) = \epsilon$.

In progress: (b) occurs with probability bounded away from zero.

Bohman/Picollelli '12: proved for **bounded** degree sequences.

N.B. If $\rho = \mu_{\mathrm{R}} = \alpha_{\mathrm{R}} = 0$ then the macroscopic epidemic occupies the entire giant component [Molloy-Reed '97].

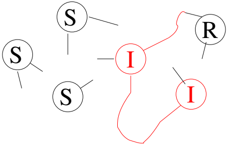
Application to vaccination

Qn. Why bother allowing $\alpha_{\mathrm{R}} > 0$?

Ans. Suppose vertices can be vaccinated prior to epidemic (or as soon as they become infective).

Vaccinated vertices behave same as recovered vertices in SIR dynamics.

Sketch proof



Epidemic stops spreading once $X_{\mathrm{I},t} = 0$. Stop the process then.

Susceptible half edges get fired at by infective half edges.

Speed everything up so that each susceptible half edge is hit at unit rate.

I.e. if there are x_I infective half edges and x half edges in total then multiply rates by $\frac{x-1}{\beta x_I}$.

Denote new time variable by τ.

Each susceptible vertex of degree $k \geq 0$ is now infected with rate k; i.e. has life time $\sim$ Exponential(k).

So $S_\tau(k) = \#$ susceptible vertices of degree $k =$

$$S_\tau(k) = \sum_{i=1}^{n_k^{\mathrm{S}}} \mathbb{1}_{L_i > \tau},$$

where L_i, $i = 1, \ldots, n_k^{\mathrm{S}}$ are i.i.d Exponential(k).

Glivenko-Cantelli Lemma[2]: the empirical CDF

$$\frac{1}{n_k^{\mathrm{S}}} \sum_{i=1}^{n_k^{\mathrm{S}}} \mathbb{1}_{L_i > \tau} \overset{\text{u.p.}}{\rightarrow} \mathbb{P}(L_i > \tau) = \exp(-k\tau)$$

as $n_k^{\mathrm{S}} \to \infty$.

[2] a corollary of law of large numbers

Then, by our assumptions,

$$S_\tau(k)/n \overset{\text{u.p.}}{\to} \alpha_{\mathrm{S}} p_k \exp(-k\tau),$$

$$S_\tau/n = \sum_k S_\tau(k)/n \overset{\text{u.p.}}{\to} \alpha_{\mathrm{S}} \sum_k p_k \exp(-k\tau),$$

and

$$X_{\mathrm{S},\tau}/n = \sum_k kS_\tau(k)/n \overset{\text{u.p.}}{\to} \alpha_{\mathrm{S}} \sum_k kp_k \exp(-k\tau) = f_{\mathrm{S}}(e^{-\tau}).$$

[The summation here relies on the uniform integrability assumption (D3)!]

Total number of half edges

Consider: $\Delta X_\tau = -2$ whenever an infective half edge fires.

In new time scale: this happens at rate

$$\beta X_{\mathrm{I},\tau} \times \left(\frac{X_\tau - 1}{\beta X_{\mathrm{I},\tau}} \right) = X_\tau - 1$$

This is near enough X_τ after dividing by n.

Glivenko-Cantelli again gives

$$X_\tau / n \overset{\mathrm{u.p.}}{\to} \mu e^{-2\tau} = f_{\mathrm{X}}(e^{-\tau}).$$

Now for $X_{\mathrm{R},\tau}$;

$$\begin{aligned} X_{\mathrm{R},\tau} &= X_{\mathrm{R},0} + \int_0^\tau \left(-\beta X_{\mathrm{I},\sigma} \frac{X_{\mathrm{R},\sigma}}{X_\sigma - 1} + \rho \sum_k k I_\sigma(k) \right) \left(\frac{X_\sigma - 1}{\beta X_{\mathrm{I},\sigma}} \right) d\sigma + M_\tau \\ &= X_{\mathrm{R},0} + \int_0^\tau \left(-\beta X_{\mathrm{I},\sigma} \frac{X_{\mathrm{R},\sigma}}{X_\sigma - 1} + \rho X_{\mathrm{I},\sigma} \right) \left(\frac{X_\sigma - 1}{\beta X_{\mathrm{I},\sigma}} \right) d\sigma + M_\tau \end{aligned}$$

$$\frac{X_{\mathrm{R},\tau}}{n} = \frac{X_{\mathrm{R},0}}{n} + \int_0^\tau \left(-\frac{X_{\mathrm{R},\sigma}}{n} + \frac{\rho}{\beta}\left(\frac{X_\sigma - 1}{n}\right)\right) d\sigma + M_\tau / n.$$

M_τ is a finite variation Martingale;

$$\begin{aligned} [M]_\tau = \sum_{\sigma \le \tau} (\Delta M_\sigma)^2 &= \sum_{\sigma \le \tau} (\Delta X_{\mathrm{R},\sigma})^2 \\ &\le X_0 + \sum_i d_i^2 \\ &\le X_0 + (\max_i d_i) \sum_i d_i = o(n^2) \end{aligned}$$

$$\frac{X_{\mathrm{R},\tau}}{n} = \frac{X_{\mathrm{R},0}}{n} + \int_0^\tau \left(-\frac{X_{\mathrm{R},\sigma}}{n} + \frac{\rho}{\beta}\left(\frac{X_\sigma - 1}{n}\right)\right) d\sigma + M_\tau/n.$$

Doob's inequality says

$$\mathbb{E}\sup_t |M_t|^2 \le 4\,\mathbb{E}[M]_\infty = o(n^2).$$

I.e. $M_\tau/n \overset{\text{u.p.}}{\to} 0$.

Gronwall's inequality then shows

$$X_{\mathrm{R},\tau} \overset{\mathrm{p}}{\to} f_{\mathrm{R}}(e^{-\tau})$$

uniformly on any bounded interval.

$X_{\mathrm{I},\tau} = X_\tau - X_{\mathrm{S},\tau} - X_{\mathrm{I},\tau}$ so $X_{\mathrm{I},\tau} \stackrel{\text{u.p.}}{\to} f_{\mathrm{I}}(e^{-\tau})$ on bounded intervals.

But then $X_{\mathrm{I},\tau} = 0$ for some $\tau < -\ln(\theta_\infty) + 1$.

Now invert the time change:

Can show $\tau(t)$ is the inverse of the (increasing) process

$$\int_0^\tau \frac{X_\sigma - 1}{\beta X_{\mathrm{I},\sigma}} d\sigma \overset{\text{u.p.}}{\to} \int_0^\tau \frac{1}{\beta \mathrm{p}_{\mathrm{I}}(e^{-\sigma})} d\sigma, \ 0 \leq \tau \leq -\ln(\theta_\infty) - \delta.$$

[This is for $\mathrm{p}_{\mathrm{I}}(e^{-0}) = \mathrm{p}_{\mathrm{I}}(1) = \mu_{\mathrm{I}}/\mu > 0$; $\mu_{\mathrm{I}} = 0$ is more delicate]

So $\tau(t) \overset{\text{u.p.}}{\to} \hat{\tau}(t)$, where $\hat{\tau}'(t) = \beta \mathrm{p}_{\mathrm{I}}(\exp(-\hat{\tau}(t)))$, and $\hat{\tau}(0) = 0$.

Thus $\theta_t = \exp(-\hat{\tau}(t))$ satisfies $\frac{d}{dt}\theta_t = -\beta \mathrm{p}_{\mathrm{I}}(\theta_t)\theta_t$.

Work in progress: describe early stages in more detail

[- If $R_0 > 1$ then a positive fraction is infected with probability bounded above zero]

- $T_{\uparrow} / \ln(n) \xrightarrow{\mathrm{p}} c > 0$

Thanks! Any questions?