Multitype GWBP (decomposable and indecomposable) (Lecture 2)

8 июня 2011 г.

Offspring generating functions for the type i particles:

$$f^i(\mathbf{s}) = f^i(s_1, ..., s_N) = \mathbf{E} s_1^{\xi_{i1}} ... s_N^{\xi_{iN}}$$

Denote

$$\mathbf{f}(\mathbf{s}) = (f^1(\mathbf{s}), ..., f^N(\mathbf{s})).$$

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The mean matrix

$$\mathbf{M} = \|m_{ij}\|_{i,j=1}^N, \ m_{ij} = \frac{\partial f^i(\mathbf{s})}{\partial s_j}|_{\mathbf{s}=\mathbf{1}} = \mathbf{E}\xi_{ij}, \qquad \mathbf{M}^n = \left\|m_{ij}^{(n)}\right\|_{i,j=1}^N.$$

$$\mathbf{F}(n,\mathbf{s}) = (F_1(n,\mathbf{s}), ..., F_N(n,\mathbf{s})),$$

where

$$F^i(n,\mathbf{s}) \quad = \quad \mathbf{E}\left[s_1^{Z_{i1}(n)}...s_N^{Z_{iN}(n)}|Z_l(0)=\delta_{il}\right]$$

and $Z_{il}(n)$ is the number of particles of type l in the process at time n steamed from a single particle of type i existing at moment 0.

In terms of the particle reproduction

$$Z_j(n) = \sum_{k=1}^{N} \sum_{r=1}^{Z_k(n-1)} \xi_{kj}(r),$$

where $Z_k(n-1)$ is the number of particles in the process at time $n-1$ and

$$(\xi_{1j}(r), ..., \xi_{Nj}(r)) \overset{dist}{=} (\xi_{1j}, ..., \xi_{Nj}), r = 1, 2, ...$$

are offspring vectors of different particles which are iid for each $j = 1, \ldots, N$.

Therefore,

$$\begin{aligned} F_i(n,\mathbf{s}) &= \mathbf{E}\left[\prod_{j=1}^{N} s_j^{Z_j(n)} | Z_l(0)=\delta_{il}\right] \\ &= \mathbf{E}\left[\prod_{k=1}^{N} \prod_{r=1}^{Z_k(n-1)} \prod_{j=1}^{N} s_j^{\xi_{kj}(r)} | Z_l(0)=\delta_{il}\right] \\ &= \mathbf{E}\left[\prod_{k=1}^{N} \left(f^k(\mathbf{s})\right)^{Z_k(n-1)} | Z_l(0)=\delta_{il}\right] = F_i(n-1,\mathbf{f}(\mathbf{s})) \\ &= \ldots = F_i(0,\mathbf{f}_n(\mathbf{s})) = f_n^i(\mathbf{s}). \end{aligned}$$

or

$$\mathbf{F}(n,\mathbf{s}) = \mathbf{f}_n(\mathbf{s}), \qquad \mathbf{f}_n(\mathbf{s}) = \mathbf{f}\left(\mathbf{f}_{n-1}(\mathbf{s})\right).$$

Therefore,

$$\begin{aligned}
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&= \mathbf{E}\left[\prod_{k=1}^{N}\left(f^k(\mathbf{s})\right)^{Z_k(n-1)} | Z_l(0)=\delta_{il}\right] = F_i(n-1,\mathbf{f}(\mathbf{s})) \\
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Hence, the vector $\mathbf{P}$ of probabilities of extinction solves the equation

$$\mathbf{P} := \lim_{n\to\infty}\mathbf{f}_{n+1}(\mathbf{0}) = \mathbf{f}\left(\lim_{n\to\infty}\mathbf{f}_n(\mathbf{0})\right) = \mathbf{f}(\mathbf{P}).$$

Mean matrix for the population size at time n

$$\mathbf{E}\left[Z_{ij}(n)|Z_l(0)=\delta_{il}, l=1,...,N\right]=\frac{\partial f_n^i(\mathbf{s})}{\partial s_j}|_{\mathbf{s}=\mathbf{1}}=m_{ij}^{(n)}.$$

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Thus, if $\mathbf{Z}(n)=(Z_1(n),...,Z_N(n))$ then

$$\mathbf{E}\left[Z_j(n)|\mathbf{Z}(n-1)\right]=\sum_{k=1}^{N}\sum_{r=1}^{Z_k(n-1)}\mathbf{E}\xi_{kj}(r)=\sum_{k=1}^{N}Z_k(n-1)m_{kj}$$

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or, in the vector form

$$\mathbf{E}\left[\mathbf{Z}(n)|\mathbf{Z}(n-1)\right]=\mathbf{Z}(n-1)\mathbf{M}$$

leading to

$$\mathbf{E}\left[\mathbf{Z}(n)\right]=\mathbf{E}\left[\mathbf{Z}(n-1)\right]\mathbf{M}=...=\mathbf{E}\left[\mathbf{Z}(0)\right]\mathbf{M}^n.$$

Indecomposable processes

$1, ..., N$-types. We write that $i \rightarrow j$ if there exists n such that

$$\mathbf{P}\left(Z_j(n) > 0 | Z_l(0) = \delta_{il}, l = 1, ..., N\right) > 0.$$

If $i \rightarrow j$ and $j \rightarrow i$ then the types are connected. If all types in the process are connected the process is called **indecomposable**.

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We assume that the process is NONperiodic and NONsingular, that is $\mathbf{f}(\mathbf{s})$ not of the form

$$\mathbf{f}(\mathbf{s}) = \mathbf{M}\mathbf{s}^T.$$

We say that a matrix $\mathbf{M}$ is strictly positive if there is $n > 0$ such that

$$\mathbf{M}^n = \left\| m_{ij}^{(n)} \right\|_{i,j=1}^{N} > \mathbf{0}.$$

Theorem

(Perron-Frobenius) Any strictly positive matrix $\mathbf{M}$ *has a maximal simple eigenvalue* ρ *which has positive right and left eigenvectors*

$$\mathbf{u} = (u_1, ..., u_d) \text{ and } \mathbf{v} = (v_1, ..., v_N) :$$

$$\mathbf{vM} = \rho\mathbf{v}, \ \mathbf{Mu} = \rho\mathbf{u}$$

If they are scaled in such a way that $(\mathbf{v}, \mathbf{u}) = 1$, *and*

$$(\mathbf{v}, \mathbf{1}) = \sum_{k=1}^{N} v_k = 1$$

then $\mathbf{M}^n = \rho^n\mathbf{S} + \mathbf{T}^n$ *where* $\mathbf{S} = \|u_i v_j\|_{i,j=1}^{N}$ *and* $\mathbf{T}^n = \left\| t_{ij}^{(n)} \right\|$

$$\mathbf{ST} = \mathbf{TS} = 0$$

and $|t_{ij}|^{(n)} \leq c\rho_0^n$ *with some* $0 < \rho_0 < \rho$.

Let $P^{(i)}$ is the probability of extinction if the process starts by one individual of type i, $\mathbf{P} = (P^{(1)}, ..., P^{(d)})$.

Theorem

Let $\mathbf{Z}(n)$ be positively regular: $\mathbf{M}^{n_0} > \mathbf{0}$, and nonsingular. Then
1) if $\rho \leq 1$ then the process dies out with probability 1
2) if $\rho > 1$ then $\mathbf{P} < \mathbf{1}$,

$$\lim_{n\to\infty} \mathbf{f}_n(\mathbf{0}) = \mathbf{P},$$

and $\mathbf{P}$ is the only solution of $\mathbf{f}(\mathbf{s}) = \mathbf{s}$ within the unit cube.

Classification:
supercritical, critical, subcritical processes depending on the value of ρ.

Example 1. Let

$$\begin{aligned} f_1(s_1, s_2) &= \tfrac{1}{3} + \tfrac{1}{3}s_1 + \tfrac{1}{3}s_2^2 \\ f_2(s_1, s_2) &= \tfrac{1}{4} + \tfrac{1}{2}s_1 + \tfrac{1}{4}s_2^2 \end{aligned}$$

Then

$$\mathbf{M} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

the process is critical $\mathbf{P} = (1, 1)$.

Example 2. Let

$$\begin{array}{rcl} f_1(s_1,s_2) & = & \frac{1}{4}+\frac{1}{4}s_1+\frac{1}{2}s_2^2 \\ f_2(s_1,s_2) & = & \frac{1}{8}+\frac{1}{8}s_1+\frac{3}{4}s_2^2 \end{array}$$

Then

$$\mathbf{M}=\begin{pmatrix} \frac{1}{4} & 1 \\ \frac{1}{8} & \frac{3}{2} \end{pmatrix}$$

the process is supercritical. The eigenvalues of the matrix

$$\rho := \frac{1}{8}\sqrt{33}+\frac{7}{8} = 1.5931 \text{ and } \frac{7}{8}-\frac{1}{8}\sqrt{33},$$

Solving

$$\begin{array}{rcl} s_1 & = & \frac{1}{4}+\frac{1}{4}s_1+\frac{1}{2}s_2^2 \\ s_2 & = & \frac{1}{8}+\frac{1}{8}s_1+\frac{3}{4}s_2^2 \end{array}$$

we get

$$(1,1) \text{ and } \left(\frac{9}{25},\frac{1}{5}\right).$$

Hence

$$\mathbf{P}=\left(\frac{9}{25},\frac{1}{5}\right).$$

Composition of population

Assume that

$$\mathbf{E}\left[\mathbf{Z}(0)\right] = c\mathbf{v}$$

Then

$$\mathbf{E}\left[\mathbf{Z}(n)\right] = \mathbf{E}\left[\mathbf{Z}(0)\right]\mathbf{M}^n = c\mathbf{v}\mathbf{M}^n = c\rho^n\mathbf{v}$$

Therefore, for each $k = 1, ..., N$

$$\lim_{n\to\infty} \frac{\mathbf{E}\left[Z_k(n)\right]}{\mathbf{E}\left[Z_1(n) + ... + Z_N(n)\right]} = \frac{v_k}{v_1 + ... + v_N} = v_k.$$

In the general case for any initial population size we have by the Perron-Frobenius theorem

$$\mathbf{E}\left[\mathbf{Z}(n)\right] = \mathbf{E}\left[\mathbf{Z}(0)\right]\mathbf{M}^n = \mathbf{E}\left[\mathbf{Z}(0)\right]\left(\rho^n \mathbf{S} + \mathbf{T}^n\right) \approx \rho^n \mathbf{E}\left[\mathbf{Z}(0)\right]\mathbf{S}.$$

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Therefore,

$$\mathbf{E}\left[Z_k(n)\right] \approx \rho^n \sum_{i=1}^{N} \mathbf{E}\left[Z_i(0)\right] u_i v_k = \rho^n v_k \sum_{i=1}^{N} \mathbf{E}\left[Z_i(0)\right] u_i.$$

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Hence, expectations of the numbers of particles of different types grow like ρ^n. In particular,

$$\lim_{n\to\infty} \frac{\mathbf{E}\left[Z_k(n)\right]}{\mathbf{E}\left[Z_1(n) + \ldots + Z_N(n)\right]} = \frac{v_k}{v_1 + \ldots + v_N} = v_k.$$

Generation overlap

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If these probabilities are assumed to be independent of age, the population can be modeled as a single-type Galton-Watson process with reproduction generating function

$$f(s) = r + qs + ps^2.$$

The expected number of offspring per individual thus equals $2p + q$ and thus the expected population size in the nth season is $(2p + q)^n$. If $2p + q > 1$ then the expected population size increases exponentially.

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The probability generating functions looks as follows

$$f^0(s_0, s_1, ..., s_N) = r + qs_0 + ps_0s_1,$$

$$f^k(s_0, s_1, ..., s_N) = r + qs_k + ps_0s_{k+1}, 1 \leq k < N,$$

$$f^N(s_0, s_1, ..., s_N) = r + (q + ps_0)s_N.$$

The mean $(N+1)\times(N+1)$ matrix of the process looks as follows:

$$\mathbf{M} = \begin{pmatrix} p+q & p & 0 & 0 & \cdots & 0 \\ p & q & p & 0 & \cdots & 0 \\ p & 0 & q & p & 0 & \cdots \\ \cdots & 0 & 0 & \cdots & p & 0 \\ \cdots & \cdots & \cdots & \cdots & q & p \\ p & 0 & 0 & \cdots & 0 & p+q \end{pmatrix}.$$

The process is indecomposable. The maximal eigenvalue of the matrix, ρ, is $2p+q$ and the corresponding (normalized) transposed right eigenvector has the form

$$\mathbf{u}^T = (1, 1, \ldots, 1)$$

while the left eigenvector (properly normalized and transposed) equals

$$\mathbf{v}^T = (v_0, \ldots, v_k, \ldots, v_{N-1}, v_N) = \left(\frac{1}{2}, \ldots, \frac{1}{2^{k+1}}, \ldots, \frac{1}{2^N}, \frac{1}{2^N}\right).$$

By the Perron-Frobenius theorem the 'parity distribution' of the population tends to

$$\lim_{n\to\infty} \frac{m_{0k}^{(n)}}{m_{00}^{(n)} + m_{01}^{(n)} + \cdots + m_{0N}^{(n)}} = \frac{v_k}{v_0 + \cdots + v_N} = \frac{1}{2^{k+1}},$$

for $k = 0, \ldots, N-1$ and

$$\lim_{n\to\infty} \frac{m_{0N}^{(n)}}{m_{00}^{(n)} + m_{01}^{(n)} + \cdots + m_{0N}^{(n)}} = \frac{1}{2^N}.$$

This result does not depend on the particular values of p and q.

Example 4. Consider a bi-annual plant species. If it were strictly bi-annual, individuals that are one year old produce no seeds and have a positive chance p_1 of survival. Two-year olds would always produce seeds and die. Since such plant species usually have enormous amounts of seeds, only few of which germinate, a Poisson offspring distribution seems natural.

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In most instances, bi-annual plant species do not follow the strict "two-years" rule , but instead there is only a chance that they flower, and then die, when they are two years old. If they do not flower, they can survive for another year (up to a certain maximum age), but as soon as they have produced flowers they die.

Model: Suppose that the maximum age of the plants is 3 years.

- 1-year old individuals survive with chance p_1.
- 2 years old individuals flower with chance q_2 and if they do not, they survive with probability p_2. If they do produce flowers they get a Poisson(λ_2) distributed number of offspring.
- 3-year old individuals always produce flowers and get a Poisson(λ_3) distributed number of offspring.

This results in the following reproduction generating functions

$$
\begin{aligned}
f_1(s_1, s_2, s_3) &= 1 - p_1 + p_1 s_2 \\
f_2(s_1, s_2, s_3) &= q_2 e^{\lambda_2 (s_1 - 1)} + (1 - q_2)\,(1 - p_2 + p_2 s_3). \\
f_3(s_1, s_2, s_3) &= e^{\lambda_3 (s_1 - 1)}.
\end{aligned}
$$

The mean reproduction matrix equals

$$
M = \begin{pmatrix} 0 & p_1 & 0 \\ q_2 \lambda_2 & 0 & (1 - q_2) p_2 \\ \lambda_3 & 0 & 0 \end{pmatrix}.
$$

More about the composition of supercritical populations

Theorem

If $\rho > 1$ and the process $\mathbf{Z}(n)$ is nonsingular and positive regular then a.s.

$$\lim_{n\to\infty} \frac{\mathbf{Z}(n)}{\rho^n} = \mathbf{v}W,$$

where W is a nonnegative random variable such that

$$\mathbf{P}\left(W > 0\right) > 0$$

if and only if for all i, j

$$\mathbf{E}\xi_{ij} \log^+ \xi_{ij} < \infty.$$

Thus, on the set of nonextinction,

$$\frac{Z_i(n)}{(\mathbf{u}, \mathbf{Z}(n))} \to v_i \quad \text{a.s.}$$

as $n \to \infty$.

Limit theorem for the critical case

Theorem

If $\mathbf{Z}(n)$ *is positively regular,* $\rho = 1$ *and for all* $i, j, k \in \{1, ..., N\}$

$$b_{ij}^{k} := \mathbf{E}\left(\xi_{ki}\xi_{kj} - \delta_{ij}\xi_{kj}\right) < \infty$$

then, as $n \to \infty$

$$\mathbf{P}\left(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{e}_i\right) \sim \frac{2u_i}{Bn}$$

where

$$B := \sum_{k,i,j=1}^{N} v_k u_i b_{ij}^{k} u_j.$$

Moreover, for any vector $\mathbf{h} = (h_1, ..., h_N)$ *such that* $(\mathbf{v}, \mathbf{h}) > 0$

$$\left\{ \frac{2}{Bn} \left(\mathbf{Z}(n), \mathbf{h}\right) | \mathbf{Z}(n) \neq \mathbf{0}; \mathbf{Z}(0) = \mathbf{e}_i \right\} \xrightarrow{d} \eta_1$$

where η_1 *is an exponential random variable with parameter* $\beta := (\mathbf{v}, \mathbf{h})$.

If under the condition above $(\mathbf{v},\mathbf{h})=0$ then

$$\left\{\frac{(\mathbf{Z}(n),\mathbf{h})}{\sqrt{n}}|\mathbf{Z}(n)\neq\mathbf{0};\mathbf{Z}(0)=\mathbf{e}_i\right\}\xrightarrow{d}\eta_2$$

where η_2 is a random variable with density

$$d(x):=\frac{\beta_2}{2}e^{-\beta_2|x|},\quad \beta_2>0.$$

Limit theorems for the subcritical case

Theorem

If $\mathbf{Z}(n)$ *is positively regular,* $\rho < 1$ *and for all* i, j

$$\mathbf{E}\xi_{ij} \log^{+} \xi_{ij} < \infty.$$

then, as $n \to \infty$

$$\mathbf{P}\left(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{e}_i\right) \sim K_i \rho^n, K_i > 0.$$

Theorem

If $\mathbf{Z}(n)$ *is positively regular and* $\rho < 1$ *then for* $\mathbf{j} = (j_1, ..., j_N) \in \mathbb{Z}_0^N$

$$\lim_{n\to\infty} \mathbf{P}\left(\mathbf{Z}(n) = \mathbf{j} | \mathbf{Z}(\mathbf{n}) \neq \mathbf{0}; \mathbf{Z}(0) = \mathbf{e}_i\right) = b(\mathbf{j})$$

is independent on i,

$$\sum_{\mathbf{j}\in\mathbb{Z}_0^N, \mathbf{j}\neq\mathbf{0}} b(\mathbf{j}) = 1$$

and

$$\sum_{\mathbf{j}\in\mathbb{Z}_0^N, \mathbf{j}\neq\mathbf{0}} \|\mathbf{j}\| \, b(\mathbf{j}) < \infty$$

if and only if for all i, j

$$\mathbf{E}\xi_{ij} \log^+ \xi_{ij} < \infty.$$

Branching processes with sibling dependence

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EXAMPLE 1. Consider a sibling-dependent Galton-Watson process where an individual can beget zero or three children, and where the dependencies are such that, in a group of three siblings, two will always reproduce while the third never will. All individuals are equally likely to be among the reproducing ones.

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Thus we have a joint probability measure on $\{0,3\}^3$ which gives equal probabilities to the points $(0,3,3)$, $(3,0,3)$ and $(3,3,0)$ and has the marginals $p_0 = 1/3$ and $p_3 = 2/3$.

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Clearly, there are sibling dependencies; if we, for instance, know that an individual has no children, we also know that her sisters have three children each.

The dependent process will have a deterministic generation size of 3×2^n individuals in the nth generation (if it starts from a full group of siblings) and hence it never becomes extinct.

Compare this process to an ordinary independent Galton-Watson process which has the same individual marginals. That is, the offspring reproduction function is

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The expected number of children is 2, the probability of extinction is the minimal nonnegative root of the equation

$$\frac{1}{3} + \frac{2}{3}s^3 = s$$

giving

$$P = \frac{\sqrt{3}-1}{2}.$$

and there is a positive probability of nonextinction, in which case the generation size tends to ∞ with the rate proportional to 2^n.

Let $Z(n)$ be the number of individuals in the nth generation in the independent process and $\zeta(n)$ be the corresponding variable in the dependent process.

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If the processes both start from a group of three siblings, we have, for some random variable w

$$\frac{Z(n)}{2^n} \to w$$

where

$$\mathbf{P}\left(w=0\right) = \left(\frac{\sqrt{3}-1}{2}\right)^3$$

and

$$\frac{\zeta(n)}{2^n} = 3$$

so that the growth rates are the same in the two processes.

How the asymptotic composition may be affected by sibling dependencies

Let A be the event that an individual has no children and define $Z^A(n)$ and $\zeta^A(n)$ the number of individuals without children in the nth generation of the independent population and the dependent population, respectively.

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Let A be the event that an individual has no children and define $Z^A(n)$ and $\zeta^A(n)$ the number of individuals without children in the nth generation of the independent population and the dependent population, respectively.

The probability of A is $1/3$ in both populations. Hence, as $n \to \infty$

$$\frac{Z^A(n)}{Z(n)} \to \frac{1}{3}$$

and

$$\frac{\zeta^A(n)}{\zeta(n)} = \frac{1}{3}.$$

The asymptotic proportion of childless individuals is thus the same in the two populations.

Let B be the event that an individual has no grandchildren.

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In the dependent population for the probability of the event B we have

$$\frac{1}{3} + \frac{2}{3} \times 0 = \frac{1}{3},$$

i.e., the same probability as the event A.

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In the dependent population for the probability of the event B we have

$$\frac{1}{3} + \frac{2}{3} \times 0 = \frac{1}{3},$$

i.e., the same probability as the event A.

In the independent population B has probability

$$\begin{aligned} f_2(0) &= f(f(0)) = \frac{1}{3} + \frac{2}{3} \times (f(0))^3 \\ &= \frac{1}{3} + \frac{2}{3} \times \left(\frac{1}{3}\right)^3 = \frac{29}{81}. \end{aligned}$$

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To compare processes with sibling dependencies with another process, a macro process. This macro process consists of sibling groups, to be called macro individuals.

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The macroprocess generated by siblings is a multitype Galton-Watson process

$$Z_1(n), Z_2(n), ..., Z_N(n),$$

the type of a macro individual being the number of siblings in that group.

We may use the classical theory for multitype Galton-Watson processes to study the macro process. And if $Z_k(n), k = 1, 2, ..., N$ is the number of macro individuals of type k of the nth generation of the macro process, then

$$\zeta(n) = \sum_{k=1}^{N} k Z_k(n).$$

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In our example

$$\zeta(n) = 3Z_3(n).$$

EXAMPLE 2. Consider a population where an individual can beget $0, 1$ or 2 children. A single individual begets $0, 1$ or 2 children with probabilities

$$\frac{17}{32}, \quad \frac{8}{32}, \quad \frac{7}{32}.$$

In a group of 2, one individual always splits into 2, the other begets either 0 or 1 child with equal probabilities. The *two* siblings are equally likely to be the splitting one.

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We wish to compare this process with the corresponding independent 2-type process that has the *same marginals*. At the level of marginals we have

$$f^1(s_1, s_2) = \frac{17}{32} + \frac{8}{32} s_1 + \frac{7}{32} s_2^2,$$

and

$$f^2(s_1, s_2) = \frac{1}{4} + \frac{1}{4} s_1 + \frac{1}{2} s_2^2.$$

As a result, the mean matrix of the new process $(Z_1^{indiv}(n), Z_2^{indiv}(n))$ with two types and independent reproduction of individuals looks as follows

$$M_{indiv} = \left(\begin{array}{cc} \frac{8}{32} & \frac{14}{32} \\ \frac{1}{4} & 1 \end{array} \right).$$

As a result, the mean matrix of the new process $(Z_1^{indiv}(n), Z_2^{indiv}(n))$ with two types and independent reproduction of individuals looks as follows

$$M_{indiv} = \begin{pmatrix} \frac{8}{32} & \frac{14}{32} \\ \frac{1}{4} & 1 \end{pmatrix}.$$

Thus, the Perron root equals $\rho = 9/8$ and the respective scaled left eigenvector is

$$\left(\frac{2}{9}, \frac{7}{9}\right),$$

the process will grow proportional to

$$\left(Z_1^{indiv}(n), Z_2^{indiv}(n)\right) \sim \left(\frac{9}{8}\right)^n \left(\frac{2}{9}, \frac{7}{9}\right) W_{indiv}$$

where $W_{individ}$ is a random variable

On the other hand, we have in the example two macro-types $\{\mathbf{1}, \mathbf{2}\}$ and in the settings above we have for the joint probability measures $\mathbf{P}(k, \cdot), k = 1, 2$

$$\mathbf{P}(\mathbf{1}, 0) = \frac{17}{32},\ \mathbf{P}(\mathbf{1}, 1) = \frac{8}{32},\ \mathbf{P}(\mathbf{1}, 2) = \frac{7}{32}$$

and

$$\mathbf{P}(\mathbf{2}, (0, 2)) = \mathbf{P}(\mathbf{2}, (1, 2)) = \mathbf{P}(\mathbf{2}, (2, 0)) = \mathbf{P}(\mathbf{2}, (2, 1)) = \frac{1}{4}.$$

On the other hand, we have in the example two macro-types $\{\mathbf{1},\mathbf{2}\}$ and in the settings above we have for the joint probability measures $\mathbf{P}\left(k,\cdot\right), k=1,2$

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Thus, at the macro-process level we have

$$F^{1}(s_1,s_2)=\frac{17}{32}+\frac{8}{32}s_1+\frac{7}{32}s_2$$

and

$$F^{2}(s_1,s_2)=\left(\frac{1}{2}+\frac{1}{2}s_1\right)s_2.$$

The mean matrix of this macro-type process is

$$M_{macro} = \begin{pmatrix} \frac{8}{32} & \frac{7}{32} \\ \frac{1}{2} & 1 \end{pmatrix}$$

The Perron root is $\rho = 9/8$, the respective scaled left eigenvector is

$$\left(\frac{4}{11}, \frac{7}{11}\right).$$

Thus,

$$(Z_1(n), Z_2(n)) \sim \left(\frac{9}{8}\right)^n \left(\frac{4}{11}, \frac{7}{11}\right) W_{macro}$$

for some random variable W_{macro}.

Therefore, the number of individuals $\zeta(n)$ in the sibling-dependent process grows like

$$\zeta(n) = Z_1(n) + 2Z_2(n) \sim \left(\frac{9}{8}\right)^n \left(\frac{4}{11} + \frac{14}{11}\right) W_{macro} = \frac{18}{11}\left(\frac{9}{8}\right)^n W_{macro}.$$

The general case of the sibling-dependent Galton- Watson process

Let

$$Z(n+1) = \sum_{j=1}^{Z(n)} \xi_j(n).$$

Thus, the $(n+1)$-th generation consists of $Z(n)$ siblings group of sizes

$$\xi_1(n), ..., \xi_{Z(n)}(n).$$

Different sibling groups are assumed to evolve independently conditioned upon what has happened up to the previous generation, i.e. for $k \neq j$ the evolution of the individuals of the groups $\xi_k(n)$ and $\xi_j(n)$ are independent.

Assume that we have a population where an individual can beget at most N children and the sibling dependencies are described by the measures

$$P(i, \cdot), i = 1, ..., N$$

with identical marginals

p_{ij} - the probability that an individual belonging to a sibling group of size i begets j children.

With this process with dependencies we associate TWO processes. The first one is a N-type branching (macro process consisting of MACRO-particles of sizes(types) $1, ..., N$ in which a macroparticle of type i produces offspring in accordance with the probability generating function

$$F_i(s_1, ..., s_N) = P(i, \mathbf{0}) + \sum_{(k_1, ..., k_i)} P(i, (k_1, ..., k_i))\, s_{k_1} ... s_{k_i}.$$

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$$F_i\left(s_1, ..., s_N\right) = P\left(i, \mathbf{0}\right) + \sum_{(k_1, ..., k_i)} P\left(i, \left(k_1, ..., k_i\right)\right) s_{k_1} ... s_{k_i}.$$

The mean matrix $M_{macro} = \left(M_{ij}\right)$ of this process has elements

$$\begin{aligned} M_{ij} &= \mathbf{E}\left[\sum_{k=1}^{i} I\left\{\xi_k = j\right\} \mid \text{the group of siblings has size } i\right] \\ &= i\mathbf{P}\left(\xi_1 = j \mid \text{the group of siblings has size } i\right) = ip_{ij} \end{aligned}$$

where

$$p_{ij} = \sum_{(k_2, ..., k_i)} P(i, (j, k_2, ..., k_i))$$

Recall that the type of an individual is the number of individuals in her sibling group; i.e., if an individual begets j children, they will all be of type j.

The second process is a N-type branching process WITHOUT dependencies in which particles of type i produce children in accordance with the probability generating function

$$f_i(s_1, ..., s_N) = p_{i0} + \sum_{j=1}^{N} p_{ij} s_j^j.$$

We call this process as an individual process.
Clearly, the mean matrix of this process has the form

$$M_{indiv} = (m_{ij}) = (jp_{ij}).$$

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Lemma

For any λ

$$\det\left(M_{macro} - \lambda I\right) = \det\left(M_{indiv} - \lambda I\right).$$

Proof. We have

$$\begin{aligned}
\det\left(M_{macro} - \lambda I\right) &= \det\left(M_{ij} - \lambda\delta_{ij}\right) = \det\left(\frac{i}{j} m_{ij} - \lambda\delta_{ij}\right) \\
&= \det\left(\frac{i}{j} m_{ij} - \frac{i}{j}\lambda\delta_{ij}\right) = \det\left(\frac{i}{j}\left(m_{ij} - \lambda\delta_{ij}\right)\right) \\
&= \det\left(m_{ij} - \lambda\delta_{ij}\right) = \det\left(M_{indiv} - \lambda I\right).
\end{aligned}$$

Therefore, the **growth rates** of both processes are equal (at least at the level of expectations).

We may go further (cousin dependent - macro-process for the first macro-process and so on...).

Only **constants** will be changed, the rate of growth not!

Assume that the matrix

$$M_{indiv} = (m_{ij})_{i,j=1}^{N}$$

has maximal eigenvalue ρ, is indecomposable and has the right and left eigenvectors

$$\mathbf{u} = (u_1, u_2, ..., u_N), \quad \mathbf{v} = (v_1, v_2, ..., v_N).$$

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Then the matrix

$$M_{macro} = (M_{ij}) = \left(\frac{i}{j} m_{ij}\right)$$

has the eigenvectors

$$\mathbf{u}^* = \left(\sum_{k=1}^{N} \frac{v_k}{k}\right) (u_1, 2u_2 ..., N u_N)$$

and

$$\mathbf{v}^* = \frac{1}{\sum_{k=1}^{N} \frac{v_k}{k}} \left(v_1, \frac{v_2}{2}, ..., \frac{v_N}{N}\right).$$

If, further, $\rho = 1$ then for the macroprocess

$$\mathbf{P}\left(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(\mathbf{0}) = \mathbf{e}_i\right) \sim \frac{u_i^*}{B^* n}, n \to \infty.$$

Therefore,

$$\begin{aligned}
\mathbf{P}\left(\zeta(n) > 0 \,|\, \zeta(0) = i\right) &= \mathbf{P}\left(\sum_{k=1}^{N} k Z_k(n) > 0 | \mathbf{Z}(\mathbf{0}) = \mathbf{e}_i\right) \\
&= \mathbf{P}\left(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(\mathbf{0}) = \mathbf{e}_i\right) \sim \frac{u_i^*}{B^* n}.
\end{aligned}$$

Decomposable processes

A BP is called decomposable if there exists a pair of types such that $i \longrightarrow j$ and $j \nrightarrow i$.

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A BP is called decomposable if there exists a pair of types such that $i \to j$ and $j \nrightarrow i$.

Particular cases Assume that

$$k \longleftrightarrow j$$

for all $k, j \leq i < N$ and

$$i \to i + 1 \to \cdots \to N.$$

Then the mean matrix of the process has the form

$$\mathbf{M} = \left\| \begin{array}{cc} \mathbf{M}_1 & \mathbf{M}_{12} \\ \mathbf{0} & \mathbf{M}_2 \end{array} \right\|$$

Here

$$\mathbf{M}_1 = \left\| \begin{array}{cccc} m_{11} & m_{12} & \cdots & m_{1i} \\ m_{21} & m_{22} & \cdots & m_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ m_{i1} & m_{i2} & \cdots & m_{ii} \end{array} \right\|, \mathbf{M}_{12} \neq \mathbf{0},$$

and

$$\mathbf{M}_2 = \left\| \begin{array}{cccc} m_{i+1,i+1} & m_{i+1,i+2} & \cdots & m_{i+1,N} \\ 0 & m_{i+2,i+2} & \cdots & m_{i+2,N} \\ 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & m_{N,N} \end{array} \right\|.$$

Classification: By the maximal of the absolute value of the eigenvalues of the matrix.

However, the asymptotic representations for the population sizes and survival probabilities are essentially different!

Consider only the case when $i = 1$. Then, assuming that

$$1 \to 2 \to \cdots \to N \text{ and } j \leftrightarrow j, j = 1, ..., N$$

and

$$f^j(s_j, 1..., 1) \neq s_j^k$$

we have for the mean matrix

$$\mathbf{M} = \left\| \begin{array}{ccccccc} m_{11} & m_{12} & m_{12} & \cdots & & & \\ 0 & m_{22} & m_{11} & & & & \\ 0 & 0 & m_{33} & m_{11} & & & \\ 0 & 0 & 0 & \cdots & & & \\ ... & ... & ... & 0 & \cdots & & \\ 0 & 0 & & & 0 & m_{N-1,N-1} & \\ 0 & 0 & & & & 0 & m_{NN} \end{array} \right\| .$$

The criticality is specified by the maximal eigenvalue: $\max_i m_{ii}$.

If $f^i(s_i, 1..., 1) \neq s_i$ for all $i = 1, ..., d$ and $\rho \leq 1$ then the probability of extinction of this process is 1.

The asymptotic behavior of the probability of survival is rather complicated.

We treat the critical case only.

Theorem

If

$$m_{11} = m_{22} = ... = m_{NN} = 1$$

and for all $i, j, k \in \{1, ..., N\}$

$$b_{ij}^{k} := \mathbf{E}\left(\xi_{ki}\xi_{kj} - \delta_{ij}\xi_{kj}\right) < \infty$$

then, as $n \to \infty$

$$\mathbf{P}\left(\mathbf{Z}\left(n\right) \neq \mathbf{0} | \mathbf{Z}\left(n\right) = \mathbf{e}_1\right) \sim Cn^{-2^{1-N}}.$$

Proof. We consider the case $N = 2$ only. Let, as $s_1 \uparrow 1$ and $s_2 \uparrow 1$

$$\begin{aligned} 1 - f^1(s_1; s_2) &= m_{11}(1 - s_1) - \sigma_1^2 (1 - s_1)^2 (1 + o(1))) \\ &\quad + m_{12}(1 - s_2)(1 + o(1)), \end{aligned}$$

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$$\begin{aligned} 1-f^1(s_1;s_2) &= m_{11}\left(1-s_1\right)+m_{12}(1-s_2)(1+o(1)) \\ &\quad -\sigma_1^2\left(1-s_1\right)^2\left(1+o(1)\right)\right), \end{aligned}$$

Denote

$$Q_1(n) := \mathbf{P}\left(\mathbf{Z}\left(n\right)\neq\mathbf{0}|\mathbf{Z}\left(0\right)=\mathbf{e}_1\right),\ Q_2(n)=\mathbf{P}\left(Z_2(n)>0|\mathbf{Z}\left(0\right)=\mathbf{e}_2\right).$$

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Denote

$$Q_1(n) := \mathbf{P}(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{e}_1), \ Q_2(n) = \mathbf{P}(Z_2(n) > 0 | \mathbf{Z}(0) = \mathbf{e}_2).$$

Then

$$\begin{aligned} Q_1(k+1) &= \mathbf{P}(\mathbf{Z}(k+1) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{e}_1) \\ &= 1 - f^1_{k+1}(\mathbf{0}) = 1 - f^1(F^1(k, \mathbf{0}); F^2(k, \mathbf{0})) \\ &= 1 - f^1(1 - Q_1(k); 1 - Q_2(k)) \\ &= Q_1(k) + m_{12} Q_2(k)(1 + o(1)) - \sigma_1^2 Q_1^2(k)(1 + o(1))) \end{aligned}$$

Hence

$$\sigma_1^2 Q_1^2(k) \sim m_{12} Q_2(k) + (Q_1(k) - Q_1(k+1))$$

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or

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Recalling that

$$Q_2(k) \sim \frac{2}{\sigma_2^2 k}, k \to \infty,$$

and summing over k from 1 to n we get

$$\begin{aligned}\sigma_1^2 \sum_{k=1}^{n} k Q_1^2(k) &\sim m_{12} \sum_{k=1}^{n} k Q_2(k) + \sum_{k=1}^{n} k\left(Q_1(k) - Q_1(k+1)\right) \\ &\sim \frac{2m_{12}}{\sigma_2^2} n + \sum_{k=1}^{n} Q_1(k) - n Q_1(n+1).\end{aligned}$$

Therefore,

$$\sum_{k=1}^{n} kQ_1^2(k) \sim \frac{2m_{12}}{(\sigma_1\sigma_2)^2} n.$$

To complete the proof we need the following statement:

Lemma

If a positive function $h(k)$ *is monotone in* k *and, as* $n \to \infty$

$$\sum_{k=1}^{n} k^{\theta} h(k) \sim n^{\theta+1-\beta} l(n)$$

for some $0 \leq \beta < \theta + 1$ *and a slowly varying function* $l(n)$ *then*

$$h(n) \sim (\theta + 1 - \beta) \, n^{-\beta} l(n), n \to \infty.$$

Thus, the representation

$$\sum_{k=1}^{n} kQ_1^2(k) \sim \frac{2m_{12}}{(\sigma_1\sigma_2)^2} n$$

implies

$$nQ_1^2(n) \sim \frac{2m_{12}}{(\sigma_1\sigma_2)^2}$$

or

$$Q_1(n) \sim \frac{\sqrt{2m_{12}}}{\sigma_1\sigma_2} n^{-1/2}.$$

Example 1. Introgression is the permanent incorporation of genes from one population into another through hybridization and backcrossing.

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Particular concern: a possible mechanism for the spread of modified crop genes to wild population. This may have severe negative environmental effects such as the spread of insectecide or herbicide resistance genes.

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Particular concern: a possible mechanism for the spread of modified crop genes to wild population. This may have severe negative environmental effects such as the spread of insectecide or herbicide resistance genes.

It may lead to transgene escape and, as a result, the production of superweeds.

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Assume that there is a large stable wild population and a random number of hybrid seeds are produced by polen flow from a nearby crop. Time period - one year. Seeds may germinate at the beginning of the year and plants grow up to be adults and may flower later in the same year.

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Hybrid formation is of type 1 and is assumed to be less fit than wild individuals. However, the hybrid formation can be followed by repeated backcrossing with wild plants and the backcrossed individuals have a positive probability of producing a permanent introgressed lineage.

The model: An individual of artificial type 0 existing in the $k-$the year of observation produces at the moment of death an individual of type 0 and a random number η of hybrid seeds. Each of these hybrid seeds survives to the adult age with probability q, becoming a type 1 individual. Thus,

$$f_0(s_0, s_1, s_2) = s_0 \mathbf{E}(1 - q + q s_1)^{\eta}.$$

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Type 1 individual flowers with probability r and produces ξ back-crossed seeds. If it does not flower (with probability $1 - r$) it may then survive to become a type 1 individual for the next year with probability p or it will die with probability $1 - p$. Each backcrossed seed germinates and survives with probability q to produce a type 2 individual.

$$f_1(s_0, s_1, s_2) = (1 - r)(1 - p) + (1 - r)ps_1 + r\mathbf{E}(1 - q + qs_2)^{\xi}.$$

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$$f_2(s_0, s_1, s_2) = \mathbf{E}s_2^\zeta.$$

Thus we have a 3-type Galton-Watson decomposable process

The mean matrix of the process is

$$M = \begin{pmatrix} 1 & m_{01} & 0 \\ 0 & (1-r)p & m_{12} \\ 0 & 0 & m_{22} \end{pmatrix}$$

where $m_{01} = q\mathbf{E}\eta$, $m_{12} = rq\mathbf{E}\xi$ and $m_{22} = \mathbf{E}\zeta$. Hence

$$M^n = \begin{pmatrix} 1 & m_{01}^{(n)} & 0 \\ 0 & (1-r)^n p^n & m_{12}^{(n)} \\ 0 & 0 & m_{22}^n \end{pmatrix}$$

where

$$m_{01}^{(n)} = m_{01} \prod_{k=0}^{n-1} \left(1 + (1-r)^k p^k\right)$$

$$m_{12}^{(n)} = m_{12} \prod_{k=0}^{n-1} \left((1-r)^k p^k + m_{22}^k\right).$$

Crump-Mode-Jagers processes counted by random characteristics

8 июня 2011 г.

Crump-Mode-Jagers process counted by random characteristics

Informal description: a particle, say, x, is characterized by three random processes

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which are iid copies of a triple $(\lambda, \xi(\cdot), \chi(\cdot))$ and have the following sense: if a particle was born at moment σ_x then

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- λ_x- is the life-length of the particle;
- $\xi_x(t-\sigma_x)$- is the number of children produced by the particle within the time-interval $[\sigma_x, t);\ \xi_x(t-\sigma_x) = 0$ if $t-\sigma_x < 0;$

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- $\chi_x(t-\sigma_x)-$ is a random characteristic of the particle within the time-interval $[\sigma_x, t); \chi_x(t-\sigma_x) = 0$ if $t-\sigma_x < 0.$

The elements of the triple $\lambda_x, \xi_x(\cdot), \chi_x(\cdot)$ may be dependent.

The stochastic process

$$Z^{\chi}(t) = \sum_{x} \chi_x(t - \sigma_x)$$

where summation is taken over all particles x born in the process up to moment t is called the branching process counted by random characteristics.

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  Then $Z^{\chi}(t) = Z(y, t)$ is the number of particles existing at moment t whose ages do not exceed y.
- $$\chi(t) = tI\{t \in [0, \lambda)\} + \lambda I\{\lambda < t\}$$

  then

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- $\chi(t) = I\{t \in [0, \lambda)\} I\{\xi(t) < \xi(\infty)\}$ - the number of fertile individuals at moment t.
- coming generation size $\chi(t) = (\xi(\infty) - \xi(t)) I\{t \in [0, \lambda)\}.$

Probability generating function

Let

$$0 \leq \delta_1 \leq \delta_2 \leq ... \leq \delta_n \leq ...$$

be the birth moments of the children of the initial particle. Then

$$\xi(t) = \#\{\delta_i : \delta_i \leq t\} = \sum_{i=1}^{\infty} I\{\delta_i \leq t\}$$

is the number of children of the initial particle born up to moment t with $N := \xi(\infty)$. Clearly,

$$Z(t) = I\{\lambda_0 > t\} + \sum_{\delta_i \leq t} Z_i(t - \delta_i)$$

where $Z_i(t) \stackrel{d}{=} Z(t)$ and are iid.

Denote

$$F(t;s) := \mathbf{E}\left[s^{Z(t)} | Z(0) = 1\right].$$

Then

$$F(t;s) := \mathbf{E}\left[s^{I\{\lambda_0 > t\} + \sum_{\delta_i \le t} Z_i(t-\delta_i)}\right] = \mathbf{E}\left[s^{I\{\lambda_0 > t\}} \prod_{i=1}^{\xi(t)} F(t-\delta_i; s)\right].$$

Let

$$P = \mathbf{P}\left(\lim_{t\to\infty} Z(t) = 0\right) = \lim_{t\to\infty} F(t;0).$$

Since $\lambda_0 < \infty$ a.s. we have by the dominated convergence theorem

$$P = \lim_{t\to\infty} \mathbf{E}\left[0^{I\{\lambda_0 > t\}} \prod_{i=1}^{\xi(t)} F(t-\delta_i; 0)\right] = \mathbf{E}\left[P^N\right] := f(P).$$

Thus, if $A := \mathbf{E}N \le 1$ then the probability of extinction equals 1.

Let us show that if $\mathbf{E}N > 1$ then P, the probability of extinction, is the smallest nonnegative root of $s = f(s)$.

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Denote ζ_n – the number of particles in generation n in the embedded Galton-Watson process.

If $Z(t) = 0$ for some t then the total number of individuals born in the process is finite. Hence $\zeta_n \to 0$ as $n \to \infty$. Therefore,

$$P = \mathbf{P}\left(\lim_{t\to\infty} Z(t) = 0\right) \leq \mathbf{P}\left(\lim_{n\to\infty} \zeta_n = 0\right)$$

as desired.

Classification

$A := \mathbf{E}N <,=,> 1$ - subcritical, critical and supercritical, respectively.

Directly Riemann integrable functions:

Let $g(t) \geq 0, t \geq 0$ be a measurable function. Let $h > 0$ and let

$$M_k(h) := \sup_{kh \leq t < (k+1)h} g(t), \quad m_k(h) := \inf_{kh \leq t < (k+1)h} g(t)$$

and

$$\Theta_h = h \sum_{k=0}^{\infty} M_k(h), \quad \theta_h = h \sum_{k=0}^{\infty} m_k(h).$$

If

$$\lim_{h \to 0} \Theta_h = \lim_{h \to 0} \theta_h < \infty$$

then $g(t)$ is called directly Riemann integrable.

Examples of directly Riemann integrable functions:

- $g(t)$ is nonnegative, bounded, continuous and

$$\sum_{k=0}^{\infty} M_k(1) < \infty;$$

- $g(t)$ is nonnegative, monotone and Riemann integrable;
- $g(t)$ is Riemann integrable and bounded (in absolute value) by a directly Riemann integrable function.

Example of NOT directly Riemann integrable function which is Riemann integrable

Let the graph of $g(t)$ is constituted buy the pieces of $X-$axis and triangulars of heights h_n with bottom-lengthes $\mu_n < 1/2,\ n = 1, 2, ...,$ with the middles located at points $n = 1, 2 \ldots$ and such that $\lim_{n\to\infty} h_n = \infty$ and

$$\int_0^\infty h(t)dt = \frac{1}{2}\sum_{n=1}^\infty h_n\mu_n < \infty.$$

It is easy to see that

$$\sum_{k=0}^\infty M_k(1) = \infty,$$

and, therefore, for any $\delta \in (0, 1]$

$$\sum_{k=0}^\infty M_k(\delta) = \infty.$$

Consider the equation

$$H(t) = g(t) + \int_0^t H(t-u)R(du), t \geq 0.$$

Theorem

If $g(t)$ is directly Riemann integrable and $R(t)$ is a nonlattice distribution (i.e. it is not concentrated on a lattice $a + kh,\ k = 0, \pm 1, \pm 2, ...$) with finite mean then

$$\lim_{t \to \infty} H(t) = \frac{\int_0^\infty g(u)du}{\int_0^\infty uR(du)}.$$

Expectation

Let $0 \leq \delta_1 \leq \delta_2 \leq ... \leq \delta_n \leq ...$be the birth moments of the children of the initial particle and $\xi_0(t) = \#\{\delta_i : \delta_i \leq t\}$. We have

$$Z^{\chi}(t) = \chi_0(t) + \sum_{x \neq 0} \chi_x(t - \sigma_x) = \chi_0(t) + \sum_{\delta_i \leq t} Z^{\chi}(t - \delta_i)$$

giving

$$\begin{aligned}
\mathbf{E}Z^{\chi}(t) &= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{\delta_i \leq t} Z^{\chi}(t - \delta_i)\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{\delta_i \leq t} \mathbf{E}\left[Z^{\chi}(t - \delta_i) | \delta_1, \delta_2, ..., \delta_n, ...\right]\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{u \leq t} \mathbf{E}\left[Z^{\chi}(t - u)\right](\xi_0(u) - \xi_0(u-))\right] \\
&= \mathbf{E}\chi(t) + \int_0^t \mathbf{E}Z^{\chi}(t - u)\mathbf{E}\xi(du)
\end{aligned}$$

Thus, we get the following renewal equation for

$$A^{\chi}(t) = \mathbf{E}Z^{\chi}(t)$$

and

$$\mu(t) = \mathbf{E}\xi(t):$$

$$A^{\chi}(t) = \mathbf{E}\chi(t) + \int_0^t A^{\chi}(t-u)\mu(du).$$

Malthusian parameter: a number α is called the Malthusian parameter of the process if

$$\int_0^\infty e^{-\alpha t}\mu(dt) = \int_0^\infty e^{-\alpha t}\mathbf{E}\xi(dt) = 1.$$

(such a solution not always exist). For the critical processes $\alpha = 0,$ for the supercritical processes $\alpha > 0$ for the subcritical processes $\alpha < 0$ (if exists).

If the Malthusian parameter exists we can rewrite the equation for $A^\chi(t)$ as

$$e^{-\alpha t}A^\chi(t) = e^{-\alpha t}\mathbf{E}\chi(t) + \int_0^t e^{-\alpha(t-u)}A^\chi(t-u)e^{-\alpha u}\mu(du).$$

Let now

$$g(t) := e^{-\alpha t}\mathbf{E}\chi(t), R(dt) := e^{-\alpha u}\mu(du)$$

If $e^{\alpha t}\mathbf{E}\chi(t)$ is directly Riemann integrable and

$$\int_0^\infty e^{-\alpha u}\mathbf{E}\chi(u)du < \infty,\ \beta := \int_0^\infty ue^{-\alpha u}\mu(du) < \infty,$$

then by the renewal theorem

$$\lim_{t\to\infty} e^{-\alpha t}A^\chi(t) = \int_0^\infty e^{-\alpha u}\mathbf{E}\chi(u)du\left(\int_0^\infty ue^{-\alpha u}\mu(du)\right)^{-1}.$$

Applications

If

$$\chi(t) = I\{t \in [0, \lambda)\}$$

then $Z^{\chi}(t) = Z(t)$ is the number of particles existing in the process up to moment t. We have

$$\mathbf{E}\chi(t) = \mathbf{E}I\{t \in [0, \lambda)\} = \mathbf{P}(t \leq \lambda) = 1 - G(t)$$

and, therefore,

$$\mathbf{E}Z(t) \sim \frac{e^{\alpha t}}{\beta} \int_0^{\infty} e^{-\alpha u}(1 - G(u))du.$$

If

$$\chi(t) = \chi(t, y) = I\{t \in [0, \min(\lambda, y))\}$$

then

$$\begin{aligned} \mathbf{E}\chi(t) &= \mathbf{E}I\{t \in [0, \min(\lambda, y))\} \\ &= \mathbf{P}(t \leq \min(\lambda, y)) = (1 - G(t)) I\{t \leq y\} \end{aligned}$$

Hence

$$A^{\chi}(t) = \mathbf{E}Z(y, t)$$

is the average number of particles existing at moment t whose ages do not exceed y. We see that

$$\mathbf{E}Z(y, t) \sim \frac{e^{\alpha t}}{\beta} \int_0^y e^{-\alpha u} (1 - G(u)) \, du$$

As a result

$$\lim_{t\to\infty}\frac{\mathbf{E}Z(y,t)}{\mathbf{E}Z(t)} = \frac{\int_0^y e^{-\alpha u}\left(1-G(u)\right)du}{\int_0^\infty e^{-\alpha u}(1-G(u))du} = \frac{\alpha}{1-m^{-1}}\int_0^y e^{-\alpha u}\left(1-G(u)\right)du$$

(the last if $m \neq 1$).
If $\chi(t) = I\left\{t \geq 0\right\}$ then

$$\mathbf{E}\chi(t) = 1.$$

Hence, for the expectation $\mathbf{E}Z^{\chi}(t)$ of the total number of particles born up to moment t in a **supercritical** process we have

$$\mathbf{E}Z^{\chi}(t) \sim \frac{e^{\alpha t}}{\beta}\int_0^\infty e^{-\alpha u}du = \frac{e^{\alpha t}}{\alpha\beta}.$$

Applications

1) Reproduction by splitting. Assume that an individual gives birth to N her daughters at once at random moment λ. Then

$$\xi(t) = NI\{\lambda \leq t\}, \ \mu(t) = \mathbf{E}[N; \lambda \leq t]$$

and

$$A = \mathbf{E}N, \ \beta = \mathbf{E}N\lambda e^{-\alpha\lambda}$$

This is the so-called **Sevastyanov** process.

If the random variables N and λ are independent then we get the so-called **Bellman-Harris** process or the **age-dependent** process.

2) Constant fertility. We assume now that time is discrete, i.e., $t = 0, 1, 2, ...$ and suppose that the offspring birth times are uniformly distributed over the fertility interval $1, 2, ..., \lambda$. Then, given $N = k$, $\lambda = j$ the number $v(t)$ individuals born at time $t \leq j$ is Binomial with parameters k and j^{-1}.
Thus,

$$\mu(t) = \mathbf{E}\left[\frac{N \min(t, \lambda)}{\lambda}\right].$$