Crump-Mode-Jagers processes counted by random characteristics

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Informal description: a particle, say, x, is characterized by three random processes

$$(\lambda_x, \xi_x(\cdot), \chi_x(\cdot))$$

which are iid copies of a triple $(\lambda, \xi(\cdot), \chi(\cdot))$ and have the following sense: if a particle was born at moment σ_x then

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- $\lambda_x -$ is the life-length of the particle;
- $\xi_x(t - \sigma_x)$- is the number of children produced by the particle within the time-interval $[\sigma_x, t)$; $\xi_x(t - \sigma_x) = 0$ if $t - \sigma_x < 0$;

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- $\lambda_x -$ is the life-length of the particle;
- $\xi_x(t-\sigma_x)$- is the number of children produced by the particle within the time-interval $[\sigma_x, t)$; $\xi_x(t-\sigma_x) = 0$ if $t - \sigma_x < 0$;
- $\chi_x(t-\sigma_x) -$ is a random characteristic of the particle within the time-interval $[\sigma_x, t); \chi_x(t-\sigma_x) = 0$ if $t - \sigma_x < 0.$

The elements of the triple $\lambda_x, \xi_x(\cdot), \chi_x(\cdot)$ may be dependent.

The stochastic process

$$Z^{\chi}(t) = \sum_{x} \chi_x(t - \sigma_x)$$

where summation is taken over all particles x born in the process up to moment t is called the branching process counted by random characteristics.

Examples of random characteristics:

- $\chi(t) = I\{t \in [0, \lambda)\}$ - in this case $Z^{\chi}(t) = Z(t)$ is the number of particles existing in the process up to moment t;

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  Then $Z^{\chi}(t) = Z(y, t)$ is the number of particles existing at moment t whose ages do not exceed y.
- $$\chi(t) = tI\{t \in [0, \lambda)\} + \lambda I\{\lambda < t\}$$

  then

  $$Z^{\chi}(t) = \int_0^t Z(u)du;$$

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- $\chi(t) = I\{t \in [0, \lambda)\} I\{\xi(t) < \xi(\infty)\}$ - the number of fertile individuals at moment t.
- coming generation size $\chi(t) = (\xi(\infty) - \xi(t)) I\{t \in [0, \lambda)\}$.

Probability generating function

Let

$$0 \leq \delta_1 \leq \delta_2 \leq ... \leq \delta_n \leq ...$$

be the birth moments of the children of the initial particle. Then

$$\xi(t) = \#\{\delta_i : \delta_i \leq t\} = \sum_{i=1}^{\infty} I\{\delta_i \leq t\}$$

is the number of children of the initial particle born up to moment t with $N := \xi(\infty)$. Clearly,

$$Z(t) = I\{\lambda_0 > t\} + \sum_{\delta_i \leq t} Z_i(t - \delta_i)$$

where $Z_i(t) \stackrel{d}{=} Z(t)$ and are iid.

Denote

$$F(t;s) := \mathbf{E}\left[s^{Z(t)} | Z(0) = 1\right].$$

Then

$$F(t;s) := \mathbf{E}\left[s^{I\{\lambda_0 > t\} + \sum_{\delta_i \le t} Z_i(t-\delta_i)}\right] = \mathbf{E}\left[s^{I\{\lambda_0 > t\}} \prod_{i=1}^{\xi(t)} F(t-\delta_i; s)\right].$$

Let

$$P = \mathbf{P}\left(\lim_{t\to\infty} Z(t) = 0\right) = \lim_{t\to\infty} F(t;0).$$

Since $\lambda_0 < \infty$ a.s. we have by the dominated convergence theorem

$$P = \lim_{t\to\infty} \mathbf{E}\left[0^{I\{\lambda_0 > t\}} \prod_{i=1}^{\xi(t)} F(t-\delta_i; 0)\right] = \mathbf{E}\left[P^N\right] := f(P).$$

Thus, if $A := \mathbf{E}N \le 1$ then the probability of extinction equals 1.

Let us show that if $\mathbf{E}N > 1$ then P, the probability of extinction, is the smallest nonnegative root of $s = f(s)$.

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Denote ζ_n – the number of particles in generation n in the embedded Galton-Watson process.

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Denote ζ_n – the number of particles in generation n in the embedded Galton-Watson process.

If $Z(t) = 0$ for some t then the total number of individuals born in the process is finite. Hence $\zeta_n \to 0$ as $n \to \infty$. Therefore,

$$P = \mathbf{P}\left(\lim_{t\to\infty} Z(t) = 0\right) \leq \mathbf{P}\left(\lim_{n\to\infty} \zeta_n = 0\right)$$

as desired.

Classification

$A := \mathbf{E}N <, =, > 1$ - subcritical, critical and supercritical, respectively.

Directly Riemann integrable functions:

Let $g(t) \geq 0, t \geq 0$ be a measurable function. Let $h > 0$ and let

$$M_k(h) := \sup_{kh \leq t < (k+1)h} g(t), \quad m_k(h) := \inf_{kh \leq t < (k+1)h} g(t)$$

and

$$\Theta_h = h \sum_{k=0}^{\infty} M_k(h), \quad \theta_h = h \sum_{k=0}^{\infty} m_k(h).$$

If

$$\lim_{h \to 0} \Theta_h = \lim_{h \to 0} \theta_h < \infty$$

then $g(t)$ is called directly Riemann integrable.

Examples of directly Riemann integrable functions:

- $g(t)$ is nonnegative, bounded, continuous and

$$\sum_{k=0}^{\infty} M_k(1) < \infty;$$

- $g(t)$ is nonnegative, monotone and Riemann integrable;
- $g(t)$ is Riemann integrable and bounded (in absolute value) by a directly Riemann integrable function.

Example of NOT directly Riemann integrable function which is Riemann integrable

Let the graph of $g(t)$ is constituted buy the pieces of $X-$axis and triangulars of heights h_n with bottom-lengthes $\mu_n < 1/2$, $n = 1, 2, ...$, with the middles located at points $n = 1, 2 \ldots$ and such that $\lim_{n\to\infty} h_n = \infty$ and

$$\int_0^\infty h(t)dt = \frac{1}{2}\sum_{n=1}^\infty h_n\mu_n < \infty.$$

It is easy to see that

$$\sum_{k=0}^\infty M_k(1) = \infty,$$

and, therefore, for any $\delta \in (0, 1]$

$$\sum_{k=0}^\infty M_k(\delta) = \infty.$$

Consider the equation

$$H(t) = g(t) + \int_0^t H(t-u)R(du), t \geq 0.$$

Theorem

If $g(t)$ is directly Riemann integrable and $R(t)$ is a nonlattice distribution (i.e. it is not concentrated on a lattice $a + kh,\ k = 0, \pm 1, \pm 2, ...$) with finite mean then

$$\lim_{t \to \infty} H(t) = \frac{\int_0^\infty g(u)du}{\int_0^\infty uR(du)}.$$

Expectation

Let $0 \leq \delta_1 \leq \delta_2 \leq ... \leq \delta_n \leq ...$be the birth moments of the children of the initial particle and $\xi_0(t) = \#\{\delta_i : \delta_i \leq t\}$. We have

$$Z^\chi(t) = \chi_0(t) + \sum_{x \neq 0} \chi_x(t - \sigma_x) = \chi_0(t) + \sum_{\delta_i \leq t} Z^\chi(t - \delta_i)$$

giving

$$\begin{aligned}
\mathbf{E}Z^\chi(t) &= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{\delta_i \leq t} Z^\chi(t - \delta_i)\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{\delta_i \leq t} \mathbf{E}\left[Z^\chi(t - \delta_i) | \delta_1, \delta_2, ..., \delta_n, ...\right]\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{u \leq t} \mathbf{E}\left[Z^\chi(t - u)\right](\xi_0(u) - \xi_0(u-))\right] \\
&= \mathbf{E}\chi(t) + \int_0^t \mathbf{E}Z^\chi(t - u)\mathbf{E}\xi(du)
\end{aligned}$$

Thus, we get the following renewal equation for

$$A^{\chi}(t) = \mathbf{E} Z^{\chi}(t)$$

and

$$\mu(t) = \mathbf{E}\xi(t) :$$

$$A^{\chi}(t) = \mathbf{E}\chi(t) + \int_0^t A^{\chi}(t-u)\mu(du).$$

Malthusian parameter: a number α is called the Malthusian parameter of the process if

$$\int_0^\infty e^{-\alpha t}\mu(dt) = \int_0^\infty e^{-\alpha t}\mathbf{E}\xi(dt) = 1.$$

(such a solution not always exist). For the critical processes $\alpha = 0,$ for the supercritical processes $\alpha > 0$ for the subcritical processes $\alpha < 0$ (if exists).
If the Malthusian parameter exists we can rewrite the equation for $A^\chi(t)$ as

$$e^{-\alpha t}A^\chi(t) = e^{-\alpha t}\mathbf{E}\chi(t) + \int_0^t e^{-\alpha(t-u)}A^\chi(t-u)e^{-\alpha u}\mu(du).$$

Let now

$$g(t) := e^{-\alpha t}\mathbf{E}\chi(t), R(dt) := e^{-\alpha u}\mu(du)$$

If $e^{\alpha t}\mathbf{E}\chi(t)$ is directly Riemann integrable and

$$\int_0^\infty e^{-\alpha u}\mathbf{E}\chi(u)du < \infty,\ \beta := \int_0^\infty ue^{-\alpha u}\mu(du) < \infty,$$

then by the renewal theorem

$$\lim_{t\to\infty} e^{-\alpha t}A^\chi(t) = \int_0^\infty e^{-\alpha u}\mathbf{E}\chi(u)du\left(\int_0^\infty ue^{-\alpha u}\mu(du)\right)^{-1}.$$

Applications

If

$$\chi(t) = I\{t \in [0, \lambda)\}$$

then $Z^{\chi}(t) = Z(t)$ is the number of particles existing in the process up to moment t. We have

$$\mathbf{E}\chi(t) = \mathbf{E}I\{t \in [0, \lambda)\} = \mathbf{P}(t \leq \lambda) = 1 - G(t)$$

and, therefore,

$$\mathbf{E}Z(t) \sim \frac{e^{\alpha t}}{\beta} \int_0^\infty e^{-\alpha u}(1 - G(u))du.$$

If

$$\chi(t) = \chi(t, y) = I\left\{t \in [0, \min(\lambda, y))\right\}$$

then

$$\begin{aligned}\mathbf{E}\chi(t) &= \mathbf{E}I\left\{t \in [0, \min(\lambda, y))\right\} \\ &= \mathbf{P}\left(t \le \min(\lambda, y)\right) = (1 - G(t))\, I\left\{t \le y\right\}\end{aligned}$$

Hence

$$A^{\chi}(t) = \mathbf{E}Z(y, t)$$

is the average number of particles existing at moment t whose ages do not exceed y. We see that

$$\mathbf{E}Z(y, t) \sim \frac{e^{\alpha t}}{\beta} \int_0^y e^{-\alpha u} \left(1 - G(u)\right) du$$

As a result

$$\lim_{t\to\infty}\frac{\mathbf{E}Z(y,t)}{\mathbf{E}Z(t)} = \frac{\int_0^y e^{-\alpha u}(1-G(u))\,du}{\int_0^\infty e^{-\alpha u}(1-G(u))du} = \frac{\alpha}{1-m^{-1}}\int_0^y e^{-\alpha u}(1-G(u))\,du$$

(the last if $m \neq 1$).
If $\chi(t) = I\{t \geq 0\}$ then

$$\mathbf{E}\chi(t) = 1.$$

Hence, for the expectation $\mathbf{E}Z^{\chi}(t)$ of the total number of particles born up to moment t in a **supercritical** process we have

$$\mathbf{E}Z^{\chi}(t) \sim \frac{e^{\alpha t}}{\beta}\int_0^\infty e^{-\alpha u}du = \frac{e^{\alpha t}}{\alpha\beta}.$$

Applications

1) Reproduction by splitting. Assume that an individual gives birth to N her daughters at once at random moment λ. Then

$$\xi(t) = NI\{\lambda \le t\}, \ \mu(t) = \mathbf{E}[N; \lambda \le t]$$

and

$$A = \mathbf{E}N, \ \beta = \mathbf{E}N\lambda e^{-\alpha\lambda}$$

This is the so-called **Sevastyanov** process.

If the random variables N and λ are independent then we get the so-called **Bellman-Harris** process or the **age-dependent** process.

2) Constant fertility. We assume now that time is discrete, i.e., $t = 0, 1, 2, ...$ and suppose that the offspring birth times are uniformly distributed over the fertility interval $1, 2, ..., \lambda$. Then, given $N = k$, $\lambda = j$ the number $v(t)$ individuals born at time $t \leq j$ is Binomial with parameters k and j^{-1}.

Thus,

$$\mu(t) = \mathbf{E}\left[\frac{N \min(t, \lambda)}{\lambda}\right].$$

Inhomogeneous Galton-Watson process

The probability generating function

$$f_n(s) := \sum_{k=0}^{\infty} p_k^{(n)} s^k$$

specifies the reproduction law of the offspring size of particles in generation $n = 0, 1, ...$ and let $Z(n)$ be the number of particles in generation n.

This Markov chain is called a branching process in varying environment.

One can show that

$$\mathbf{E}\left[s^{Z(n)}; Z(0) = 1\right] = f_0(f_1(...(f_{n-1}(s)...)).$$

If $p_0^{(n)} > 0$ for each $n = 0, 1, 2, ...$ then

$$\lim_{n\to\infty} Z(n)$$

exists and is nonrandom with probability 1 (and may be equal to $+\infty$).

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It is known (**Lindvall T., Almost sure convergence of branching processes in varying and random environments, Ann. Probab., 2(1974), N2, 344-346**) that the limit is equal to a positive natural number with a positive probability if and only if

$$\sum_{n=1}^{\infty}\left(1-p_1^{(n)}\right) < +\infty.$$

If this is not the case, then

$$\mathbf{P}(\lim_{n\to\infty} Z(n)=0) + \mathbf{P}(\lim_{n\to\infty} Z(n)=\infty) = 1$$

Let now

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be a probability measure on the set of nonnegative integers and

$$\Omega := \{\Pi\}$$

be the set of all such probability measures.

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we introduce the distance of total variation

$$d(\Pi_1, \Pi_2) = \frac{1}{2}\sum_{k=0}^{\infty}\left|p_k^{(1)} - p_k^{(2)}\right|.$$

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Thus, Ω becomes a metric space and on the Borel σ-algebra $\mathcal{F}$ of the sets of Ω we may introduce a probability measure $\mathbf{P}$ and consider the probability space

$$(\Omega, \mathcal{F}, \mathbf{P}) .$$

BP in random environemnt

Let

$$\Pi_n = \left(p_0^{(n)}, p_1^{(n)}, ..., p_k^{(n)}, ...\right), \quad n = 0, 1, ...$$

be a sequence of random elements selected from $\mathbf{\Omega}$ in iid manner. The sequence

$$\Pi_0, \Pi_1, ..., \Pi_n, ...$$

is called a random environment. Clearly

$$f_n(s) := \sum_{k=0}^{\infty} p_k^{(n)} s^k \longleftrightarrow \Pi_n = \left(p_0^{(n)}, p_1^{(n)}, ..., p_k^{(n)}, ...\right).$$

BP in random environment is specified by the relationship

$$\mathbf{E}\left(s^{Z(n)} | \Pi_0, \Pi_1, ..., \Pi_{n-1}; Z(0) = 1\right) = f_0(f_1(...(f_{n-1}(s)...)).$$

Now we let

$$\hat{\mathbf{P}}(...) = \mathbf{P}(...|\Pi_0, \Pi_1, ..., \Pi_n, ...)$$

and

$$\hat{\mathbf{E}}(...) = \mathbf{E}(...|\Pi_0, \Pi_1, ..., \Pi_n, ...).$$

Clearly,

$$\mathbf{P}(Z(n) \in B) = \mathbf{E}\hat{\mathbf{P}}(Z(n) \in B).$$

This leads to TWO different approaches to study BPRE:

Quenched approach: the study the behavior of characteristics of a BPRE for typical realizations of the environment $\Pi_0, \Pi_1, ..., \Pi_n, ...$.

This means that, for instance

$$\hat{\mathbf{P}}\left(Z(n) > 0\right)$$

is a random variable on the space of realizations of the environment and

$$\hat{\mathbf{P}}\left(Z(n) \in B\right)$$

is a random law and

$$\hat{\mathbf{P}}\left(Z(n) \in B | Z(n) > 0\right)$$

is a random conditional law.

Annealed approach: the study the behavior of characteristics of a BPRE performing averaging over possible scenarios $\Pi_0, \Pi_1, ..., \Pi_n, ...$ on the space of realizations of the environment:

$$\mathbf{P}\left(Z(n) > 0\right) = \mathbf{E}\hat{\mathbf{P}}\left(Z(n) > 0\right)$$

is a number.

Introduce a sequence of random variables

$$X_n = \log f'_{n-1}(1), n = 1, 2, ...$$

and set

$$S_0 = 0, S_k = X_1 + ... + X_n, n = 1, 2, ...$$

The sequence $\{S_n, n \geq 0\}$ is called an associated RW for our BPRE. Clearly,

$$\hat{\mathbf{E}}Z(n) = f'_0(1)f'_1(1)...f'_{n-1}(1) = e^{S_n}, n = 0, 1, ...$$

and

$$\mathbf{E}\left(\hat{\mathbf{E}}Z(n)\right) = \mathbf{E}e^{S_n}.$$

We assume in what follows that the random variables $p_0^{(n)}$ and $p_1^{(n)}$ are positive with probability 1 and $p_0^{(n)} + p_1^{(n)} < 1$.

Theorem in Feller, Volume 2, Chapter XII, Section 2 :
There are only four types of random walks with $S_0 = 0$:

- $$\lim_{n\to\infty} S_n = +\infty \qquad \text{with probability 1;} \tag{1}$$
- $$\lim_{n\to\infty} S_n = -\infty \qquad \text{with probability 1;} \tag{2}$$
- $$\lim_{n\to\infty} \sup S_n = +\infty, \quad \lim_{n\to\infty} \inf S_n = -\infty, \tag{3}$$
  with probability 1;
- $S_n \equiv 0$.

Classification:
BPRE are called supercritical if (1) is valid, subcritical, if (2) is valid and critical, if (3) is valid.

For the critical and subcritical cases

$$\begin{aligned}\hat{\mathbf{P}}\left(Z(n)>0\right) &= \hat{\mathbf{P}}\left(Z(n)\geq 1\right)=\min_{0\leq k\leq n}\hat{\mathbf{P}}\left(Z(k)\geq 1\right)\\ &\leq \min_{0\leq k\leq n}\hat{\mathbf{E}}Z(k)=e^{\min_{0\leq k\leq n}S_k}\to 0\end{aligned}$$

with probability 1 as $n\to\infty$. This means that the critical and subcritical processes die out for almost all realizations of the environment.

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with probability 1 as $n\to\infty$. This means that the critical and subcritical processes die out for almost all realizations of the environment.
In particular, if

$$\mathbf{E}X=\mathbf{E}\log f'(1)=0,\qquad \mathbf{E}\left(\log f'(1)\right)^2>0$$

then the process is critical, and if

$$\mathbf{E}X=\mathbf{E}\log f'(1)<0$$

then the process is subcritical.

Our aim is to study the asymptotic behavior of the probabilities

$$\hat{\mathbf{P}}\left(Z(n)>0\right) \text{ and } \mathbf{P}\left(Z(n)>0\right)$$

as $n\to\infty$ for the critical and subcritical processes and to prove the conditional theorems of the form

$$\mathbf{P}\left(Z(n)\in B|Z(n)>0\right)$$

and

$$\hat{\mathbf{P}}\left(Z(n)\in B|Z(n)>0\right)$$

for such processes.

Main steps

1) To express the needed characteristics in terms of some reasonable functionals and the associated random walks

2) To prove conditional limit theorems for the associated random walks

3) To make a change of measures in an appropriate way

4) To apply the results established for the associated random walks

Sparre-Anderson and Spitzer identities

Let

$$\tau = \tau_1 = \min \{n > 0 : S_n \leq 0\}$$

be the first **weak descending** ladder epoch, and

$$\tau_j := \min \left\{n > \tau_{j-1} : S_n \leq S_{\tau_{j-1}}\right\}, j = 2, 3, ...$$

(PICTURE).
Clearly,

$$(\tau_1, S_{\tau_1}), (\tau_2 - \tau_1, S_{\tau_2} - S_{\tau_1}), ..., \left(\tau_j - \tau_{j-1}, S_{\tau_j} - S_{\tau_{j-1}}\right)$$

are iid.

Strong descending ladder epochs :

$$\tau' = \tau'_1 = \min\{n > 0 : S_n < 0\}$$

and

$$\tau'_j := \min\left\{n > \tau'_{j-1} : S_n < S_{\tau'_{j-1}}\right\}$$

Introduce also **strong and weak ascending** ladder epochs:

$$T = T_1 = \min\{n > 0 : S_n > 0\}$$

and

$$T_j := \min\left\{n > T_{j-1} : S_n > S_{T_{j-1}}\right\}, j = 2, 3, ...$$

and

$$T' = T'_1 = \min\{n > 0 : S_n \geq 0\}$$

and

$$T'_j := \min\left\{n > T'_{j-1} : S_n \geq S_{T'_{j-1}}\right\}, j = 2, 3, ...$$

Sparre-Anderson identity

Theorem

For $\lambda > 0$ *and* $|s| < 1$

$$1 - \sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n}; T = n\right] = \exp\left\{-\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right]\right\}.$$

Recall

$$T = \min\{n > 0 : S_n > 0\}$$

Proof. Along with

$$X_1, X_2, ..., X_n$$

consider the permutations

$$X_i, X_{i+1}, ..., X_n X_1, X_2, ..., X_{i-1}$$

for $i = 2, 3, ..., n$.

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consider the permutations

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for $i = 2, 3, ..., n$. Let

$$S_0^{(i)} = 0, \text{ and } S_k^{(i)} = X_i + X_{i+1} + ...$$

the permutable random walks.
Clearly,

$$\left\{ S_k^{(i)}, k = 0, 1, ..., n \right\} \overset{d}{=} \{ S_k, k = 0, 1, ..., n \} .$$

Let $T_r^{(i)}$ be the rth strict ascending epoch for $\left\{S_k^{(i)}, k = 0, 1, ..., n\right\}$.

If $T_r = n$ for some r then $T_r^{(i)} = n$ for exactly $r - 1$ sequences $\left\{S_k^{(i)}, k = 0, 1, ..., n\right\}, i = 2, 3, ..., n$

(PROOF by picture!!!)

Besides,

$$S_n = S_n^{(2)} = ... = S_n^{(n)}.$$

Consider for a positive a the probability

$$\mathbf{P}\left(T_r = n, 0 < S_n \leq a\right)$$

and let

$$\eta_i = I\left\{T_r^{(i)} = n, 0 < S_n^{(i)} \leq a\right\}, i = 1, 2, ..., n$$

be a sequence of identically distributed RW.

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and let

$$\eta_i = I\left\{T_r^{(i)} = n, 0 < S_n^{(i)} \leq a\right\}, i = 1, 2, ..., n$$

be a sequence of identically distributed RW. Hence

$$\mathbf{P}\left(T_r = n, 0 < S_n \leq a\right) = \mathbf{E}\eta_1 = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\eta_i.$$

Consider for a positive a the probability

$$\mathbf{P}\left(T_r = n, 0 < S_n \leq a\right)$$

and let

$$\eta_i = I\left\{T_r^{(i)} = n, 0 < S_n^{(i)} \leq a\right\}, i = 1, 2, ..., n$$

be a sequence of identically distributed RW. Hence

$$\mathbf{P}\left(T_r = n, 0 < S_n \leq a\right) = \mathbf{E}\eta_1 = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\eta_i.$$

In view of the remark about the number of strong ascending epochs

$$\sum_{i=1}^{n}\eta_i$$

takes only two values: either 0 or r. This gives

$$\sum_{i=1}^{n}\mathbf{E}\eta_i = r\mathbf{P}\left(\sum_{i=1}^{n}\eta_i = r\right).$$

Let $S_n > 0$ and let i_0 be the first moment when the maximal value of the sequence $S_0, S_1, ..., S_n$ is attained. Then

$$S_n^{(i_0+1)} > S_i^{(i_0+1)}$$

for all $i = 1, 2, ..., n-1$ and, therefore, for the sequence

$$\left\{ S_i^{(i_0+1)}, i = 0, 1, ..., n \right\}$$

the moment n is a strict ascending epoch for some r.

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$$\{0 < S_n \le a\} = \left\{ 0 < S_n^{(i_0+1)} \le a \right\} = \cup_{r=1}^{\infty} \{\eta_1 + ... + \eta_n = r\}$$

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Therefore,

$$\mathbf{P}(0 < S_n \leq a) = \sum_{r=1}^{\infty} \mathbf{P}(\eta_1 + ... + \eta_n = r).$$

Thus,

$$\frac{1}{n}\mathbf{P}\left(0 < S_n \le a\right) = \sum_{r=1}^{\infty} \frac{1}{rn} r \mathbf{P}\left(\eta_1 + \dots + \eta_n = r\right)$$

$$= \sum_{r=1}^{\infty} \frac{1}{rn} \sum_{i=1}^{n} \mathbf{E}\eta_i = \sum_{r=1}^{\infty} \frac{1}{r} \mathbf{P}\left(T_r = n, 0 < S_n \le a\right).$$

Thus,

$$\frac{1}{n}\mathbf{P}\left(0<S_n\le a\right)=\sum_{r=1}^{\infty}\frac{1}{rn}r\mathbf{P}\left(\eta_1+...+\eta_n=r\right)$$
$$=\sum_{r=1}^{\infty}\frac{1}{rn}\sum_{i=1}^{n}\mathbf{E}\eta_i=\sum_{r=1}^{\infty}\frac{1}{r}\mathbf{P}\left(T_r=n,0<S_n\le a\right).$$

Passing to the Laplace transforms we get

$$\sum_{r=1}^{\infty}\frac{1}{r}\mathbf{E}\left(e^{-\lambda S_n};T_r=n\right)=\frac{1}{n}\mathbf{E}\left(e^{-\lambda S_n};S_n>0\right)$$

Thus,

$$\frac{1}{n}\mathbf{P}\left(0<S_n\le a\right)=\sum_{r=1}^{\infty}\frac{1}{rn}r\mathbf{P}\left(\eta_1+...+\eta_n=r\right)$$
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$$\sum_{r=1}^{\infty}\frac{1}{r}\mathbf{E}\left(e^{-\lambda S_n};T_r=n\right)=\frac{1}{n}\mathbf{E}\left(e^{-\lambda S_n};S_n>0\right)$$

Multiplying by s^n and summing over $n=1,2,...$ we obtain

$$\sum_{r=1}^{\infty}\frac{1}{r}\sum_{n=1}^{\infty}s^n\mathbf{E}\left(e^{-\lambda S_n};T_r=n\right)=\sum_{n=1}^{\infty}\frac{s^n}{n}\mathbf{E}\left(e^{-\lambda S_n};S_n>0\right).$$

Further,

$$\begin{aligned}
\sum_{n=1}^{\infty} s^n \mathbf{E}\left(e^{-\lambda S_n}; T_r = n\right) &= \mathbf{E}\left(s^{T_r} e^{-\lambda S_{T_r}}; T_r < \infty\right) \\
&= \left(\mathbf{E}\left(s^{T} e^{-\lambda S_{T}}; T < \infty\right)\right)^r \\
&= \left(\sum_{n=1}^{\infty} s^n \mathbf{E}\left(e^{-\lambda S_n}; T = n\right)\right)^r
\end{aligned}$$

and, therefore,

$$\sum_{r=1}^{\infty} \frac{1}{r} \sum_{n=1}^{\infty} s^n \mathbf{E}\left(e^{-\lambda S_n}; T_r = n\right) = -\log\left(1 - \mathbf{E}\left(s^{T} e^{-\lambda S_{\tau}}; T < \infty\right)\right).$$

As a result

$$-\log\left(1-\mathbf{E}\left(s^T e^{-\lambda S_\tau}; T<\infty\right)\right)=\sum_{n=1}^{\infty}\frac{s^n}{n}\mathbf{E}\left(e^{-\lambda S_n}; S_n>0\right)$$

or

$$1-\sum_{n=1}^{\infty}s^n\mathbf{E}\left[e^{-\lambda S_n}; T=n\right]=\exp\left\{-\sum_{n=1}^{\infty}\frac{s^n}{n}\mathbf{E}\left[e^{-\lambda S_n}; S_n>0\right]\right\}.$$

Theorem

For $\lambda > 0$ *and* $|s| < 1$

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n}; \tau > n\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right]\right\}$$

and

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{\lambda S_n}; T > n\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{\lambda S_n}; S_n \le 0\right]\right\}$$

Proof is omitted.

Spitzer identity.

Let

$$M_n = \max_{0 \le k \le n} S_n.$$

Theorem

For $\lambda, \mu > 0$ *and* $|s| < 1$

$$\sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda M_n - \mu(M_n - S_n)}\right]$$
$$= \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n}\left(\mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right] + \mathbf{E}\left[e^{\mu S_n}; S_n \le 0\right]\right)\right\}.$$

In particular,

$$\sum_{n=1}^{\infty} s^n \mathbf{E} e^{-\lambda M_n} = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E} e^{-\lambda \max(0, S_n)}\right\}.$$

Proof. Let

$$R_n := \min\{k : S_k = M_n\}.$$

We have

$$\mathbf{E}\left[e^{-\lambda M_n - \mu(M_n - S_n)}\right] = \sum_{k=0}^{n} \mathbf{E}\left[e^{-\lambda M_n - \mu(M_n - S_n)}; R_n = k\right]$$

$$= \sum_{k=0}^{n} \mathbf{E}\left[e^{-\lambda S_k - \mu(S_k - S_n)}; R_n = k\right]$$

$$= \sum_{k=0}^{n} \mathbf{E}\left[e^{-\lambda S_k - \mu(S_k - S_n)}; R_k = k, S_k \geq S_j, j = k+1, ..., n\right]$$

$$= \sum_{k=0}^{n} \mathbf{E}\left[e^{-\lambda S_k}; R_k = k\right] \mathbf{E}\left[e^{-\mu(S_k - S_n)}; S_k \geq S_j, j = k+1, ..., n\right]$$

$$= \sum_{k=0}^{n} \mathbf{E}\left[e^{-\lambda S_k}; \tau > k\right] \mathbf{E}\left[e^{\mu S_{n-k}}; T > n-k\right].$$

Now multiplying by s^n and summing over $n = 0, 1, ...$ gives

$$\begin{aligned}
&\sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda M_n - \mu(M_n - S_n)}\right] \\
&= \sum_{k=0}^{\infty} s^k \mathbf{E}\left[e^{-\lambda S_k}; \tau > k\right] \sum_{l=0}^{\infty} s^l \mathbf{E}\left[e^{\mu S_l}; T > l\right] \\
= \quad & \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \left(\mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right] + \mathbf{E}\left[e^{\mu S_n}; S_n \leq 0\right]\right)\right\}.
\end{aligned}$$

Application of Sparre-Anderson and Spitzer identities

Recall that a function $L(t)$, $t > 0$ is called slowly varying if

$$\lim_{t\to+\infty} \frac{L(tx)}{L(t)} = 1 \text{ for any } x > 0.$$

Theorem

(Tauberian theorem). Assume $a_n \geq 0$ *and the series* $R(s) = \sum_{n=0}^{\infty} a_n s^n$ *converges for* $s \in [0,1)$. *Then the following statements are equivalent for* $\rho \in [0,\infty)$:

$$R(s) \sim \frac{1}{(1-s)^{\rho}} L\left(\frac{1}{1-s}\right) \text{ as } s \uparrow 1 \tag{4}$$

and

$$R_n := \sum_{k=0}^{n} a_k \sim \frac{1}{\Gamma(\rho+1)} n^{\rho} L(n) \text{ as } n \to \infty.$$

If a_n *is monotone and* $\rho \in (0,\infty)$ *then (4) is equivalent to*

$$a_n \sim \frac{1}{\Gamma(\rho)} n^{\rho-1} L(n) \text{ as } n \to \infty.$$

Theorem

Let $\mathbf{E}X = 0, \sigma^2 := \mathbf{E}X^2 \in (0, \infty)$. *Then*
1) the random variables τ, τ', T *and* T' *are proper random variables*
2)

$$\sum_{n=1}^{\infty} \frac{\mathbf{P}(S_n = 0)}{n} := c_0 < \infty \text{ and } \sum_{n=1}^{\infty} \frac{1}{n}\left[\mathbf{P}(S_n > 0) - \frac{1}{2}\right] := c$$

3)

$$\mathbf{E}S_T = \frac{\sigma}{\sqrt{2}} e^{-c}, \quad \mathbf{E}S_{T'} = \frac{\sigma}{\sqrt{2}} e^{-c-c_0}$$

and

$$\mathbf{E}S_\tau = \frac{\sigma}{\sqrt{2}} e^{c}, \quad \mathbf{E}S_{\tau'} = \frac{\sigma}{\sqrt{2}} e^{c+c_0}.$$

Proof. By Sparre-Anderson identity

$$1-\sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n}; T=n\right] = \exp\left\{-\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right]\right\} \quad (5)$$

and

$$1-\sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n}; T'=n\right] = \exp\left\{-\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n \geq 0\right]\right\}. \quad (6)$$

Hence, letting $\lambda \downarrow 0$ we get

$$1-\sum_{n=1}^{\infty} s^n \mathbf{P}\left(T'=n\right) = \exp\left\{-\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n \geq 0\right)\right\}.$$

This, in turn, gives as $s \uparrow 1$

$$1 - \mathbf{P}\left(T' < \infty\right) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{1}{n} \mathbf{P}\left(S_n \geq 0\right) \right\} = 0$$

since $\mathbf{P}\left(S_n \geq 0\right) \sim 2^{-1}$ as $n \to \infty$. Hence $\mathbf{P}\left(T' < \infty\right) = 1$. The arguments for τ, T and τ' are similar.
Now using (6) we conclude by letting $\lambda \to \infty$ that

$$1 - \sum_{n=1}^{\infty} s^n \mathbf{P}\left(S_{T'} = 0; T' = n\right) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n = 0\right) \right\}$$

and now as $s \uparrow 1$

$$1-\sum_{n=1}^{\infty} \mathbf{P}\left(S_{T'}=0; T'=n\right)=1-\mathbf{P}\left(S_{T'}=0\right)=\exp\left\{-\sum_{n=1}^{\infty} \frac{1}{n} \mathbf{P}\left(S_n=0\right)\right\}>0$$

since $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$. Hence

$$c_0 := \sum_{n=1}^{\infty} \frac{1}{n} \mathbf{P}\left(S_n=0\right)<\infty .$$

Further, differentiating (5) with respect to λ we get

$$\begin{aligned} & \sum_{n=1}^{\infty} s^n \mathbf{E}\left[S_n e^{-\lambda S_n}; T=n\right] \\ & =\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[S_n e^{-\lambda S_n}; S_n>0\right] \exp\left\{-\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n>0\right]\right\}. \end{aligned}$$

This allows us to pass to the limit as $\lambda \downarrow 0$ to get

$$\sum_{n=1}^{\infty} s^n \mathbf{E}\left[S_n; T = n\right] = \sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[S_n; S_n > 0\right] \exp\left\{-\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n > 0\right)\right\}$$

or

$$\sum_{n=1}^{\infty} s^n \mathbf{E}\left[S_n; T = n\right] = \frac{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[S_n; S_n > 0\right] \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \left[\mathbf{P}\left(S_n > 0\right) - \frac{1}{2}\right]\right\}}{\sqrt{1-s}}.$$

Note, that

$$\mathbf{E}\left[S_n; S_n > 0\right] = \sigma\sqrt{n}\mathbf{E}\left[\frac{S_n}{\sigma\sqrt{n}}; \frac{S_n}{\sigma\sqrt{n}} > 0\right]$$
$$\sim \sigma\sqrt{n}\frac{1}{\sqrt{2\pi}}\int_0^\infty xe^{-x^2/2}dx = \frac{\sigma\sqrt{n}}{\sqrt{2\pi}}.$$

If

$$a_n := \frac{1}{n}\mathbf{E}\left[S_n; S_n > 0\right] \sim \frac{\sigma}{\sqrt{2\pi}}n^{-1/2}$$

then

$$\sum_{k=1}^{n} a_k \sim \frac{\sigma}{\sqrt{2\pi}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \sim \frac{2\sigma}{\sqrt{2\pi}}\sqrt{n}$$

implying by Tauberian theorem

$$\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[S_n; S_n > 0\right] \sim \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{1-s}} \frac{2\sigma}{\sqrt{2\pi}} = \frac{1}{\sqrt{1-s}} \frac{\sigma}{\sqrt{2}} \text{ as } s \uparrow 1.$$

Thus, as $s \uparrow 1$

$$\sum_{n=1}^{\infty} s^n \mathbf{E}\left[S_n; T = n\right] \sim \frac{\sigma}{\sqrt{2}} \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \left[\mathbf{P}\left(S_n > 0\right) - \frac{1}{2}\right]\right\}$$

In view of

$$\lim_{s\uparrow 1}\sum_{n=1}^{\infty} s^n \mathbf{E}\left[S_n; T=n\right] = \mathbf{E}\left[S_T; T<+\infty\right] = \mathbf{E}S_T > 0.$$

for **ANY random walk** there exists the limit

$$\lim_{s\uparrow 1}\exp\left\{\sum_{n=1}^{\infty}\frac{s^n}{n}\left[\mathbf{P}\left(S_n>0\right)-\frac{1}{2}\right]\right\} := b > 0. \qquad (7)$$

and

$$\mathbf{E}S_\tau = \frac{\sigma}{\sqrt{2}}b.$$

We show that $b < \infty$.

Assume the opposite. Then

$$\lim_{s\uparrow 1}\sum_{n=1}^{\infty}\frac{s^n}{n}\left[-\frac{1}{2}+\mathbf{P}\left(S_n>0\right)\right]=+\infty$$

Hence

$$\lim_{s\uparrow 1}\sum_{n=1}^{\infty}\frac{s^n}{n}\left[-\frac{1}{2}+\mathbf{P}\left(S_n<0\right)\right]=-\infty$$

For the random walk $\{S_n^*\}$ with steps $-X_1,-X_2,...,-X_n,...$ we get

$$\lim_{s\uparrow 1}\sum_{n=1}^{\infty}\frac{s^n}{n}\left[-\frac{1}{2}+\mathbf{P}\left(S_n^*>0\right)\right]=-\infty$$

and this contradicts (7) applied to $\{S_n^*\}$. Thus,

$$b=\exp\left\{-\sum_{n=1}^{\infty}\frac{1}{n}\left[\mathbf{P}\left(S_n>0\right)-\frac{1}{2}\right]\right\}=e^{-c}$$

implying

$$\mathbf{E}S_T=\frac{\sigma}{\sqrt{2}}e^{-c}.$$

Theorem

Let $\mathbf{E}X = 0, \sigma^2 := \mathbf{E}X^2 \in (0, \infty)$. *Then as* $n \to \infty$

$$\mathbf{P}\left(\tau > n\right) \sim \frac{1}{\sqrt{\pi}} e^{c} \frac{1}{\sqrt{n}}, \quad \mathbf{P}\left(T > n\right) \sim \frac{1}{\sqrt{\pi}} e^{-c} \frac{1}{\sqrt{n}}.$$

Proof. Only the first statement. By Sparre-Anderson identity we have

$$1+\sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n};\tau>n\right]=\exp\left\{\sum_{n=1}^{\infty}\frac{s^n}{n}\mathbf{E}\left[e^{-\lambda S_n};S_n>0\right]\right\}$$

or, passing to the limit as $\lambda \downarrow 0$

$$\begin{aligned}1+\sum_{n=1}^{\infty} s^n \mathbf{P}\left(\tau>n\right) &= \exp\left\{\sum_{n=1}^{\infty}\frac{s^n}{n}\mathbf{P}\left(S_n>0\right)\right\}\\ &=\frac{1}{\sqrt{1-s}}\exp\left\{\sum_{n=1}^{\infty}\frac{s^n}{n}\left[\mathbf{P}\left(S_n>0\right)-\frac{1}{2}\right]\right\}.\end{aligned}$$

Therefore, as $s \uparrow 1$

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{P}\left(\tau > n\right) \sim \frac{1}{\sqrt{1-s}} e^c$$

or, by monotonicity of $\mathbf{P}\left(\tau > n\right)$

$$\mathbf{P}\left(\tau > n\right) \sim \frac{1}{\Gamma(\frac{1}{2})} e^c \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{\pi}} e^c \frac{1}{\sqrt{n}}.$$

The rest is similar.

Let

$$M_n = \max_{1\le k\le n} S_k, \qquad L_n = \min_{1\le k\le n} S_k$$

We evaluate the probabilities

$$\mathbf{P}\left(M_n \le x\right), \qquad \mathbf{P}\left(L_n \ge -x\right).$$

Recall

$$T = T_1 = \min\left\{n > 0 : S_n > 0\right\}$$

and

$$T_j := \min\left\{n > T_{j-1} : S_n > S_{T_{j-1}}\right\}, j = 2, 3, \ldots$$

and

$$\tau = \tau_1 = \min\left\{n > 0 : S_n \le 0\right\}$$

and

$$\tau_j := \min\left\{n > \tau_{j-1} : S_n \le S_{\tau_{j-1}}\right\}, j = 2, 3, \ldots$$

Theorem

Let $\mathbf{E}X = 0, \sigma^2 := \mathbf{E}X^2 \in (0,\infty)$. *Then for any* $x \geq 0$ *as* $n \to \infty$

$$\mathbf{P}\left(M_n \leq x\right) \sim \frac{e^{-c}}{\sqrt{\pi}} U(x) \frac{1}{\sqrt{n}}, \qquad \mathbf{P}\left(L_n \geq -x\right) \sim \frac{e^{c}}{\sqrt{\pi}} V(x) \frac{1}{\sqrt{n}}$$

where

$$U(x) = 1 + \sum_{i=1}^{\infty} \mathbf{P}\left(S_{T_i} \leq x\right) \qquad V(x) = 1 + \sum_{i=1}^{\infty} \mathbf{P}\left(S_{\tau_i} \geq -x\right).$$

Proof. Only the first. By Spitzer identity

$$\sum_{n=1}^{\infty} s^n \mathbf{E} e^{-\lambda M_n} = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E} e^{-\lambda \max(0, S_n)}\right\}$$
$$= \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right]\right\} \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n \le 0\right)\right\}.$$

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$$= \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right]\right\} \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n \leq 0\right)\right\}.$$

By a Sparre -Anderson identity

$$\exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right]\right\} = 1 + \sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n}; \tau > n\right] = \int_0^{+\infty} e^{-\lambda x} U_s(dx)$$

where

$$U_s(x) = \sum_{n=0}^{\infty} s^n \mathbf{P}\left(S_n \leq x; \tau > n\right).$$

Therefore,

$$\sum_{n=1}^{\infty} s^n \mathbf{P}\left(M_n \leq x\right) = U_s(x) \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n \leq 0\right)\right\}$$

Therefore,

$$\sum_{n=1}^{\infty} s^n \mathbf{P}\left(M_n \leq x\right) = U_s(x) \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n \leq 0\right)\right\}$$

Note that

$$\begin{aligned} \lim_{s \uparrow 1} U_s(x) &= \sum_{n=0}^{\infty} \mathbf{P}\left(S_n \leq x; \tau > n\right) \\ &= 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(S_n \leq x; S_n > S_j, j = 0, 1, ..., n-1\right) \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} s^n \mathbf{P}\left(M_n \leq x\right) = U_s(x) \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n \leq 0\right)\right\}$$

Note that

$$\begin{aligned}
\lim_{s \uparrow 1} U_s(x) &= \sum_{n=0}^{\infty} \mathbf{P}\left(S_n \leq x; \tau > n\right) \\
&= 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(S_n \leq x; S_n > S_j, j = 0, 1, ..., n-1\right) \\
&= 1 + \sum_{n=1}^{\infty} \sum_{r=1}^{n} \mathbf{P}\left(S_n \leq x; T_r = n\right)
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} s^n \mathbf{P}\left(M_n \le x\right) = U_s(x) \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n \le 0\right)\right\}$$

Note that

$$\begin{aligned}
\lim_{s\uparrow 1} U_s(x) &= \sum_{n=0}^{\infty} \mathbf{P}\left(S_n \le x; \tau > n\right) \\
&= 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(S_n \le x; S_n > S_j, j = 0, 1, ..., n-1\right) \\
&= 1 + \sum_{n=1}^{\infty} \sum_{r=1}^{n} \mathbf{P}\left(S_n \le x; T_r = n\right) \\
&= 1 + \sum_{r=1}^{\infty} \sum_{n=r}^{\infty} \mathbf{P}\left(S_n \le x; T_r = n\right) = 1 + \sum_{r=1}^{\infty} \mathbf{P}\left(S_{T_r} \le x\right) = U(x)
\end{aligned}$$

is a renewal function!

Clearly, as $s \uparrow 1$

$$\exp\left\{\sum_{n=1}^{\infty}\frac{s^n}{n}\mathbf{P}\left(S_n \le 0\right)\right\} = \frac{1}{\sqrt{1-s}}\exp\left\{\sum_{n=1}^{\infty}\frac{s^n}{n}\left[\mathbf{P}\left(S_n \le 0\right)-\frac{1}{2}\right]\right\}$$
$$\sim \frac{e^{-c}}{\sqrt{1-s}}$$

Thus, as $s \uparrow 1$

$$\sum_{n=1}^{\infty} s^n \mathbf{P}\left(M_n \le x\right) \sim \frac{U(x)}{\sqrt{1-s}}e^{-c}$$

and, by monotonicity of $\mathbf{P}\left(M_n \le x\right)$ we get

$$\mathbf{P}\left(M_n \le x\right) \sim \frac{U(x)}{\sqrt{\pi}}e^{-c}\frac{1}{\sqrt{n}}.$$

Corollary

Let $\mathbf{E}X = 0, \sigma^2 := \mathbf{E}X^2 \in (0, \infty)$. *Then there exists a constant* $K < \infty$ *such that for any* $x \geq 0$

$$\mathbf{P}\left(M_n \leq x\right) \leq \frac{KU(x)}{\sqrt{n}}, \qquad \mathbf{P}\left(L_n \geq -x\right) \leq \frac{KV(x)}{\sqrt{n}}.$$

Proof. We have

$$\sum_{n=1}^{\infty} s^n \mathbf{P}\left(M_n \leq x\right) \leq U(x) \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}\left(S_n \leq 0\right)\right\} = \frac{U(x)}{\sqrt{1-s}} h(s)$$

where

$$h(s) := \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n}\left[\mathbf{P}\left(S_n \leq 0\right) - \frac{1}{2}\right]\right\}.$$

Clearly,

$$\begin{aligned}\frac{n}{2}\left(1-\frac{1}{n}\right)^n \mathbf{P}\left(M_n \leq x\right) &\leq \sum_{n/2\leq k\leq n}\left(1-\frac{1}{n}\right)^k \mathbf{P}\left(M_k \leq x\right)\\ &\leq U(x)\sqrt{n}h\left(1-\frac{1}{n}\right)\end{aligned}$$

implying the desired statement as $s \uparrow 1$, i.e. $n \to \infty$.

Properties of some renewal functions

As we know

$$U(x) = 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(S_n \le x; L_n \ge 0\right) = 1 + \sum_{i=1}^{\infty} \mathbf{P}\left(S_{T_i} \le x\right)$$

where

$$T = T_1 = \min\left\{n > 0 : S_n > 0\right\}$$

and

$$T_j := \min\left\{n > T_{j-1} : S_n > S_{T_{j-1}}\right\}, j = 2, 3, ...$$

Thus, $U(x)$ is a renewal function (we assume that $U(x) = 0$, $x < 0$). Therefore, if $\mathbf{E}X = 0$ and $\mathbf{E}X^2 < \infty$ then, as $x \to \infty$

$$U(x) = \frac{x}{\mathbf{E}S_T} + o(x).$$

Let us show that $U(x)$ is a harmonic function, that is,

$$\mathbf{E}\left[U(x-X);x-X\geq 0\right]=U(x),x\geq 0.$$

where X has the same distribution as $X_1, ..., X_n, ...$

We have

$$\mathbf{E}\left[U(x-X); x-X \geq 0\right] = \mathbf{P}\left(X \leq x\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_k \leq x - X; L_k \geq 0\right)$$

We have

$$\begin{aligned}
\mathbf{E}\left[U(x-X);x-X\geq 0\right] &= \mathbf{P}\left(X\leq x\right)+\sum_{k=1}^{\infty}\mathbf{P}\left(S_k\leq x-X;L_k\geq 0\right)\\
&= \mathbf{P}\left(X_1\leq x;L_1\geq 0\right)\\
&\quad +\sum_{k=1}^{\infty}\mathbf{P}\left(S_{k+1}\leq x;L_{k+1}\geq 0\right)\\
&\quad +\mathbf{P}\left(X_1<0\right)+\sum_{k=1}^{\infty}\mathbf{P}\left(S_{k+1}<0;L_k\geq 0\right)
\end{aligned}$$

We have

$$\begin{aligned}
\mathbf{E}\left[U(x-X); x-X \geq 0\right] &= \mathbf{P}\left(X \leq x\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_k \leq x - X; L_k \geq 0\right) \\
&= \mathbf{P}\left(X_1 \leq x; L_1 \geq 0\right) \\
&\quad + \sum_{k=1}^{\infty} \mathbf{P}\left(S_{k+1} \leq x; L_{k+1} \geq 0\right) \\
&\quad + \mathbf{P}\left(X_1 < 0\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_{k+1} < 0; L_k \geq 0\right)
\end{aligned}$$

Clearly,

$$\mathbf{P}\left(X_1 < 0\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_{k+1} < 0; L_k \geq 0\right) = \mathbf{P}\left(S_k < 0 \text{ for some } k\right) = 1.$$

Hence the statement follows.

Consider strong descending ladder epochs :

$$\tau' = \tau_1' = \min\{n > 0 : S_n < 0\}$$

and

$$\tau_j' := \min\left\{n > \tau_{j-1} : S_n < S_{\tau_{j-1}'}\right\}$$

and consider the renewal function

$$V(x) := \begin{cases} 1 + \sum_{i=1}^{\infty} \mathbf{P}\left(S_{\tau_i'} \geq -x\right) & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Again as $x \to \infty$

$$V(x) = \frac{x}{\mathbf{E} S_{\tau'}} + o(x).$$

Besides, $V(x)$ is a harmonic function:

$$\mathbf{E}[V(x+X); x+X > 0] = V(x), x \geq 0.$$

Let $\mathcal{F}_n = \sigma\left(\Pi_0, \Pi_1, ..., \Pi_{n-1}; Z(0), Z(1), ...Z(n-1)\right)$ and let

$$\mathcal{F} := \vee_{n=1}^{\infty} \mathcal{F}_n$$

be a filtration. As earlier, denote

$$L_n := \min_{0 \le i \le n} S_n, \qquad M_n = \max_{1 \le i \le n} S_n.$$

Lemma

The sequences

$$V(S_n) I\left\{L_n \ge 0\right\}$$

and

$$U(-S_n) I\left\{M_n < 0\right\}$$

are martingales with respect to filtration $\mathcal{F}$.

Proof. Only the first statement. Observe that

$$V(S_{n+1})I\left\{L_{n+1} \geq 0\right\} = V(S_n + X_{n+1})I\left\{L_n \geq 0\right\}$$

Therefore,

$$\begin{aligned} \mathbf{E}\left[V(S_{n+1})I\left\{L_{n+1} \geq 0\right\}|\mathcal{F}_n\right] &= \mathbf{E}\left[V(S_n + X_{n+1})|\mathcal{F}_n\right] I\left\{L_n \geq 0\right\} \\ &= V(S_n)I\left\{L_n \geq 0\right\} \end{aligned}$$

as desired.

Introduce two sequence of probability measures

$$d\mathbf{P}_n^+ = V(S_n) I\{L_n \geq 0\} \, d\mathbf{P}, \quad n \in \mathbb{N}$$

and

$$d\mathbf{P}_n^- = U(-S_n) I\{M_n < 0\} \, d\mathbf{P}, \quad n \in \mathbb{N}$$

on $\mathcal{F}_n$ or, what is the same, for any nonnegative random variable Y_n measurable with respect to $\mathcal{F}_n$

$$\mathbf{E}_n^+ [Y_n] = \mathbf{E} [Y_n V(S_n) I\{L_n \geq 0\}]$$

and

$$\mathbf{E}_n^- [Y_n] = \mathbf{E} [Y_n U(-S_n) I\{M_n < 0\}] .$$

They are consistent since, for instance, for any $Y_n \in \mathcal{F}_n$

$$\begin{aligned} \mathbf{E}_{n+1}^{+}\left[Y_n\right] &= \mathbf{E}\left[Y_n V(S_{n+1}) I\left\{L_{n+1} \geq 0\right\}\right] \\ &= \mathbf{E}\left[Y_n V(S_n) I\left\{L_n \geq 0\right\}\right] = \mathbf{E}_n^{+}\left[Y_n\right]. \end{aligned}$$

Hence, there exists a probability measure $\mathbf{P}^{+}$ on $\mathcal{F}$ such that

$$\mathbf{P}^{+}|\mathcal{F}_n = \mathbf{P}_n^{+}, \quad n \geq 0.$$

or,

$$\mathbf{E}^{+}\left[Y_n\right] = \mathbf{E}\left[Y_n V(S_n) I\left\{L_n \geq 0\right\}\right].$$

Similarly, we have a measure $\mathbf{P}^{-}$ on $\mathcal{F}$ such that

$$\mathbf{P}^{-}|\mathcal{F}_n = \mathbf{P}_n^{-}, \quad n \geq 0.$$

We know that

$$\mathbf{P}\left(L_n \geq -x\right) \sim \frac{cV(x)}{\sqrt{n}}$$

and there exists a constant $K > 0$ such that

$$\mathbf{P}\left(L_n \geq -x\right) \leq \frac{KV(x)}{\sqrt{n}}$$

for all n and $x \geq 0$.

Lemma

Let $\mathbf{E}X = 0$ *and* $\sigma^2 := \mathbf{E}X^2 \in (0,\infty)$. *Then for any* $\mathcal{F}_k$*-measurable bounded random variable* $\psi_\kappa, k \in \mathbb{N}$

$$\lim_{n\to\infty} \mathbf{E}\left[\psi_\kappa | L_n \geq 0\right] = \mathbf{E}^+\left[\psi_\kappa\right] = \mathbf{E}\left[\psi_\kappa V(S_k) I\left\{L_k \geq 0\right\}\right],$$

$$\lim_{n\to\infty} \mathbf{E}\left[\psi_\kappa | M_n < 0\right] = \mathbf{E}^-\left[\psi_\kappa\right] = \mathbf{E}\left[\psi_\kappa U(-S_k) I\left\{M_k < 0\right\}\right].$$

If the sequence $\psi_1, \psi_2, ..., \psi_n, ...$ *is uniformly bounded and is adopted to filtration* $\mathcal{F}$ *and*

$$\lim_{n\to\infty} \psi_n := \psi_\infty$$

$\mathbf{P}^+$ *a.s., (*$\mathbf{P}^-$ *a.s.) then*

$$\lim_{n\to\infty} \mathbf{E}\left[\psi_n | L_n \geq 0\right] = \mathbf{E}^+\left[\psi_\infty\right]$$

and

$$\lim_{n\to\infty} \mathbf{E}\left[\psi_n | M_n < 0\right] = \mathbf{E}^-\left[\psi_\infty\right]$$

Proof. Only the first. Let

$$L_{k,n} := \min_{k \le i \le n} (S_i - S_k)$$

We have

$$\begin{aligned} \mathbf{E}\left[\psi_k | L_n \ge 0\right] &= \frac{\mathbf{E}\left[\psi_k I\left\{L_n \ge 0\right\}\right]}{\mathbf{P}\left(L_n \ge 0\right)} = \frac{\mathbf{E}\left[\psi_k I\left\{L_k \ge 0\right\} \mathbf{P}\left(L_{k,n} \ge -S_k\right)\right]}{\mathbf{P}\left(L_n \ge 0\right)} \\ &= \mathbf{E}\left[\psi_k I\left\{L_k \ge 0\right\} \frac{\mathbf{P}\left(L_{k,n} \ge -S_k\right)}{\mathbf{P}\left(L_n \ge 0\right)}\right]. \end{aligned}$$

By theorems of the previous part

$$\mathbf{P}\left(L_n \ge -x\right) \le \frac{KV(x)}{\sqrt{n}}$$

for all $x \ge 0$ and

$$\mathbf{P}\left(L_n \ge -x\right) \sim \frac{CV(x)}{\sqrt{n}}$$

for any fixed x.

This and the bounded convergence theorem imply

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}\left[\psi_k I\left\{L_k \geq 0\right\} \frac{\mathbf{P}\left(L_{k,n} \geq -S_k\right)}{\mathbf{P}\left(L_n \geq 0\right)}\right] \\
&\quad = \mathbf{E}\left[\psi_k I\left\{L_k \geq 0\right\} \lim_{n\to\infty} \frac{\mathbf{P}\left(L_{k,n} \geq -S_k\right)}{\mathbf{P}\left(L_n \geq 0\right)}\right] \\
&\quad = \mathbf{E}\left[\psi_k V(S_k) I\left\{L_k \geq 0\right\}\right]
\end{aligned}$$

proving the first part of the lemma.

For the second we fix $\gamma > 1$ and observe that

$$\begin{aligned}\mathbf{E}\left[\psi_k | L_{n\gamma} \geq 0\right] - \mathbf{E}\left[|\psi_n - \psi_k| \, | L_{n\gamma} \geq 0\right] \leq \mathbf{E}\left[\psi_n | L_{n\gamma} \geq 0\right] \\ \leq \mathbf{E}\left[\psi_k | L_{n\gamma} \geq 0\right] + \mathbf{E}\left[|\psi_n - \psi_k| \, | L_{n\gamma} \geq 0\right]\end{aligned}$$

and

$$\begin{aligned}&\mathbf{E}\left[|\psi_n - \psi_k| \, | L_{n\gamma} \geq 0\right] \\ &\quad = \frac{\mathbf{E}\left[|\psi_n - \psi_k| \, I\left\{L_{n\gamma} \geq 0\right\}\right]}{\mathbf{P}\left(L_{n\gamma} \geq 0\right)} \\ &\quad = \mathbf{E}\left[|\psi_n - \psi_k| \, I\left\{L_n \geq 0\right\} \frac{\mathbf{P}\left(L_{n,n\gamma} \geq -S_n\right)}{\mathbf{P}\left(L_{n\gamma} \geq 0\right)}\right] \\ &\quad \leq K \frac{\mathbf{P}\left(L_{n(\gamma-1)} \geq 0\right)}{\mathbf{P}\left(L_{n\gamma} \geq 0\right)} \mathbf{E}\left[|\psi_n - \psi_k| \, I\left\{L_n \geq 0\right\} V(S_n)\right] \\ &\quad \leq K_1 \mathbf{E}\left[|\psi_n - \psi_k| \, I\left\{L_n \geq 0\right\} V(S_n)\right] = K_1 \mathbf{E}^+\left[|\psi_n - \psi_k|\right].\end{aligned}$$

Now first we let $n \to \infty$ and then $k \to \infty$ we get

$$\lim_{n\to\infty} \mathbf{E}\left[\psi_n | L_{n\gamma} \geq 0\right] = \lim_{k\to\infty} \lim_{n\to\infty} \mathbf{E}\left[\psi_k | L_{n\gamma} \geq 0\right] = \mathbf{E}^{+}\psi_{\infty}.$$

Further we have

$$
\begin{aligned}
& \left|\mathbf{E}\left[\psi_n | L_n \geq 0\right] - \mathbf{E}\left[\psi_n | L_{n\gamma} \geq 0\right]\right| \\
= & \left|\frac{\mathbf{E}\left[\psi_n I\left\{L_n \geq 0\right\}\right]}{\mathbf{P}\left(L_n \geq 0\right)} - \frac{\mathbf{E}\left[\psi_n I\left\{L_{n\gamma} \geq 0\right\}\right]}{\mathbf{P}\left(L_{n\gamma} \geq 0\right)}\right| \\
= & \left|\frac{\mathbf{E}\left[\psi_n\left(I\left\{L_n \geq 0\right\} - I\left\{L_{n\gamma} \geq 0\right\}\right)\right]}{\mathbf{P}\left(L_n \geq 0\right)}\right. \\
& \left. - \left(\frac{1}{\mathbf{P}\left(L_{n\gamma} \geq 0\right)} - \frac{1}{\mathbf{P}\left(L_n \geq 0\right)}\right) \mathbf{E}\left[\psi_n I\left\{L_{n\gamma} \geq 0\right\}\right]\right| \\
\leq & \left|\frac{\mathbf{E}\left[\psi_n\left(I\left\{L_n \geq 0\right\} - I\left\{L_{n\gamma} \geq 0\right\}\right)\right]}{\mathbf{P}\left(L_n \geq 0\right)}\right| \\
& + K\left(\frac{1}{\mathbf{P}\left(L_{n\gamma} \geq 0\right)} - \frac{1}{\mathbf{P}\left(L_n \geq 0\right)}\right) \mathbf{P}\left(L_{n\gamma} \geq 0\right) \\
\leq & K_1 \frac{\mathbf{P}\left(L_n \geq 0, L_{n\gamma} < 0\right)}{\mathbf{P}\left(L_n \geq 0\right)} + K\left(1 - \frac{\mathbf{P}\left(L_{n\gamma} \geq 0\right)}{\mathbf{P}\left(L_n \geq 0\right)}\right) \leq K_2\left(1 - \frac{\mathbf{P}\left(L_{n\gamma} \geq\right.}{\mathbf{P}\left(L_n \geq\right.}\right.
\end{aligned}
$$

and, therefore, in view of

$$\mathbf{P}(L_n \geq 0) \sim \frac{C}{\sqrt{n}}$$

we have

$$\begin{aligned} & \limsup_{\gamma \downarrow 1} \lim \sup_{n \to \infty} \left| \mathbf{E}[\psi_n | L_n \geq 0] - \mathbf{E}[\psi_n | L_{n\gamma} \geq 0] \right| \\ \leq \quad & K_2 \limsup_{\gamma \downarrow 1} \left(1 - \frac{1}{\sqrt{\gamma}} \right) = 0. \end{aligned}$$

from which the statement of the lemma follows.