Sparre-Anderson identity

Theorem

For $\lambda > 0$ *and* $|s| < 1$

$$1 - \sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n}; T = n\right] = \exp\left\{-\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right]\right\}.$$

Recall

$$T = \min\{n > 0 : S_n > 0\}$$

Proof. Along with

$$X_1, X_2, ..., X_n$$

consider the permutations

$$X_i, X_{i+1}, ..., X_n X_1, X_2, ..., X_{i-1}$$

for $i = 2, 3, ..., n$.

Proof. Along with

$$X_1, X_2, ..., X_n$$

consider the permutations

$$X_i, X_{i+1}, ..., X_n X_1, X_2, ..., X_{i-1}$$

for $i = 2, 3, ..., n$. Let

$$S_0^{(i)} = 0, \text{ and } S_k^{(i)} = X_i + X_{i+1} + ...$$

the permutable random walks.
Clearly,

$$\left\{ S_k^{(i)}, k = 0, 1, ..., n \right\} \stackrel{d}{=} \{ S_k, k = 0, 1, ..., n \}.$$

Let $T_r^{(i)}$ be the rth strict ascending epoch for $\left\{S_k^{(i)}, k = 0, 1, ..., n\right\}$.
If $T_r = n$ for some r then $T_r^{(i)} = n$ for exactly $r - 1$ sequences $\left\{S_k^{(i)}, k = 0, 1, ..., n\right\}, i = 2, 3, ..., n$

(PROOF by picture!!!)

Besides,

$$S_n = S_n^{(2)} = ... = S_n^{(n)}.$$

Consider for a positive a the probability

$$\mathbf{P}\left(T_r = n, 0 < S_n \leq a\right)$$

and let

$$\eta_i = I\left\{T_r^{(i)} = n, 0 < S_n^{(i)} \leq a\right\}, i = 1, 2, ..., n$$

be a sequence of identically distributed RW.

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$$\mathbf{P}\left(T_r = n, 0 < S_n \le a\right)$$

and let

$$\eta_i = I\left\{T_r^{(i)} = n, 0 < S_n^{(i)} \le a\right\}, i = 1, 2, ..., n$$

be a sequence of identically distributed RW. Hence

$$\mathbf{P}\left(T_r = n, 0 < S_n \le a\right) = \mathbf{E}\eta_1 = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\eta_i.$$

Consider for a positive a the probability

$$\mathbf{P}\left(T_r = n, 0 < S_n \leq a\right)$$

and let

$$\eta_i = I\left\{T_r^{(i)} = n, 0 < S_n^{(i)} \leq a\right\}, i = 1, 2, ..., n$$

be a sequence of identically distributed RW. Hence

$$\mathbf{P}\left(T_r = n, 0 < S_n \leq a\right) = \mathbf{E}\eta_1 = \frac{1}{n}\mathbf{E}\Big[\sum_{i=1}^{n}\eta_i\Big].$$

In view of the remark about the number of strong ascending epochs

$$\sum_{i=1}^{n}\eta_i$$

takes only two values: either 0 or r. This gives

$$\mathbf{E}\Big[\sum_{i=1}^{n}\eta_i\Big] = r\mathbf{P}\left(\sum_{i=1}^{n}\eta_i = r\right).$$

Let $S_n > 0$ and let i_0 be the first moment when the maximal value of the sequence $S_0, S_1, ..., S_n$ is attained. Then

$$S_n^{(i_0+1)} > S_i^{(i_0+1)}$$

for all $i = 1, 2, ..., n-1$ and, therefore, for the sequence

$$\left\{ S_i^{(i_0+1)}, i = 0, 1, ..., n \right\}$$

the moment n is a strict ascending epoch for some r.

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the moment n is a strict ascending epoch for some r. Thus,

$$\{0 < S_n \leq a\} = \left\{ 0 < S_n^{(i_0+1)} \leq a \right\} = \cup_{r=1}^{\infty} \{\eta_1 + ... + \eta_n = r\}$$

Let $S_n > 0$ and let i_0 be the first moment when the maximal value of the sequence $S_0, S_1, ..., S_n$ is attained. Then

$$S_n^{(i_0+1)} > S_i^{(i_0+1)}$$

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Therefore,

$$\mathbf{P}(0 < S_n \leq a) = \sum_{r=1}^{\infty} \mathbf{P}(\eta_1 + ... + \eta_n = r).$$

Thus,

$$\frac{1}{n}\mathbf{P}\left(0 < S_n \le a\right) = \sum_{r=1}^{\infty} \frac{1}{r}\frac{r}{n}\mathbf{P}\left(\eta_1 + ... + \eta_n = r\right)$$

$$= \sum_{r=1}^{\infty} \frac{1}{r} \times \frac{1}{n}\mathbf{E}\Big[\sum_{i=1}^{n} \eta_i\Big] = \sum_{r=1}^{\infty} \frac{1}{r}\mathbf{P}\left(T_r = n, 0 < S_n \le a\right).$$

Thus,

$$\frac{1}{n}\mathbf{P}\left(0 < S_n \le a\right) = \sum_{r=1}^{\infty} \frac{1}{rn} r \mathbf{P}\left(\eta_1 + ... + \eta_n = r\right)$$

$$= \sum_{r=1}^{\infty} \frac{1}{rn} \sum_{i=1}^{n} \mathbf{E}\eta_i = \sum_{r=1}^{\infty} \frac{1}{r}\mathbf{P}\left(T_r = n, 0 < S_n \le a\right).$$

Passing to the Laplace transforms we get

$$\sum_{r=1}^{\infty} \frac{1}{r}\mathbf{E}\left(e^{-\lambda S_n}; T_r = n\right) = \frac{1}{n}\mathbf{E}\left(e^{-\lambda S_n}; S_n > 0\right)$$

Thus,

$$\frac{1}{n}\mathbf{P}\left(0 < S_n \leq a\right) = \sum_{r=1}^{\infty} \frac{1}{rn} r\mathbf{P}\left(\eta_1 + ... + \eta_n = r\right)$$

$$= \sum_{r=1}^{\infty} \frac{1}{rn} \sum_{i=1}^{n} \mathbf{E}\eta_i = \sum_{r=1}^{\infty} \frac{1}{r}\mathbf{P}\left(T_r = n, 0 < S_n \leq a\right).$$

Passing to the Laplace transforms we get

$$\sum_{r=1}^{\infty} \frac{1}{r}\mathbf{E}\left(e^{-\lambda S_n}; T_r = n\right) = \frac{1}{n}\mathbf{E}\left(e^{-\lambda S_n}; S_n > 0\right)$$

Multiplying by s^n and summing over $n = 1, 2, ...$ we obtain

$$\sum_{r=1}^{\infty} \frac{1}{r} \sum_{n=1}^{\infty} s^n \mathbf{E}\left(e^{-\lambda S_n}; T_r = n\right) = \sum_{n=1}^{\infty} \frac{s^n}{n}\mathbf{E}\left(e^{-\lambda S_n}; S_n > 0\right).$$

Further,

$$\begin{aligned}
\sum_{n=1}^{\infty} s^n \mathbf{E}\left(e^{-\lambda S_n}; T_r = n\right) &= \mathbf{E}\left(s^{T_r} e^{-\lambda S_{T_r}}; T_r < \infty\right) \\
&= \left(\mathbf{E}\left(s^{T} e^{-\lambda S_{T}}; T < \infty\right)\right)^r \\
&= \left(\sum_{n=1}^{\infty} s^n \mathbf{E}\left(e^{-\lambda S_n}; T = n\right)\right)^r
\end{aligned}$$

Further,

$$\begin{aligned}\sum_{n=1}^{\infty} s^n \mathbf{E}\left(e^{-\lambda S_n}; T_r = n\right) &= \mathbf{E}\left(s^{T_r} e^{-\lambda S_{T_r}}; T_r < \infty\right) \\ &= \left(\mathbf{E}\left(s^T e^{-\lambda S_T}; T < \infty\right)\right)^r \\ &= \left(\sum_{n=1}^{\infty} s^n \mathbf{E}\left(e^{-\lambda S_n}; T = n\right)\right)^r\end{aligned}$$

and, therefore,

$$\sum_{r=1}^{\infty} \frac{1}{r} \sum_{n=1}^{\infty} s^n \mathbf{E}\left(e^{-\lambda S_n}; T_r = n\right) = -\log\left(1 - \mathbf{E}\left(s^T e^{-\lambda S_\tau}; T < \infty\right)\right).$$

As a result

$$-\log\left(1-\mathbf{E}\left(s^{T}e^{-\lambda S_{\tau}};T<\infty\right)\right)=\sum_{n=1}^{\infty}\frac{s^{n}}{n}\mathbf{E}\left(e^{-\lambda S_{n}};S_{n}>0\right)$$

or

$$1-\sum_{n=1}^{\infty}s^{n}\mathbf{E}\left[e^{-\lambda S_{n}};T=n\right]=\exp\left\{-\sum_{n=1}^{\infty}\frac{s^{n}}{n}\mathbf{E}\left[e^{-\lambda S_{n}};S_{n}>0\right]\right\}.$$

Theorem

For $\lambda > 0$ *and* $|s| < 1$

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n}; \tau > n\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right]\right\}$$

and

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{\lambda S_n}; T > n\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{\lambda S_n}; S_n \le 0\right]\right\}$$

Proof is omitted.

Spitzer identity.

Let

$$M_n = \max_{0 \le k \le n} S_n.$$

Theorem

For $\lambda > 0$

$$\sum_{n=1}^{\infty} s^n \mathbf{E} e^{-\lambda M_n} = \exp\left\{ \sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E} e^{-\lambda \max(0, S_n)} \right\}.$$

Proof. Omitted.

Application of Sparre-Anderson and Spitzer identities

Theorem

Let $\mathbf{E}X = 0, \sigma^2 := \mathbf{E}X^2 \in (0, \infty)$. *Then*
1) the random variables τ, τ', T *and* T' *are proper random variables*
2)

$$\sum_{n=1}^{\infty} \frac{\mathbf{P}(S_n = 0)}{n} := c_0 < \infty \text{ and } \sum_{n=1}^{\infty} \frac{1}{n}\left[\mathbf{P}(S_n > 0) - \frac{1}{2}\right] := c < \infty$$

Proof is omitted.

Theorem

Let $\mathbf{E}X = 0, \sigma^2 := \mathbf{E}X^2 \in (0,\infty)$. *Then as* $n \to \infty$

$$\mathbf{P}(\tau > n) \sim \frac{1}{\sqrt{\pi}} e^{c} \frac{1}{\sqrt{n}}, \quad \mathbf{P}(T > n) \sim \frac{1}{\sqrt{\pi}} e^{-c} \frac{1}{\sqrt{n}}.$$

Proof. Only the first statement. By Sparre-Anderson identity we have

$$1+\sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n}; \tau > n\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n > 0\right]\right\}$$

Proof. Only the first statement. By Sparre-Anderson identity we have

$$1+\sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{-\lambda S_n};\tau>n\right]=\exp\left\{\sum_{n=1}^{\infty}\frac{s^n}{n}\mathbf{E}\left[e^{-\lambda S_n};S_n>0\right]\right\}$$

or, passing to the limit as $\lambda\downarrow 0$

$$\begin{aligned}1+\sum_{n=1}^{\infty} s^n \mathbf{P}\left(\tau>n\right) &= \exp\left\{\sum_{n=1}^{\infty}\frac{s^n}{n}\mathbf{P}\left(S_n>0\right)\right\}\\ &=\frac{1}{\sqrt{1-s}}\exp\left\{\sum_{n=1}^{\infty}\frac{s^n}{n}\left[\mathbf{P}\left(S_n>0\right)-\frac{1}{2}\right]\right\}.\end{aligned}$$

Therefore, as $s \uparrow 1$

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{P}(\tau > n) \sim \frac{1}{\sqrt{1-s}} e^c$$

or, by monotonicity of $\mathbf{P}(\tau > n)$

$$\mathbf{P}(\tau > n) \sim \frac{1}{\Gamma(\frac{1}{2})} e^c \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{\pi}} e^c \frac{1}{\sqrt{n}}.$$

The rest is similar.

Properties of some renewal functions

Recall

$$T = T_1 = \min\{n > 0 : S_n > 0\}$$

and

$$T_j := \min\left\{n > T_{j-1} : S_n > S_{T_{j-1}}\right\}, j = 2, 3, ...$$

and

$$\tau = \tau_1 = \min\{n > 0 : S_n \leq 0\}$$

and

$$\tau_j := \min\left\{n > \tau_{j-1} : S_n \leq S_{\tau_{j-1}}\right\}, j = 2, 3, ...$$

Let

$$U(x) = 1 + \sum_{i=1}^{\infty} \mathbf{P}\left(S_{T_i} \leq x\right), \qquad V(x) = 1 + \sum_{i=1}^{\infty} \mathbf{P}\left(S_{\tau_i} > -x\right).$$

Let

$$M_n = \max_{1\leq k\leq n} S_k, \qquad L_n = \min_{1\leq k\leq n} S_k$$

One can show that

$$U(x) = 1 + \sum_{i=1}^{\infty} \mathbf{P}\left(S_{T_i} \leq x\right) = 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(S_n \leq x; L_n \geq 0\right)$$

and

$$V(x) = 1 + \sum_{i=1}^{\infty} \mathbf{P}\left(S_{\tau_i} > -x\right) = 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(S_n > x; M_n < 0\right)$$

Since $U(x)$ and $V(x)$ are renewal functions (we assume that $U(x) = V(x) = 0,\ x < 0$) we know that if $\mathbf{E}X = 0$ and $\mathbf{E}X^2 < \infty$ then, as $x \to \infty$

$$U(x) = \frac{x}{\mathbf{E}S_T} + o(x), \qquad V(x) = \frac{x}{\mathbf{E}S_\tau} + o(x).$$

Lemma

The functions $U(x)$ *and* $V(x)$ *are a harmonic function, that is, for all* $x > 0$

$$\mathbf{E}\left[U(x - X); x - X \geq 0\right] = U(x), \qquad \mathbf{E}[V(x + X); x + X > 0] = V(x).$$

where X *has the same distribution as* $X_1, ..., X_n, ...$

Proof. Only the first. We have

$$\mathbf{E}\left[U(x-X); x-X \geq 0\right] = \mathbf{P}\left(X \leq x\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_k \leq x-X; L_k \geq 0\right)$$

Proof. Only the first. We have

$$\begin{aligned}
\mathbf{E}\left[U(x-X); x-X \geq 0\right] &= \mathbf{P}\left(X \leq x\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_k \leq x - X; L_k \geq 0\right) \\
&= \mathbf{P}\left(X_1 \leq x; L_1 \geq 0\right) \\
&\quad + \sum_{k=1}^{\infty} \mathbf{P}\left(S_{k+1} \leq x; L_{k+1} \geq 0\right) \\
&\quad + \mathbf{P}\left(X_1 < 0\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_{k+1} < 0; L_k \geq 0\right)
\end{aligned}$$

Proof. Only the first. We have

$$\begin{aligned}\mathbf{E}\left[U(x-X); x-X \geq 0\right] &= \mathbf{P}\left(X \leq x\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_k \leq x - X; L_k \geq 0\right) \\ &= \mathbf{P}\left(X_1 \leq x; L_1 \geq 0\right) \\ &\quad + \sum_{k=1}^{\infty} \mathbf{P}\left(S_{k+1} \leq x; L_{k+1} \geq 0\right) \\ &\quad + \mathbf{P}\left(X_1 < 0\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_{k+1} < 0; L_k \geq 0\right)\end{aligned}$$

Clearly,

$$\mathbf{P}\left(X_1 < 0\right) + \sum_{k=1}^{\infty} \mathbf{P}\left(S_{k+1} < 0; L_k \geq 0\right) = \mathbf{P}\left(S_k < 0 \text{ for some } k\right) = 1.$$

Hence the statement follows.

Let $\mathcal{F}_n = \sigma\left(\Pi_0, \Pi_1, ..., \Pi_{n-1}; Z(0), Z(1), ...Z(n-1)\right)$ and let

$$\mathcal{F} := \vee_{n=1}^{\infty} \mathcal{F}_n$$

be a filtration. As earlier, denote

$$L_n := \min_{0 \le i \le n} S_n, \qquad M_n = \max_{1 \le i \le n} S_n.$$

Lemma

The sequences

$$V(S_n) I\left\{L_n \ge 0\right\}$$

and

$$U(-S_n) I\left\{M_n < 0\right\}$$

are martingales with respect to filtration $\mathcal{F}$*.*

Proof. Only the first statement. Observe that

$$V(S_{n+1})I\left\{L_{n+1} \geq 0\right\} = V(S_n + X_{n+1})I\left\{L_n \geq 0\right\}$$

Therefore,

$$\begin{aligned} \mathbf{E}\left[V(S_{n+1})I\left\{L_{n+1} \geq 0\right\}|\mathcal{F}_n\right] &= \mathbf{E}\left[V(S_n + X_{n+1})|\mathcal{F}_n\right]I\left\{L_n \geq 0\right\} \\ &= V(S_n)I\left\{L_n \geq 0\right\} \end{aligned}$$

as desired.

Introduce on $\mathcal{F}_n$ two sequence of probability measures

$$d\mathbf{P}_n^+ = V(S_n) I\{L_n \geq 0\} d\mathbf{P}, \quad n \in \mathbb{N}$$

and

$$d\mathbf{P}_n^- = U(-S_n) I\{M_n < 0\} d\mathbf{P}, \quad n \in \mathbb{N}$$

Introduce on $\mathcal{F}_n$ two sequence of probability measures

$$d\mathbf{P}_n^+ = V(S_n) I\left\{L_n \geq 0\right\} d\mathbf{P}, \quad n \in \mathbb{N}$$

and

$$d\mathbf{P}_n^- = U(-S_n) I\left\{M_n < 0\right\} d\mathbf{P}, \quad n \in \mathbb{N}$$

or, what is the same, for any nonnegative random variable Y_n measurable with respect to $\mathcal{F}_n$

$$\mathbf{E}_n^+\left[Y_n\right] = \mathbf{E}\left[Y_n V(S_n) I\left\{L_n \geq 0\right\}\right]$$

and

$$\mathbf{E}_n^-\left[Y_n\right] = \mathbf{E}\left[Y_n U(-S_n) I\left\{M_n < 0\right\}\right].$$

They are consistent since, for instance, for any $Y_n \in \mathcal{F}_n$

$$\begin{aligned}\mathbf{E}_{n+1}^{+}\left[Y_n\right] &= \mathbf{E}\left[Y_n V(S_{n+1}) I\left\{L_{n+1} \geq 0\right\}\right] \\ &= \mathbf{E}\left[Y_n V(S_n) I\left\{L_n \geq 0\right\}\right] = \mathbf{E}_n^{+}\left[Y_n\right].\end{aligned}$$

Hence, there exists a probability measure $\mathbf{P}^{+}$ on $\mathcal{F}$ such that

$$\mathbf{P}^{+}|\mathcal{F}_n = \mathbf{P}_n^{+}, \quad n \geq 0.$$

or,

$$\mathbf{E}^{+}\left[Y_n\right] = \mathbf{E}\left[Y_n V(S_n) I\left\{L_n \geq 0\right\}\right].$$

Similarly, we have a measure $\mathbf{P}^{-}$ on $\mathcal{F}$ such that

$$\mathbf{P}^{-}|\mathcal{F}_n = \mathbf{P}_n^{-}, \quad n \geq 0.$$

Lemma

Let $\mathbf{E}X = 0$ *and* $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$. *Then for any* $\mathcal{F}_k$*-measurable bounded random variable* $\psi_\kappa, k \in \mathbb{N}$

$$\lim_{n\to\infty} \mathbf{E}\left[\psi_\kappa | L_n \geq 0\right] = \mathbf{E}^+\left[\psi_\kappa\right] = \mathbf{E}\left[\psi_\kappa V(S_k) I\left\{L_k \geq 0\right\}\right],$$

$$\lim_{n\to\infty} \mathbf{E}\left[\psi_\kappa | M_n < 0\right] = \mathbf{E}^-\left[\psi_\kappa\right] = \mathbf{E}\left[\psi_\kappa U(-S_k) I\left\{M_k < 0\right\}\right].$$

If the sequence $\psi_1, \psi_2, ..., \psi_n, ...$ *is uniformly bounded and is adopted to filtration* $\mathcal{F}$ *and*

$$\lim_{n\to\infty} \psi_n := \psi_\infty$$

$\mathbf{P}^+$ *a.s., (*$\mathbf{P}^-$ *a.s.) then*

$$\lim_{n\to\infty} \mathbf{E}\left[\psi_n | L_n \geq 0\right] = \mathbf{E}^+\left[\psi_\infty\right]$$

and

$$\lim_{n\to\infty} \mathbf{E}\left[\psi_n | M_n < 0\right] = \mathbf{E}^-\left[\psi_\infty\right]$$

Lemma

Let $\mathbf{E}X = 0$ *and* $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$. *Then*

$$\sum_{k=0}^{\infty} e^{-S_k} < \infty \quad \mathbf{P}^{+} \text{ - a.s.} \quad and \quad \sum_{k=0}^{\infty} e^{S_k} < \infty \quad \mathbf{P}^{-} \text{ - a.s.}$$

Only the first. We have

$$\begin{aligned}
\mathbf{E}^+\left[\sum_{k=0}^{\infty} e^{-S_k}\right] &= \sum_{k=0}^{\infty} \mathbf{E}^+\left[e^{-S_k}\right] = \sum_{k=0}^{\infty} \mathbf{E}\left[e^{-S_k} V(S_k) I\left\{L_k \geq 0\right\}\right] \\
&= \sum_{k=0}^{\infty} \int_0^{\infty} e^{-x} V(x) \mathbf{P}\left(S_k \in dx; L_k \geq 0\right) \\
&= \int_0^{\infty} e^{-x} V(x) \sum_{k=0}^{\infty} \mathbf{P}\left(S_k \in dx; L_k \geq 0\right) \\
&= \int_0^{\infty} e^{-x} V(x) U(dx) \\
&\leq \sum_{j=0}^{\infty} e^{-j} V(j+1) U(j+1) \leq C \sum_{j=0}^{\infty} e^{-j} (j+1)^2.
\end{aligned}$$

Let

$$f_{k,n}(s) := f_{k+1}(f_{k+2}(...f_n(s)...)), 0 \leq k < n.$$

Then

$$\mathbf{P}\left(Z(n) > 0 | Z(k) = 1, \Pi_k, ..., \Pi_n\right) = 1 - f_{k,n}(0).$$

Lemma

In the case when

$$f_n(s) = \frac{q_n}{1 - p_n s}$$

we have

$$1 - f_{k,n}(s) = \left(\frac{e^{-(S_n - S_k)}}{1 - s} + \sum_{j=k}^{n-1} e^{-(S_j - S_k)} \right)^{-1}$$

where

$$S_j = \log \frac{p_0}{q_0} + \ldots + \log \frac{p_{j-1}}{q_{j-1}} = \log f_0'(1) + \ldots + \log f_{j-1}'(1).$$

Proof. We have

$$\begin{aligned}
\frac{1}{1-f_{k,n}(s)} &= \frac{1}{1-f_{k+1}(f_{k+1,n}(s))} - \frac{1}{f'_{k+1}(1)\,(1-f_{k+1,n}(s))} \\
&\quad + \frac{1}{f'_{k+1}(1)\,(1-f_{k+1,n}(s))} \\
&= e^{-(S_k - S_k)} + \frac{e^{-(S_{k+1}-S_k)}}{1-f_{k+1,n}(s)} \cdots .
\end{aligned}$$

The critical case: Quenched approach, Probability of survival

Theorem

Let $\mathbf{E}X = 0$ and $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$ and $\tau(n)$ is the left-most point of minimum of the associated random walk on $[0, n]$. Then in the pure geometric case

$$\frac{\mathbf{P}_\pi (Z(n) > 0)}{e^{S_{\tau(n)}}} \stackrel{d}{\to} \zeta^{-1}$$

where $\zeta \in (1, \infty)$ with probability 1 and

$$\mathbf{E}\left[e^{-\lambda\zeta}\right] = \mathbf{E}^- \left[e^{-\lambda\zeta^-}\right] \mathbf{E}^+ \left[e^{-\lambda\zeta^+}\right]$$

where

$$\zeta^- = \sum_{i=1}^{\infty} e^{S_i}, \quad \zeta^+ = \sum_{i=0}^{\infty} e^{-S_i}.$$

Proof. We have

$$\frac{e^{S_{\tau(n)}}}{\mathbf{P}_\pi\left(Z(n)>0\right)}=\sum_{k=0}^{n}e^{S_{\tau(n)}-S_k}=\zeta^-(\tau(n))+\zeta^+(\tau(n))$$

where

$$\zeta^-(\tau(n)):=\sum_{i=0}^{\tau(n)-1}e^{S_{\tau(n)}-S_i},\quad \zeta^+(\tau(n)):=\sum_{i=\tau(n)}^{n}e^{S_{\tau(n)}-S_i}.$$

Proof. We have

$$\frac{e^{S_{\tau(n)}}}{\mathbf{P}_\pi\left(Z(n)>0\right)} = \sum_{k=0}^{n} e^{S_{\tau(n)}-S_k} = \zeta^-(\tau(n)) + \zeta^+(\tau(n))$$

where

$$\zeta^-(\tau(n)) := \sum_{i=0}^{\tau(n)-1} e^{S_{\tau(n)}-S_i}, \quad \zeta^+(\tau(n)) := \sum_{i=\tau(n)}^{n} e^{S_{\tau(n)}-S_i}.$$

Introduce two random walks

$$S'_l = S_k - S_{k-l}, l \geq 0, \quad and \quad S''_l = S_{k+l} - S_k, l \geq 0$$

Now

$$\begin{aligned}
&\mathbf{E}\left[e^{-\lambda\left(\zeta^-(n)+\zeta^+(n)\right)}\right] \\
= &\sum_{k=0}^{n}\mathbf{E}\left[e^{-\lambda\left(\zeta^-(n)+\zeta^+(n)\right)};\tau(n)=k\right] \\
= &\sum_{k=0}^{n}\mathbf{E}\left[e^{-\lambda\left(\zeta^-(k)+\zeta^+(n-k)\right)};M_k^{'}<0,L_{n-k}^{''}\geq 0\right] \\
= &\sum_{k=0}^{n}\mathbf{E}\left[e^{-\lambda\left(\zeta^-(k)+\zeta^+(n-k)\right)}|M_k^{'}<0,L_{n-k}^{''}\geq 0\right]\mathbf{P}\left(M_k^{'}<0,L_{n-k}^{''}\geq 0\right),
\end{aligned}$$

where

$$\zeta^-(k):=\sum_{i=1}^{k}e^{S_i^{'}},\quad \zeta^+(n-k):=\sum_{i=0}^{n-k}e^{-S_i^{''}}.$$

By independency,

$$\mathbf{E}\left[e^{-\lambda\left(\zeta^-(k)+\zeta^+(n-k)\right)}; M_k^{'} < 0, L_{n-k}^{''} \geq 0\right]$$
$$= \mathbf{E}\left[e^{-\lambda\zeta^-(k))}|M_k^{'} < 0\right]\mathbf{E}\left[e^{-\lambda\zeta^+(n-k)}|L_{n-k}^{''} \geq 0\right]$$

Hence, applying previous lemmas we get

$$\lim_{\min(k,n-k)\to\infty} \mathbf{E}\left[e^{-\lambda\left(\zeta^-(k)+\zeta^+(n-k)\right)}; M_k^{'} < 0, L_{n-k}^{''} \geq 0\right]$$
$$= \mathbf{E}^-\left[e^{-\lambda\zeta^-}\right]\mathbf{E}^+\left[e^{-\lambda\zeta^+}\right].$$

To complete the proof it remains to observe that by the arcsine law

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup \mathbf{E}\left[e^{-\lambda\left(\zeta^-(n)+\zeta^+(n)\right)}; \tau(n) \notin [n\varepsilon, n(1-\varepsilon)]\right]$$
$$\leq \lim_{\varepsilon\to 0}\lim_{n\to\infty} \mathbf{P}\left(\tau(n) \notin [n\varepsilon, n(1-\varepsilon)]\right) = 0.$$

Corollary

Let $\mathbf{E}X = 0$ and $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$ and $\tau(n)$ is the left-most point of minimum of the associated random walk on $[0, n]$. Then for the pure geometric case

$$\mathbf{P}_\pi \left(Z(n) > 0 | Z(\tau(n)) = 1 \right) \xrightarrow{d} \frac{1}{\zeta_+} > 0.$$

Proof. We have by known formulas that

$$\frac{1}{\mathbf{P}_\pi \left(Z(n) > 0 | Z(\tau(n)) = 1 \right)} = \sum_{k=\tau(n)}^{n} e^{-(S_k - S_{\tau(n)})} = \sum_{k=\tau(n)}^{n} e^{S_{\tau(n)} - S_k}$$

and the statement follows.

Yaglom limit theorems

Theorem

Let $\mathbf{E}X = 0$ *and* $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$, *and* $\tau(n)$ *be the left-most point of minimum of the associated random walk on* $[0, n]$. *Then in the pure geometric case for any* $\lambda > 0$

$$\mathbf{E}_\pi \left[\exp \left\{ -\lambda \frac{Z(n)}{\mathbf{E}_\pi \left[Z(n) | Z(n) > 0 \right]} \right\} | Z(n) > 0 \right] \xrightarrow{d} \frac{1}{1 + \lambda}.$$

Proof. Note that

$$\frac{\mathbf{E}_\pi\left[Z(n)|Z(n)>0\right]}{e^{S_n-S_{\tau(n)}}}=\frac{e^{S_{\tau(n)}}}{\mathbf{P}_\pi\left(Z(n)>0\right)}\xrightarrow{d}\zeta^-+\zeta^+.$$

We have

$$\begin{aligned}
&\mathbf{E}_\pi\left[\exp\left\{-\lambda\frac{Z(n)}{e^{S_n-S_{\tau(n)}}}\right\}|Z(n)>0\right] \\
&\quad=1-\frac{1-f_{0,n}\Big(\exp\left\{-\lambda e^{S_{\tau(n)}-S_n}\right\}\Big)}{1-f_{0,n}(0)} \\
&\quad=1-\frac{e^{S_{\tau(n)}}}{1-f_{0,n}(0)}\frac{1-f_{0,n}\Big(\exp\left\{-\lambda e^{S_{\tau(n)}-S_n}\right\}\Big)}{e^{S_{\tau(n)}}}.
\end{aligned}$$

By direct calculations one can show that

$$\frac{e^{S_{\tau(n)}}}{1 - f_{0,n}(\exp\{-\lambda e^{S_{\tau(n)} - S_n}\})} = \sum_{i=0}^{\tau(n)-1} e^{S_{\tau(n)} - S_i} + \sum_{i=\tau(n)}^{n-1} e^{S_{\tau(n)} - S_i} + \frac{e^{S_{\tau(n)} - S_n}}{1 - \exp\{-\lambda e^{S_{\tau(n)} - S_n}\}}.$$

Hence, by the arguments used earlier and the fact that, as $n \to \infty$

$$e^{S_{\tau(n)} - S_n} \xrightarrow{p} 0$$

with probability close to 1 we get

$$\frac{e^{S_{\tau(n)}}}{1 - f_{0,n}(\exp\{-\lambda e^{S_{\tau(n)} - S_n}\})} \xrightarrow{d} \zeta^- + \zeta^+ + \frac{1}{\lambda}.$$

It follows that

$$1-\frac{1-f_{0,n}(\exp\left\{-\lambda e^{S_{\tau(n)}-S_n}\right\})}{1-f_{0,n}(0)} \quad\to\quad 1-\frac{\zeta^-+\zeta^+}{\zeta^-+\zeta^++\lambda^{-1}}$$
$$= \frac{1}{1+\lambda\left(\zeta^-+\zeta^+\right)}.$$

From this on account of

$$e^{S_{\tau(n)}-S_n}\mathbf{E}_\pi\left[Z(n)|Z(n)>0\right]\xrightarrow{d}\zeta^-+\zeta^+$$

one can deduce that

$$1-\frac{1-f_{0,n}(\exp\left\{-\lambda/\mathbf{E}_\pi\left[Z(n)|Z(n)>0\right]\right\})}{1-f_{0,n}(0)}\xrightarrow{d}\frac{1}{1+\lambda}$$

Now we consider the number of particles in the process at a random moment

$$\tau(n) = \min\{i \leq n : S_i = L_n\}.$$

Theorem

Let $\mathbf{E}X = 0$ *and* $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$ *Then in the pure geometric case for any* $s \in [0,1]$

$$\mathbf{E}_\pi\left[s^{Z(\tau(n))}|Z(n) > 0\right] \xrightarrow{d} s\phi(s),$$

where

$$\phi(s) := \frac{\zeta^- + \zeta^+}{(1 + \zeta^- - s\zeta^-)\left(\zeta^+(1 + \zeta^-) - s\zeta^-(\zeta^+ - 1)\right)}.$$

Proof. Direct calculation shows that for $m < n$

$$\mathbf{E}_\pi \left[s^{Z(m)} | Z(n) > 0 \right] = \frac{f_{0,m}(s) - f_{0,m}(s f_{m.n}(0))}{1 - f_{0,n}(0)}$$
$$= \frac{f_{0,m}(s) - f_{0,m}(s f_{m.n}(0))}{e^{S_m}} \frac{e^{S_m}}{1 - f_{0,n}(0)}.$$

Now in the geometric case

$$\frac{e^{S_{\tau(n)}}}{1 - f_{0,n}(0)} \xrightarrow{d} \zeta^- + \zeta^+, \quad f_{\tau(n),n}(0) \xrightarrow{d} 1 - \frac{1}{\zeta^+}$$

while

$$\frac{1 - f_{0,\tau(n)}(s)}{e^{S_{\tau(n)}}} = \left(\sum_{k=0}^{\tau(n)-1} e^{S_{\tau(n)} - S_k} + \frac{1}{1-s} \right)^{-1}$$

$$\xrightarrow{d} \left(\zeta^{-} + \frac{1}{1-s} \right)^{-1}$$

while

$$\frac{1-f_{0,\tau(n)}(s)}{e^{S_{\tau(n)}}} = \left(\sum_{k=0}^{\tau(n)-1} e^{S_{\tau(n)}-S_k} + \frac{1}{1-s}\right)^{-1}$$

$$\xrightarrow{d} \left(\zeta^- + \frac{1}{1-s}\right)^{-1}$$

and, therefore,

$$\frac{f_{0,\tau(n)}(s) - f_{0,\tau(n)}(s f_{\tau(n),n}(0))}{e^{S_{\tau(n)}}}$$

$$\xrightarrow{d} \left(\zeta^- + \frac{1}{1-s\left(1-\left(\zeta^+\right)^{-1}\right)}\right)^{-1} - \left(\zeta^- + \frac{1}{1-s}\right)^{-1}$$

$$= \frac{1}{\left(1+\zeta^- - s\zeta^-\right)\left(\zeta^+\left(1+\zeta^-\right) - s\zeta^-\left(\zeta^+ - 1\right)\right)}.$$

while

$$\frac{1-f_{0,\tau(n)}(s)}{e^{S_{\tau(n)}}} = \left(\sum_{k=0}^{\tau(n)-1} e^{S_{\tau(n)}-S_k} + \frac{1}{1-s}\right)^{-1}$$

$$\stackrel{d}{\to} \left(\zeta^- + \frac{1}{1-s}\right)^{-1}$$

and, therefore,

$$\frac{f_{0,\tau(n)}(s) - f_{0,\tau(n)}(sf_{\tau(n),n}(0))}{e^{S_{\tau(n)}}}$$

$$\stackrel{d}{\to} \left(\zeta^- + \frac{1}{1-s\left(1-\left(\zeta^+\right)^{-1}\right)}\right)^{-1} - \left(\zeta^- + \frac{1}{1-s}\right)^{-1}$$

$$= \frac{1}{\left(1+\zeta^- - s\zeta^-\right)\left(\zeta^+\left(1+\zeta^-\right) - s\zeta^-\left(\zeta^+ - 1\right)\right)}.$$

Bottleneck!

Now we consider the point $\tau(nt)$ and assume that $\tau(nt) < \tau(n)$.

Theorem

Let $\mathbf{E}X = 0$ and $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$ and $\tau(nt)$ be the left-most point of minimum of the associated random walk on $[0, nt]$. Then in the pure geometric case for any $s \in [0, 1]$

$$\mathbf{E}_\pi \left[s^{Z(\tau(nt))} | Z(n) > 0 \right] I\left\{\tau(n) > nt\right\} \xrightarrow{d} s\psi(s) I\left\{\tau_{arc} > t\right\},$$

where τ_{arc} is a random variable subject to the arcsine law and

$$\psi(s) := \frac{1}{(1 + \zeta^- - s\zeta^-)^2}.$$

Proof. On the event $\{\tau(nt) < \tau(n)\}$

$$f_{\tau(nt),n}(0) \xrightarrow{d} 1.$$

Thus

$$\begin{aligned}
\mathbf{E}_\pi\left[s^{Z(\tau(nt))}|Z(n) > 0\right] &= \frac{f_{0,\tau(nt)}(s) - f_{0,\tau(nt)}(s f_{\tau(nt),n}(0))}{1 - f_{0,n}(0)} \\
&\approx \frac{s f'_{0,\tau(nt)}(s)\left(1 - f_{\tau(nt),n}(0)\right)}{1 - f_{0,n}(0)} \\
&= s\frac{f'_{0,\tau(nt)}(s)}{e^{S_{\tau(nt)}}}\frac{1 - f_{\tau(nt),n}(0)}{e^{S_{\tau(n)} - S_{\tau(nt)}}}\frac{e^{S_{\tau(n)}}}{1 - f_{0,n}(0)}.
\end{aligned}$$

In the pure geometric case

$$\frac{f'_{0,\tau(nt)}(s)}{e^{S_{\tau(nt)}}} = \left(1 + (1-s)\sum_{k=0}^{\tau(nt)-1} e^{S_{\tau(nt)}-S_k}\right)^{-2} \xrightarrow{d} \left(1 + (1-s)\zeta^{-}\right)^{2}$$

while

$$\frac{1 - f_{\tau(nt),n}(0)}{e^{S_{\tau(n)}-S_{\tau(nt)}}} \frac{e^{S_{\tau(n)}}}{1 - f_{0,n}(0)} \xrightarrow{d} 1.$$

Theorem

Let $\mathbf{E}X = 0$ *and* $\sigma^2 := \mathbf{E}X^2 \in (0,\infty)$*. Then, in the pure geometric case, as* $n \to \infty$*, for any* $\lambda > 0$

$$\mathbf{E}_\pi \left[\exp\left\{ -\lambda \frac{Z(nt)}{\mathbf{E}_\pi\left[Z(nt)|Z(nt)>0\right]} \right\} |Z(n)>0 \right] I\left\{\tau(n) < nt\right\} \xrightarrow{d} \frac{1}{1+\lambda} I\left\{\tau_{arc} < t\right\}$$

and

$$\mathbf{E}_\pi \left[\exp\left\{ -\lambda \frac{Z(nt)}{\mathbf{E}_\pi\left[Z(nt)|Z(nt)>0\right]} \right\} |Z(n)>0 \right] I\left\{\tau(n) > nt\right\} \xrightarrow{d} \frac{1}{(1+\lambda)^2} I\left\{\tau_{arc} > t\right\}.$$

Note that

$$\mathbf{E}_\pi\left[Z(nt)|Z(nt)>0\right] = \frac{e^{S_{nt}}}{\mathbf{P}_\pi\left(Z(n)>0\right)} \asymp e^{S_{nt}-S_{\tau(nt)}}.$$

Critical processes dying at a fixed moment

The next theorem deals with the distribution of the number of particles at moments $nt,\ 0 < t < 1$ in the case when the event

$$\mathcal{A}_n := \{Z(n-1) > 0, Z(n) = 0\}$$

occurs.

$$O_{m,n} := \frac{1 - f_{0,n}(0)}{1 - f_{m,n}(0)} \frac{f_{m,n}(0)}{1 - f_{0,m}(0)}. \tag{1}$$

Theorem

Let $\mathbf{E}X = 0$ and $\sigma^2 :\in (0, \infty)$. Then, in the pure geometric case for any $t \in (0,1)$ and $\lambda \in (0, \infty)$

$$\mathbf{E}\pi \left[\exp\left\{ -\lambda \frac{Z_{nt}}{O_{nt,n}} \right\} | \mathcal{A}_n \right] \xrightarrow{d} \frac{1}{(1+\lambda)^2} \quad \textit{as} \;\; n \to \infty.$$

It is necessary to note that despite of the unique form of the limit in the cases $\{\tau(n) \geq nt\}$ and $\{\tau(n) < nt\}$, the behavior of the scaling function $O_{nt,n}$ as $n \to \infty$ is different for the mentioned situations:

It is necessary to note that despite of the unique form of the limit in the cases $\{\tau(n) \geq nt\}$ and $\{\tau(n) < nt\}$, the behavior of the scaling function $O_{nt,n}$ as $n \to \infty$ is different for the mentioned situations: in the first case

$$O_{nt,n} I\left\{\tau(n) \geq nt\right\} \asymp e^{S_{nt} - S_{\tau(nt)}} I\left\{\tau(n) \geq nt\right\},$$

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$$O_{nt,n} I\left\{\tau(n) \geq nt\right\} \asymp e^{S_{nt}-S_{\tau(nt)}} I\left\{\tau(n) \geq nt\right\},$$

i.e., is, essentially, specified by the **past** behavior of the associated random walk while in the second case

$$O_{nt,n} I\left\{\tau(n) < nt\right\} \asymp e^{S_{nt}-S_{\tau(nt,n)}} I\left\{\tau(n) < nt\right\}$$

i.e., is, essentially, specified by the **future** behavior of the associated random walk.

Annealed approach

Theorem

Let $\mathbf{E}X = 0$ *and* $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$ *Then, in the pure geometric case, as* $n \to \infty$

$$\mathbf{P}\left(Z(n) > 0\right) \sim \theta \mathbf{P}\left(\min\left(S_0, S_1, ..., S_n\right) \geq 0\right) \sim \frac{C}{\sqrt{n}}$$

and for any $t \in (0, 1)$

$$\lim_{n \to \infty} \mathbf{E}\left[\exp\left\{-\lambda \frac{Z(nt)}{\mathbf{E}_\pi\left[Z(nt)|Z(nt) > 0\right]}\right\}|Z(n) > 0\right] = \frac{1}{1+\lambda}.$$

Properties of the prospective minima

Introduce the random variable

$$\nu := \min\{m \geq 1 : S_{m+n} \geq S_m \text{ for any } n \in \mathbb{N}_0\}$$

Lemma

Let $\mathbf{E}X = 0$ and $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$ and the measure $\mathbf{P}$ is replaced by $\mathbf{P}^+$. Then $\nu < \infty$ $\mathbf{P}^+$-a.s. Moreover,
1) the sequences $\{S_n\}$ and $\{S_n^* = S_{n+\nu} - S_n\}$ have the same distribution;
2) the random sequences $\{\nu, S_1, ..., S_\nu\}$ and $\{S_n^*\}$ are independent.
3) for all $k \in \mathbb{N}$ and $x \geq 0$

$$\mathbf{P}^+(\nu = k, S_\nu \in dx) = \mathbf{P}(T' = k, S_{T'} \in dx)$$

where

$$T' = \min\{j \geq 1 : S_j \geq 0\}.$$

Proof omitted.

Recall that

$$\hat{\mathbf{P}}^{+}(...) = \mathbf{P}^{+}(...|\Pi_0, \Pi_1, ...)$$

Lemma

Let $\mathbf{E}X = 0$ and $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$. Then in the pure geometric case $\mathbf{P}^{+}$ a.s.

$$\hat{\mathbf{P}}^{+}(Z(n) > 0 \text{ for all } n > 0) > 0.$$

In particular,

$$\mathbf{P}^{+}(Z(n) > 0 \text{ for all } n > 0) > 0.$$

Moreover, as $n \to \infty$ we have $\mathbf{P}^{+}$ a.s.

$$\frac{Z(n)}{e^{S_n}} \xrightarrow{d} Z^{+}$$

where the random variable Z^{+} has the following property $\mathbf{P}^{+}$ a.s.:

$$\left\{Z^{+} > 0\right\} = \left\{Z(n) > 0 \text{ for all } n > 0\right\}.$$

Proof. We know that

$$\hat{\mathbf{P}}^{+}(Z(n) > 0) = 1 - f_{0,n}(0).$$

Hence,

$$\hat{\mathbf{P}}^{+}(Z(n) > 0) = \left(\sum_{j=0}^{n} e^{-S_j} \right)^{-1}$$

Proof. We know that

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Hence,

$$\hat{\mathbf{P}}^{+}(Z(n) > 0) = \left(\sum_{j=0}^{n} e^{-S_j} \right)^{-1}$$

and in view of the previous results $\mathbf{P}^{+}$ a.s.

$$\lim_{n\to\infty} \hat{\mathbf{P}}^{+}(Z(n) > 0) = \left(\sum_{j=0}^{\infty} e^{-S_j} \right)^{-1} > 0$$

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In view of

$$\mathbf{P}^{+}(Z(n) > 0) = \mathbf{E}\hat{\mathbf{P}}^{+}(Z(n) > 0)$$

the second statement follows.

Since $Z(n)e^{-S_n}$ is a nonnegative martingale, we conclude

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Let us show that

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Since

$$\{Z(n) = 0 \text{ for some } n > 0\} \subset \left\{Z^+ = 0\right\}$$

it follows that

$$\mathbf{P}^+ \left(Z^+ = 0\right) \geq \mathbf{P}^+ \left(Z(n) = 0 \text{ for some } n > 0\right).$$

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First we demonstrate that

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To this aim it suffices to show that $\mathbf{P}^{+}$ a.s.

$$\hat{\mathbf{P}}^{+}\left(Z(n)=0 \text{ for some } n>0\right)+\hat{\mathbf{P}}^{+}\left(Z(n) \rightarrow \infty\right)=1.$$

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We check that $\mathbf{P}^{+}$ a.s.

$$\sum_{k=0}^{\infty}\left(1-p_1^{(k)}\right)=\infty$$

(it is sufficient according to Lindvall).

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The random variables

$$p_1^{(v(k)+1)}, k \in \mathbb{N}_0$$

are iid according to the previous results.

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$$\sum_{k=0}^{\infty}\left(1 - p_1^{(k)}\right) \geq \sum_{k=0}^{\infty}\left(1 - p_1^{(v(k)+1)}\right) = \infty.$$

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from this it follows that $\mathbf{P}^+$ a.s.

$$\hat{\mathbf{P}}^+\left(Z(n) = 0 \text{ for some } n > 0\right) + \hat{\mathbf{P}}^+\left(Z(n) \to \infty\right) = 1$$

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from this it follows that $\mathbf{P}^+$ a.s.

$$\hat{\mathbf{P}}^+\left(Z(n) = 0 \text{ for some } n > 0\right) + \hat{\mathbf{P}}^+\left(Z(n) \to \infty\right) = 1$$

implying

$$\mathbf{P}^+\left(Z(n) = 0 \text{ for some } n > 0\right) + \mathbf{P}^+\left(Z(n) \to \infty\right) = 1.$$

Further we have $\mathbf{P}^+$ a.s.

$$\hat{\mathbf{E}}^+ \left[\exp\left\{ -\frac{\lambda Z(n)}{e^{S_n}} \right\} | Z(k) = 1 \right] = f_{k,n}\left(\exp\left\{ -\lambda e^{-S_n} \right\} \right)$$

$$= 1 - \left(\frac{e^{-(S_n - S_k)}}{1 - \exp\left\{ -\lambda e^{-S_n} \right\}} + \sum_{j=k}^{n-1} e^{-(S_j - S_k)} \right)^{-1}$$

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Now we let $n \to \infty$. Then $S_n \to +\infty$ $\mathbf{P}^+$ a.s. and

$$\frac{Z(n)}{e^{S_n}} \xrightarrow{d} Z^+$$

$\mathbf{P}^+$ a.s. Therefore, letting $n \to \infty$ we see that

$$\hat{\mathbf{E}}^+ \left[\exp\left\{ -\lambda Z^+ \right\} |Z(k) = 1 \right] = 1 - \left(\frac{e^{S_k}}{\lambda} + \sum_{j=k}^{\infty} e^{-(S_j - S_k)} \right)^{-1}$$

As $\lambda \to \infty$ we get $\mathbf{P}^+$ a.s.

$$\hat{\mathbf{P}}^+\left(Z^+ = 0 | Z(k) = 1\right) = 1 - \left(\sum_{j=k}^{\infty} e^{-(S_j - S_k)}\right)^{-1}.$$

As $\lambda \to \infty$ we get $\mathbf{P}^+$ a.s.

$$\hat{\mathbf{P}}^+ \left(Z^+ = 0 | Z(k) = 1\right) = 1 - \left(\sum_{j=k}^{\infty} e^{-(S_j - S_k)}\right)^{-1}.$$

Further, we get $\mathbf{P}^+$ a.s.

$$\hat{\mathbf{P}}^+ \left(Z^+ = 0 | Z(\nu(k)) = 1\right) = 1 - \left(\sum_{j=\nu(k)}^{\infty} e^{-\left(S_j - S_{\nu(k)}\right)}\right)^{-1}.$$

As $\lambda \to \infty$ we get $\mathbf{P}^+$ a.s.

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Besides, for any $l \in \mathbb{N}$

$$\hat{\mathbf{P}}^+\left(Z^+ = 0 | Z(\nu(k)) = l\right) = \left(\hat{\mathbf{P}}^+\left(Z^+ = 0 | Z(\nu(k)) = 1\right)\right)^l.$$

As a result we get $\mathbf{P}^+$ a.s

$$\begin{aligned}
&\hat{\mathbf{P}}^+\left(Z^+ = 0 | Z(0) = 1\right) \\
&\quad = \hat{\mathbf{E}}^+\left[\hat{\mathbf{P}}^+\left(Z^+ = 0 | Z(\nu(k))\right)\right] \\
&\quad = \hat{\mathbf{E}}^+\left[1 - \left(\sum_{j=\nu(k)}^{\infty} e^{-\left(S_j - S_{\nu(k)}\right)}\right)^{-1}\right]^{Z(\nu(k))} \\
&\quad \leq \hat{\mathbf{P}}^+\left(Z(\nu(k)) \leq z\right) + \hat{\mathbf{E}}^+\left[1 - \left(\sum_{j=\nu(k)}^{\infty} e^{-\left(S_j - S_{\nu(k)}\right)}\right)^{-1}\right]^{z}.
\end{aligned}$$

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&\quad = \hat{\mathbf{E}}^+\left[1 - \left(\sum_{j=\nu(k)}^{\infty} e^{-\left(S_j - S_{\nu(k)}\right)}\right)^{-1}\right]^{Z(\nu(k))} \\
&\quad \le \hat{\mathbf{P}}^+\left(Z(\nu(k)) \le z\right) + \hat{\mathbf{E}}^+\left[1 - \left(\sum_{j=\nu(k)}^{\infty} e^{-\left(S_j - S_{\nu(k)}\right)}\right)^{-1}\right]^{z}.
\end{aligned}
$$

By the properties of prospective minima

$$
\hat{\mathbf{E}}^+\left[1 - \left(\sum_{j=\nu(k)}^{\infty} e^{-\left(S_j - S_{\nu(k)}\right)}\right)^{-1}\right]^{z} = \hat{\mathbf{E}}^+\left[1 - \left(\sum_{j=0}^{\infty} e^{-S_j}\right)^{-1}\right]^{z}
$$

Thus,

$$\hat{\mathbf{P}}^{+}\left(Z^{+}=0|Z(0)=1\right) \leq \hat{\mathbf{P}}^{+}\left(Z(\nu(k)) \leq z\right) + \hat{\mathbf{E}}^{+}\left[1-\left(\sum_{j=0}^{\infty} e^{-S_j}\right)^{-1}\right]^{z}$$

Thus,

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Now we let $k \to \infty$ to get

$$\hat{\mathbf{P}}^{+}\left(Z^{+}=0|Z(0)=1\right) \leq \hat{\mathbf{P}}^{+}\left(Z(n)=0 \text{ for some } n>0\right) + \hat{\mathbf{E}}^{+}\left[1-\left(\sum_{j=0}^{\infty} e^{-S_j}\right)^{-1}\right]^{z}$$

Thus,

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Now letting $z \to \infty$ we see that

$$\hat{\mathbf{P}}^{+}\left(Z^{+}=0|Z(0)=1\right) \leq \hat{\mathbf{P}}^{+}\left(Z(n)=0 \text{ for some } n>0\right).$$

Since the inequality

$$\hat{\mathbf{P}}^{+}\left(Z^{+}=0|Z(0)=1\right) \geq \hat{\mathbf{P}}^{+}\left(Z(n)=0 \text{ for some } n>0\right)$$

is evident, the statement of the lemma follows.

Lemma

If $\mathbf{E}X = 0$ *and* $\sigma^2 := \mathbf{E}X^2 \in (0, \infty)$ *then for any* $\varepsilon > 0$ *there exists* $l \in \mathbb{N}$ *such that*

$$\begin{aligned}\mathbf{E}\left[e^{S_{\tau(n)}}; \tau(n) > l\right] &= \sum_{k=l+1}^{n} \mathbf{E}\left[e^{S_k}; \tau(k) = k\right] \mathbf{P}\left(L_{n-k} \geq 0\right) \\ &\leq \varepsilon \mathbf{P}\left(L_n \geq 0\right).\end{aligned}$$

Proof. Let

$$T' := \min\{i > 0 : S_i \geq 0\}.$$

We know that

$$\mathbf{E}\left[e^{S_k}; \tau(k) = k\right] = \mathbf{E}\left[e^{S_k}; T' > k\right].$$

By Sparre-Anderson identity we have

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}\left[e^{S_n}; T' > n\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{S_n}; S_n < 0\right]\right\}.$$

We know by the local limit theorem for sums of iid random variables with zero mean and finite variance $\sigma^2 > 0$ that (for nonlattice case)

$$\mathbf{P}\left(S_n \in [kh, k(h+1))\right) = \frac{1}{\sigma\sqrt{2\pi}n^{3/2}} e^{-(kh)^2/2n\sigma^2} + o\left(\frac{1}{n^{3/2}}\right).$$

Therefore,

$$\begin{aligned} \mathbf{E}\left[e^{S_n}; S_n < 0\right] &= \int_{-\infty}^{0} e^x d\mathbf{P}\left(S_n \le x\right) \\ &\le h\sum_{k=-\infty}^{0} e^{kh} \frac{1}{\sigma\sqrt{2\pi}n^{3/2}} e^{-(kh)^2/2n\sigma^2} + o\left(\frac{1}{n^{3/2}}\right). \end{aligned}$$

An similar estimate is valid from below. Hence

$$\mathbf{E}\left[e^{S_n}; S_n < 0\right] \sim \frac{1}{\sigma\sqrt{2\pi}n^{3/2}}$$

and, therefore, as $n \to \infty$

$$\mathbf{E}\left[e^{S_n}; T' > n\right] \sim \frac{C}{n^{3/2}}$$

Hence we get for sufficiently large l and $\delta \in (0,1)$:

$$\begin{aligned}
&\sum_{k=l+1}^{n} \mathbf{E}\left[e^{S_k}; \tau(k) = k\right] \mathbf{P}\left(L_{n-k} \geq 0\right) \\
= \quad & \left(\sum_{k=l+1}^{n(1-\delta)} + \sum_{k=n(1-\delta)+1}^{n}\right) \mathbf{E}\left[e^{-S_k}; \tau(k) = k\right] \mathbf{P}\left(L_{n-k} \geq 0\right) \\
\leq \quad & C \sum_{k=l+1}^{n(1-\delta)} \frac{1}{k^{3/2}} \mathbf{P}\left(L_{n\delta} \geq 0\right) + \frac{1}{n^{3/2}} \sum_{k=n(1-\delta)+1}^{n} \mathbf{P}\left(L_{n-k} \geq 0\right) \\
\leq \quad & \varepsilon_1 \mathbf{P}\left(L_n \geq 0\right) + \frac{1}{n^{1/2}} \mathbf{P}\left(L_{n\delta} \geq 0\right) \leq \varepsilon \mathbf{P}\left(L_n \geq 0\right).
\end{aligned}$$

We show that

$$\mathbf{P}\left(Z(n) > 0\right) \sim \frac{C}{\sqrt{n}}.$$

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We show that

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We have

$$\mathbf{P}\left(Z(n)>0\right)=\mathbf{P}\left(Z(n)>0;\tau(n)\leq l\right)+\mathbf{P}\left(Z(n)>0;\tau(n)>l\right).$$

It is easy to see that in view of the inequality

$$\begin{aligned}\mathbf{P}_{\pi}\left(Z(n)>0\right) &= 1-f_{0,n}(0)=\min_{0\leq k\leq n}\mathbf{P}_{\pi}\left(Z(n)>0\right)\\ &\leq \min_{0\leq k\leq n}e^{S_k}=e^{S_{\tau(n)}}\end{aligned}$$

that

$$\begin{aligned}\mathbf{P}\left(Z(n)>0;\tau(n)>l\right) &= \mathbf{E}\left[1-f_{0,n}(0);\tau(n)>l\right]\\ &\leq \mathbf{E}\left[e^{S_{\tau(n)}};\tau(n)>l\right]\leq\varepsilon\mathbf{P}\left(L_n\geq 0\right).\end{aligned}$$

On the other hand, for each fixed k

$$\begin{aligned}
&\mathbf{P}\left(Z(n)>0;\tau(n)=k\right)\\
=\;&\sum_{j=1}^{\infty}\mathbf{P}\left(Z(n)>0;Z(k)=j;\tau(n)=k\right)\\
=\;&\sum_{j=1}^{\infty}\mathbf{P}\left(Z(k)=j;\tau(k)=k\right)\mathbf{P}\left(Z(n-k)>0;\tau>n-k|Z(k)=j\right)
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\mathbf{P}\left(Z(n-k)>0;\tau>n-k|Z(k)=j\right)\\
=\;&\mathbf{E}\left[1-f_{0,n-k}^{j}(0);\tau>n-k\right]\\
=\;&\mathbf{E}\left[1-f_{0,n-k}^{j}(0)|\tau>n-k\right]\mathbf{P}\left(\tau>n-k\right).
\end{aligned}$$

We know that, as $n \to \infty$

$$\mathbf{P}\left(\tau > n-k\right) \sim \frac{C_1}{\sqrt{n}}$$

while

$$\mathbf{E}\left[1 - f_{0,n-k}^{j}(0) | \tau > n-k\right] \to \mathbf{E}^{+}\left[1 - f_{0,\infty}^{j}(0)\right]$$

where, in the pure geometric case

$$\mathbf{E}^{+}\left[1 - f_{0,\infty}^{j}(0)\right] = \mathbf{E}^{+}\left[1 - \left(1 - \frac{1}{\zeta^{+}}\right)^{j}\right].$$

from this it follows that

$$\mathbf{P}\left(Z(n) > 0\right) \sim \frac{C}{\sqrt{n}}.$$

Proof of the Yaglom limit theorem, the annealed case

We need to show that

$$\lim_{n\to\infty} \mathbf{E}\left[\exp\left\{-\lambda\frac{Z(nt)}{\mathbf{E}_{\pi}\left[Z(nt)|Z(nt)>0\right]}\right\}|Z(n)>0\right] = \Psi(\lambda).$$

We consider $t=1$ only and prove that

$$\lim_{n\to\infty} \mathbf{E}\left[\exp\left\{-\lambda\frac{Z(n)}{e^{S_n - S_{\tau(n)}}}\right\}|Z(n)>0\right] = \Psi_1(\lambda).$$

It suffices to consider for a fixed k

$$\mathbf{E}\left[\left(1-\exp\left\{-\lambda\frac{Z(n)}{e^{S_n-S_{\tau(nt)}}}\right\}\right);\tau(n)=k\right]$$

$$=\sum_{j=1}^{\infty}\mathbf{E}\left[\left(1-\exp\left\{-\lambda\frac{Z(n)}{e^{S_n-S_{\tau(nt)}}}\right\}\right);\tau(n)=k;Z(k)=j\right]$$

$$=\sum_{j=1}^{\infty}\mathbf{E}\left[\left(1-\exp\left\{-\lambda\frac{Z(n-k)}{e^{S_{n-k}}}\right\}\right);\tau>n-k|Z(0)=j\right]\times$$

$$\times\mathbf{P}\left(\tau(k)=k;Z(k)=j\right).$$

Now, as $n\to\infty$

$$\mathbf{E}\left[\left(1-\exp\left\{-\lambda\frac{Z(n-k)}{e^{S_{n-k}}}\right\}\right);\tau>n-k|Z(0)=j\right]$$

$$\sim\mathbf{E}^{+}\left[\left(1-\exp\left\{-\lambda(Z_1^{+}+...+Z_j^{+}\}\right)\right]\mathbf{P}\left(\tau>n\right)$$

and the desired statement follows.