Moments, moderate and large deviations for a branching process in a random environment

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Ecole Polytechnique

ANR workshop, February 8-9, 2011, Paris

Description of BPRE

- Random environment

  Let $\xi = (\xi_n)_{n \geq 0}$ be a sequence of i.i.d. random variables (in space Θ) indexed by time $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$.

  Each ξ_n corresponds to a probability on $\mathbb{N}$, denoted by $p(\xi_n) = \{p_k(\xi_n) : k \geq 0\}$, where

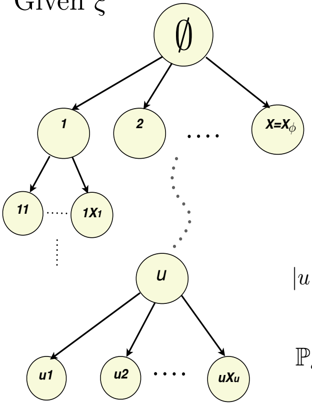
  $$0 \leq p_k(\xi_n) \leq 1 \quad \text{and} \quad \sum_k p_k(\xi_n) = 1.$$

  We call ξ a random environment.

Description of BPRE

- Branching Process in a Random Environment (BPRE)

Given ξ



$$X \sim p(\xi_0)$$

$$|u| = n, \quad X_u \sim p(\xi_n)$$

$$\mathbb{P}_\xi(X_u = k) = p_k(\xi_n)$$

Description of BPRE

Denote

Z_n – the population size of the n-th generation,

X_u– the number of offspring of u.

By definition,

$$Z_0 = 1, \quad Z_{n+1} = \sum_{|u|=n} X_u, \quad (n \geq 0).$$

where given ξ, $\{X_u : |u| = n\}$ are conditionally independent of each other and have a common distribution $p(\xi_n) = \{p_k(\xi_n) : k \geq 0\}$.

Description of BPRE

- Quenched and annealed laws

  Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment ξ is fixed. As usual, $\mathbb{P}_\xi$ is called quenched law.

  The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}_\xi \otimes \tau$, and τ is the law of the environment ξ. The total probability $\mathbb{P}$ is called annealed law.

  $\mathbb{P}_\xi$ can be considered to be the conditional probability of $\mathbb{P}$ given ξ.

Description of BPRE

- The martingale in BPRE
  Denote
  $$m_n = \sum_k k p_k(\xi_n)$$
  $$P_0 = 1, \qquad P_n = m_0 \cdots m_{n-1} \text{ for } n \geq 1.$$
  Then the normalized population size
  $$W_n = \frac{Z_n}{P_n}$$
  is a nonnegative martingale and converges a.s. to a nonnegative random variable:
  $$W = \lim_{n\to\infty} W_n \quad a.s.$$
  with $\mathbb{E}W \leq 1$.

Description of BPRE

- Supercritical BRPE
  We consider the supercritical case where

$$\mathbb{E} \log m_0 \in (0, \infty) \quad \text{and} \quad \mathbb{E} \frac{Z_1}{m_0} \log^+ Z_1 < \infty.$$

For simplicity, let $p_k = p_k(\xi_0)$ and assume that

$$p_0 = 0 \quad a.s.,$$

Therefore

$$W > 0 \quad \text{and} \quad Z_n \to \infty \; a.s..$$

Objective

- Law of large numbers
  It is well known [see e.g. Tanny(1977)] that

$$\lim_{n\to\infty} \frac{\log Z_n}{n} = \mathbb{E} \log m_0 \quad a.s. \text{ (on } \{Z_n \to \infty\}).$$

We are interested in the asymptotic properties of the corresponding deviation probabilities.

Objective

Notice that

$$\log Z_n = \log P_n + \log W_n$$

and

$$W_n \to W > 0 \qquad a.s..$$

Certain asymptotic properties of $\log Z_n$ can be determined by those of $\log P_n$.

We show that $\log Z_n$ and $\log P_n$ satisfy the same limit theorems under suitable conditions.

Central Limit Theorem

- Central Limit Theorem
  At first, $\log Z_n$ and $\log P_n = \log m_0 + ... + \log m_{n-1}$ satisfy the same central limit theorem.

Theorem 1 (Central Limit Theorem)

If $\sigma^2 = var(\log m_0) \in (0, \infty)$, then

$$\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \to \mathcal{N}(0,1) \quad \text{in law.}$$

Large Deviations

The central limit theorem suggests that $\log Z_n$ and $\log P_n$ would satisfy the same large deviation principle.

- Rate function
  Let
  $$\Lambda(t) = \log \mathbb{E} m_0^t,$$
  and
  $$\Lambda^*(x) = \sup_{t \in \mathbb{R}} \{xt - \Lambda(t)\}$$
  be the Fenchel-Legendre transform of Λ.

Large Deviations

We need the following assumption.

Assumption (H)

There exist constants $\delta > 0$ and $A > A_1 > 1$ such that a.s.

$$A_1 \le \mathbb{E}_\xi Z_1, \qquad \mathbb{E}_\xi Z_1^{1+\delta} \le A^{1+\delta}.$$

So $A_1 \le m_0 \le A$ a.s..

Large Deviations

- Large Deviation Principle

Theorem 2 (Large Deviation Principle)

Assume (H). If $\mathbb{E}Z_1^s < \infty$ for all $s > 1$ and $p_1 = 0$ *a.s.*, then for any measurable subset B of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x\in B^o} \Lambda^*(x) &\leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \\ &\leq -\inf_{x\in \bar{B}} \Lambda^*(x). \end{aligned}$$

Large Deviations

- Probabilities of tail events
  From LDP, we obtain the following corollary.

Corollary (Bansaye and Berestycki (2009))

Assume (H). If $\mathbb{E}Z_1^s < \infty$ for all $s > 1$ and $p_1 = 0$ *a.s.*, then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \le x\right) = -\Lambda^*(x) \quad \text{for } x < \mathbb{E}\log m_0,$$
$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \ge x\right) = -\Lambda^*(x) \quad \text{for } x > \mathbb{E}\log m_0.$$

This result has been obtained by Bansaye and Berestycki in 2009. Our approach is different, and it is also available to get the following result.

Large Deviations

Theorem 3

(i) Let $a > 0$. If $a \in (0, 1]$ and $\mathbb{E} m_0^{a-1} Z_1 \log^+ Z_1 < \infty$, or $a > 1$ and $\mathbb{E} Z_1^a < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \geq x\right) = -\Lambda^*(x), \quad \forall x \in (\mathbb{E} \log m_0, \Lambda'(a)).$$

(ii) Let $a < 0$. Assume (H) and $\|p_1\|_\infty := \mathit{esssup}\ p_1 < 1$. If $\mathbb{E} p_1 < \mathbb{E} m_0^a$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \leq x\right) = -\Lambda^*(x), \quad \forall x \in (\Lambda'(a), \mathbb{E} \log m_0).$$

Moderate Deviations

- Moderate Deviation Principle
  - Large deviation: $\frac{\log Z_n - n\mathbb{E}\log m_0}{n}$
  - Central limit theorem: $\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}}$
  - Moderate deviation: $\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n}$

Let $\{a_n\}$ be a sequence of positive numbers satisfying

$$\frac{a_n}{n} \to 0 \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \to \infty, \text{ as } n \to \infty.$$

Moderate Deviations

Theorem 4 (Moderate Deviation Principle)

Assume (H) and write $\sigma^2 = var(\log m_0) \in (0, \infty)$. Then for any measurable subset B of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x \in B^o} \frac{x^2}{2\sigma^2} &\leq \liminf_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B \right) \\ &\leq \limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B \right) \\ &\leq -\inf_{x\in \bar{B}} \frac{x^2}{2\sigma^2}. \end{aligned}$$

Proof – Large deviations

- Proof for large deviations

  Notice that the Laplace transform of $\log Z_n$ is

$$\mathbb{E}e^{t\log Z_n} = \mathbb{E}Z_n^t.$$

Our results about large deviations (Theorems 2 and 3) are consequences of the Gärtner-Ellis theorem and the following result.

Theorem 5 (Moments of Z_n)

Under certain moment conditions, we have

$$\lim_{n\to\infty} \frac{\mathbb{E}Z_n^t}{\left(\mathbb{E}m_0^t\right)^n} = C(t) \in (0,\infty), \qquad \forall t \in \mathbb{R}.$$

This is an extension of a result of Ney and Vidyashankar (2003) on the Galton-Watson process.

Harmonic moments of W

To study the moments of Z_n, we need to consider the moments of W.

- Moments of positive orders

  Guivarc'h and Liu (2001) showed that for $p > 1$,

$$\mathbb{E}W^p \in (0, \infty) \quad \text{iff} \quad \mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < \infty \text{ and } \mathbb{E}m_0^{1-p} < 1.$$

Harmonic moments of W

- Harmonic moments (moments of negative orders)

Theorem 6 (Harmonic moments of W)

Assume (H).
(i) (General case) There exists a constant $a > 0$ such that

$$\mathbb{E}W^{-a} < \infty.$$

(ii) (Special case) If $\|p_1\|_\infty < 1$, then $\forall a > 0$,

$$\mathbb{E}W^{-a} < \infty \quad \text{if and only if} \quad \mathbb{E}p_1 m_0^a < 1.$$

Harmonic moments of W

Corollary (Critical value)

Assume (H) and $\|p_1\|_\infty < 1$. If $\mathbb{E} p_1 m_0^{a_0} = 1$, then

$$\begin{aligned} \mathbb{E} W^{-a} &< \infty \quad \text{if } 0 < a < a_0, \\ \mathbb{E} W^{-a} &= \infty \quad \text{if } a \geq a_0. \end{aligned}$$

Remark

Hambly (1992) proved that under some assumption similar to (H), the number $\alpha_0 := -\frac{\mathbb{E} \log p_1}{\mathbb{E} \log m_0}$ is the critical value for the a.s. existence of the quenched moments $\mathbb{E}_\xi W^{-a} (a > 0)$. By Jensen's inequality, it is easy to see that $a_0 \leq \alpha_0$.

Harmonic moments of W

Corollary (Critical value)

Assume (H) and $\|p_1\|_\infty < 1$. If $\mathbb{E} p_1 m_0^{a_0} = 1$, then

$$\mathbb{E} W^{-a} < \infty \quad \text{if } 0 < a < a_0,$$
$$\mathbb{E} W^{-a} = \infty \quad \text{if } a \geq a_0.$$

Remark

Hambly (1992) proved that under some assumption similar to (H), the number $\alpha_0 := -\frac{\mathbb{E} \log p_1}{\mathbb{E} \log m_0}$ is the critical value for the a.s. existence of the quenched moments $\mathbb{E}_\xi W^{-a} (a > 0)$. By Jensen's inequality, it is easy to see that $a_0 \leq \alpha_0$.

Proof – Harmonic moments of W

- Proof for Harmonic moments of W (Theorem 6)
  (ii) Special case where $\|p_1\|_\infty < 1$.
  - Necessity

$$W = \frac{1}{m_0}\sum_{i=1}^{Z_1} W_i^{(1)} \qquad a.s.,$$

where $W_i^{(1)}$ is the limit related to the i-th particle of the first generation. Since $\mathbb{P}(Z_1 \geq 2) > 0$,

$$\mathbb{E}W^{-a} > \mathbb{E}m_0^a \left(W_1^{(1)}\right)^{-a} \mathbf{1}_{\{Z_1=1\}} = \mathbb{E}p_1 m_0^a \mathbb{E}W^{-a}.$$

Therefore $\mathbb{E}p_1 m_0^a < 1$.

Proof – Harmonic moments of W

- Sufficiency
  Set

$$\phi_\xi(t) = \mathbb{E}_\xi e^{-tW} \quad \text{and} \quad \phi(t) = \mathbb{E}\phi_\xi(t) \; (t > 0).$$

Lemma (Liu, 2001)

Let A be a positive random variable such that for some $0 < p < 1$, $t_0 \geq 0$ and all $t > t_0$,

$$\phi(t) \leq p\mathbb{E}\phi(At).$$

If $p\mathbb{E}A^{-a_1} < 1$ for some $a_1 > 0$, then

$$\phi(t) = O(t^{-a_1})(t \to \infty) \text{ and } \mathbb{E}W^{-a} < \infty \text{ for all } a \in (0, a_1).$$

Proof – Harmonic moments of W

Lemma (Upper bound for ϕ_ξ)

Assume (H). Then there exist constants $\beta \in (0,1)$ and $K > 0$ such that a.s.

$$\phi_\xi(t) \leq \beta \quad \forall t \geq 1/K.$$

If additionally $\|p_1\|_\infty < 1$, then for some constants $a > 0$ and $C > 0$, a.s.

$$\phi_\xi(t) \leq Ct^{-a} \quad \forall t \geq 1/K.$$

By this lemma, $\forall \varepsilon > 0$, there exists a constant $t_\varepsilon > 0$ such that a.s.

$$\phi_\xi(t) \leq \varepsilon, \quad \forall t \geq t_\varepsilon.$$

Proof – Harmonic moments of W

Notice that ϕ_ξ satisfies the functional equation

$$\phi_\xi(t) = f_0(\phi_{T\xi}(\frac{t}{m_0})),$$

where $f_0(s) = \sum_{i=0}^{\infty} p_i(\xi_0)s^i, \; s \in [0,1]$, is the generating function of $p(\xi_0)$. We therefore have a.s.

$$\phi_\xi(t) \leq (p_1 + (1-p_1)\varepsilon)\phi_{T\xi}(\frac{t}{m_0}), \quad \forall t \geq At_\varepsilon.$$

Proof – Harmonic moments of W

Taking expectation, we obtain for $t \geq At_\varepsilon$,

$$\phi(t) \leq \mathbb{E}(p_1 + (1 - p_1)\varepsilon)\phi(\frac{t}{m_0}) = p_\varepsilon \mathbb{E}\phi(\tilde{A}_\varepsilon t),$$

where $p_\varepsilon = \mathbb{E}(p_1 + (1 - p_1)\varepsilon) < 1$ and $\tilde{A}_\varepsilon$ is a positive random variable whose distribution is determined by

$$\mathbb{E}g(\tilde{A}_\varepsilon) = \frac{1}{p_\varepsilon}\mathbb{E}(p_1 + (1 - p_1)\varepsilon)g(\frac{1}{m_0})$$

for all bounded and measurable function g.

Proof – Harmonic moments of W

Since $\mathbb{E}p_1 m_0^a < 1$, we can take $a_1 > a$ such that $\mathbb{E}p_1 m_0^{a_1} < 1$. Take $\varepsilon > 0$ small enough such that

$$p_\varepsilon \mathbb{E}\tilde{A}_\varepsilon^{-a_1} = \mathbb{E}(p_1 + (1-p_1)\varepsilon)m_0^{a_1} < 1.$$

Then by the lemma of Liu (2001),

$$\phi(t) = O(t^{-a_1})(t \to \infty) \quad \text{and} \quad \mathbb{E}W^{-a} < \infty.$$

(i) General case
Notice that $\phi_\xi(t) \le \beta$ a.s. for $t \ge t_\beta = \frac{1}{K}$. It suffices to repeat the proof of sufficiency of (ii) with β in place of ε.

Proof – Harmonic moments of W

Since $\mathbb{E}p_1 m_0^a < 1$, we can take $a_1 > a$ such that $\mathbb{E}p_1 m_0^{a_1} < 1$. Take $\varepsilon > 0$ small enough such that

$$p_\varepsilon \mathbb{E}\tilde{A}_\varepsilon^{-a_1} = \mathbb{E}(p_1 + (1 - p_1)\varepsilon)m_0^{a_1} < 1.$$

Then by the lemma of Liu (2001),

$$\phi(t) = O(t^{-a_1})(t \to \infty) \quad \text{and} \quad \mathbb{E}W^{-a} < \infty.$$

(i) General case

Notice that $\phi_\xi(t) \leq \beta$ a.s. for $t \geq t_\beta = \frac{1}{K}$. It suffices to repeat the proof of sufficiency of (ii) with β in place of ε.

Proof – Moments of Z_n

Theorem 5 (Moments of Z_n)

Under certain moment conditions, we have

$$\lim_{n\to\infty} \frac{\mathbb{E} Z_n^t}{(\mathbb{E} m_0^t)^n} = C(t) \in (0,\infty), \quad \forall t \in \mathbb{R}.$$

Proof. Denote the distribution of ξ_0 by τ_0. Fix $t \in \mathbb{R}$ and define a new distribution $\tilde{\tau}_0$ as

$$\tilde{\tau}_0(dx) = \frac{m(x)^t \tau_0(dx)}{\mathbb{E} m_0^t},$$

where $m(x) = \mathbb{E}[Z_1 | \xi_0 = x] = \sum_{i=0}^{\infty} i p_i(x)$.

Proof – Moments of Z_n

Consider the new BPRE whose environment distribution is $\tilde{\tau} = \tilde{\tau}_0^{\otimes \mathbb{N}}$ instead of $\tau = \tau_0^{\otimes \mathbb{N}}$. The corresp. total probability and expectation are denoted by $\tilde{\mathbb{P}} = \mathbb{P}_\xi \otimes \tilde{\tau}$ and $\tilde{\mathbb{E}}$.

Then

$$\frac{\mathbb{E} Z_n^t}{\left(\mathbb{E} m_0^t\right)^n} = \tilde{\mathbb{E}} W_n^t.$$

We distinguish three cases: $t \in (0,1)$, $t > 1$ and $t < 0$. For each case, under certain moment conditions,

$$\lim_{n \to \infty} \tilde{\mathbb{E}} W_n^t = \tilde{\mathbb{E}} W^t \in (0, \infty).$$

Take $C(t) = \tilde{\mathbb{E}} W^t$.

Proof – Moderate deviations

- Proof of MDP (Theorem 3)
  Similar to the proof of LDP (Theorem 2), the proof of MDP is a combination of the Gärtner-Ellis theorem and the following result.

Theorem 7

Assume (H). Let $\Lambda_n(t) = \log \mathbb{E} \exp\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} t\right)$ and $\tilde{\Lambda}_n(t) = \log \mathbb{E} \exp\left(\frac{\log P_n - n\mathbb{E}\log m_0}{a_n} t\right)$. Then

$$\lim_{n\to\infty} \frac{\Lambda_n(\frac{a_n^2}{n} t)}{\tilde{\Lambda}_n(\frac{a_n^2}{n} t)} = 1, \qquad \forall t \neq 0.$$

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Thank you !

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