Branching Brownian motion seen from the tip

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Joint work with Elie Aidekon, Eric Brunet and Zhan Shi

Outline

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1. BBM and FKPP

2. BBM seen from the tip

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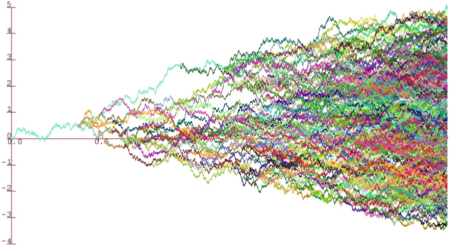
BBM

The model = Particles $X_1(t), \ldots, X_{N(t)}(t)$ on $\mathbb{R}$



- Start with one particle at 0.
- Movement = independent Brownian motions
- Branching = at rate 1 into two new particles (more general possible).

Rightmost particle

Define $M(t) = \max_{i=1,\dots,N(t)} X_i(t)$.

Theorem (Rightmost particle $M(t)$)

$c^* = \sqrt{2}$.

- $\lim_{t\to\infty} \frac{M(t)}{t} = c^*$.
- $c^* t - M(t) \to \infty$ *a.s.*

Proof by martingale techniques

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The map $u(t,x) = \mathbb{P}(M(t) < x)$ solves

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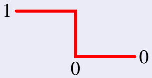
avec $1 - u(x,0) = \Big($ 

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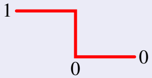
avec $1 - u(x,0) = \Bigg($ 

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Idea : Initial particle in 0. After dt

- with proba $(1-dt)$ no split but diffuse : $u(t+dt,x) \mapsto u(t,x-\xi_t)$
- with proba dt branch $u(t+dt,x) \mapsto u(t,x)^2$

Bramson's result

Define m_t by $u(t, m_t) = 1/2$. i.e. m_t is the median of $M(t)$ (results still valid if we take the expectation).

KPP '37

$$u(t, m_t + x) \to w(x) \text{ unif. in } x \text{ as } t \to \infty$$

where $m_t = \sqrt{2}t + a(t)$ and $a(t) \to -\infty$. (McKean shows that $a(t) \ll -2^{-3/2} \log t$) and w solution to $\frac{1}{2}w'' + \sqrt{2}w' + (w^2 - w) = 0$.

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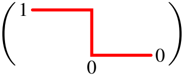
If change initial condition from 

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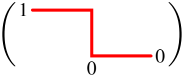
If change initial condition from 

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Hence $\exists C_B \in \mathbb{R}$ such that

$$M_t - (\sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + C_B) \to_{\text{dist}} W$$

with $\mathbb{P}(W \leq x) = w(x)$.

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Suppose it holds. Then $t^{-1}\int_0^t \mathbf{1}_{M_s^x < m_s + x} ds \to w(0)$ a.s. . Start two independent BBM, one at x and one at 0. Positive probability that they meet before any branching $\to$ successfull coupling so $w(0) = w(x)$. Contradiction.

The derivative martingale

The derivative martingale

$$Z(t) = \sum_{u \in N(t)} (\sqrt{2}t - X_u(t)) e^{\sqrt{2}X_u(t) - 2t}$$

(additive martingale $W_{-\sqrt{2}}(t) = \sum_{u \in N(t)} e^{\sqrt{2}X_u(t) - 2t} \to 0$)

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$\exists C > 0$ *s.t.* $\forall x \in \mathbb{R}$

$$\lim_{s \to \infty} \lim_{t \to \infty} \mathbb{P}\left(M(t+s) - m(t+s) \le x | \mathcal{F}_s\right) = \exp\{-CZe^{-\sqrt{2}x}\}, \ a.s.$$

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$1 - w(x) \sim Cxe^{-\sqrt{2}x}$.

Structure of W

Suggests that, once we "know" Z,

$$\mathbb{P}\left(M(t) - m(t) \leq x\right) \sim \exp\{-e^{-\sqrt{2}x + \log CZ}\}, \text{ a.s.}$$

so that $\mathbb{P}(\sqrt{2}(M(t) - m(t)) - \log(CZ) \leq x) \to \exp(-e^{-x})$ and hence

$$M(t) - m(t) - 2^{-1/2}\log(CZ) \to_{\text{dist}} 2^{-1/2} G$$

where G is a Gumbel variable. $W =_{\text{dist}} (G + \log(CZ))/\sqrt{2}$

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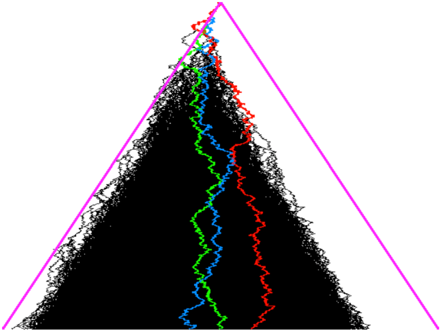

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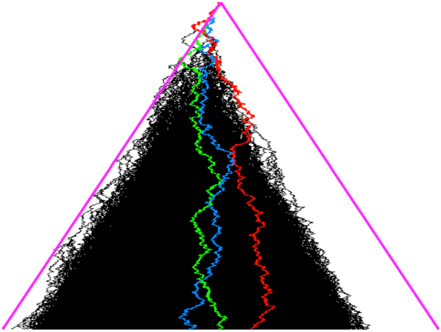

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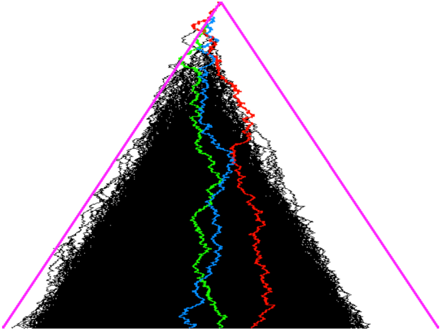

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- Because of early fluctuation a random delay is created $(2^{-1/2}\log(CZ))$
- Fluctuations around $m(t) + 2^{-1/2}\log(cZ)$ are Gumbel.

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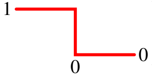
converges to an equilibrium state.
First we show a simple argument due to Brunet Derrida to show that the PP $X_i(t) - m(t)$ converges. Uses Bramson and McKean representation.

McKean Representation

Recall : $u(t,x) := \mathbb{P}(M(t) < x)$ solves $\partial_t u = \frac{1}{2}\partial_x^2 u + u(u-1)$ with

$1 - u(x,0) = \Big($ 

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More generally, let $g : \mathbb{R} \mapsto [0,1]$ then

Theorem (McKean, 1975)

If $u : \mathbb{R} \times \mathbb{R}_+ \mapsto [0,1]$ *solves the FKPP equation*

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with initial condition $u(0,x) = g(x)$*, then*

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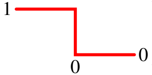
$1 - u(x,0) = \Big($ 

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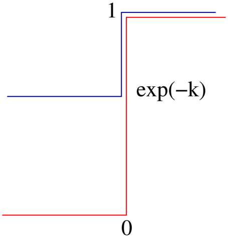
In general $\partial_t u = \frac{1}{2}\partial_x^2 u + \beta(f(u) - u)$.

Brunet Derrida argument (seen from m_t)



$H_\phi(x,t) = \mathbb{E}[\prod \phi(x - X_u(t))]$, H_ϕ solves KPP with $H_\phi(x,0) = \phi(x)$.

- If ϕ $H_\phi(x,t) = \mathbb{P}(M(t) < x)$
- If ϕ $H_\phi(x,t) = \mathbb{E}[e^{-kN(x,t)}]$ with $N(x,t) = \# u : X_u(t) > x$.

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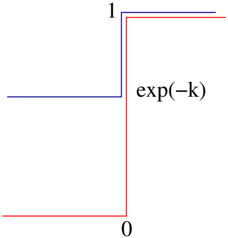

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Bramson : $H_\phi(m(t)+z,t) \to w(z+\delta(\phi))$.
For any Borel set $A \subset \mathbb{R}$, the Laplace transform of $\#\{u : X_u(t) \in m(t) + A\}$ converges.

Theorem (Brunet Derrida 2010)

The point process of the particles seen from $m(t)$ converges in distribution as $t \to \infty$.

Not too hard to show : the limit point process $(X_i, i = 1, 2, \dots)$ has the superposition property, i.e.

$$\forall, \alpha, \beta \text{ s.t. } e^{\alpha} + e^{\beta} = 1 : X^{\alpha} + X^{\beta} =_d X.$$

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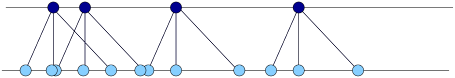

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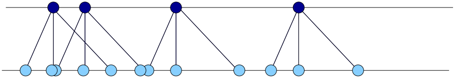

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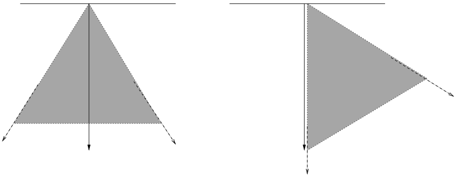
Independently, Bovier, Arguin and Kestler 2010 obtain results that are similar to part of what follows. Seem to use $\neq$ techniques.

Normalization

We do the following change of coordinates : we suppose that the Brownian motions have diffusion $\sigma^2 = 2$ and drift $\rho = 2$. Instead of rightmost focus on leftmost.



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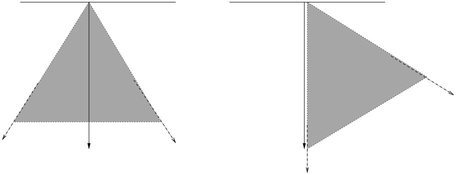

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In this framework

$$Z(t) := \sum_{u \in N(t)} X_u(t) e^{-X_u(t)}$$

is the derivative martingale. Recall its limit exists and is positive a.s.

BBM seen from the tip

$$Y_i(t) := X_i(t) - m_t + \log(CZ), \qquad 1 \le i \le N(t).$$

Theorem (Aidekon, B., Brunet, Shi '11)

As $t \to \infty$ the point process $(Y_i(t), 1 \le i \le N(t))$ converges in distribution to the point process $\mathcal{L}$ obtained as follows

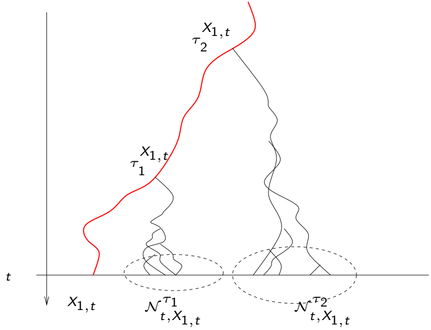
(i) *Define $\mathcal{P}$ a Poisson point process on $\mathbb{R}_+$, with intensity measure $\frac{\sqrt{2}}{\sigma} e^{\sqrt{2}x/\sigma} dx$ where a is a constant.*

(ii) *For each atom x of $\mathcal{P}$, we attach a point process $x + \mathcal{Q}^{(x)}$ where $\mathcal{Q}^{(x)}$ are i.i.d. copies of a certain point process $\mathcal{Q}$.*

(iii) *$\mathcal{L}$ is then the superposition of all the point processes $x + \mathcal{Q}^{(x)}$: $\mathcal{L} := \{x + y : x \in \mathcal{P} \cup \{0\}, y \in \mathcal{Q}^{(x)}\}$*

Bovier Arguin and Kistler (2010) obtain a very similar result (and much more).

Structure of $\mathcal{Q}$



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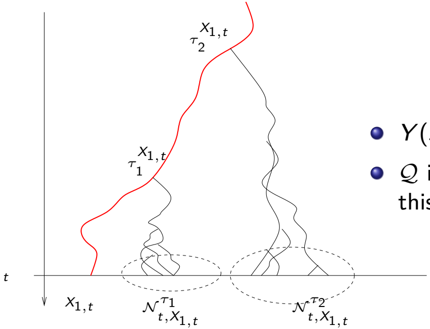

- $Y(s) := X_{1,t}(t-s) - X_{1,t}(t)$ is the red path.
- $\mathcal{Q}$ is the superposition of the BBM that branch on this path.

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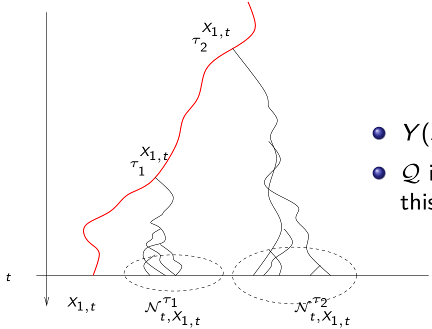


- $Y(s) := X_{1,t}(t-s) - X_{1,t}(t)$ is the red path.
- $\mathcal{Q}$ is the superposition of the BBM that branch on this path.

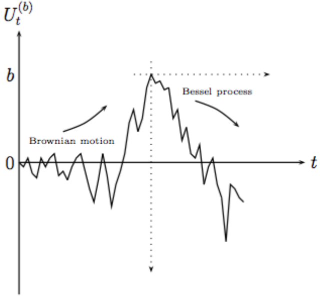
$\mathcal{N}_x(t) =$ BBM at time t started from one ptc at x.
$\mathcal{N}_x^*(t) =$ BBM at time t conditioned to $\min \mathcal{N}_x^*(t) > 0$ started from one ptc at x.

$$G_t(x) \quad := \quad \mathbb{P}\{\min \mathcal{N}_0(t) \leq x\},$$

so that $G_t(x + m_t) \to \mathbb{P}(\sigma W \leq x)$ by Bramson.

Law of Y.

$$U_v^{(b)} := \begin{cases} B_v, & \text{if } v \in [0,\ T_b], \\ b - R_{v-T_b}, & \text{if } v \geq T_b. \end{cases}$$



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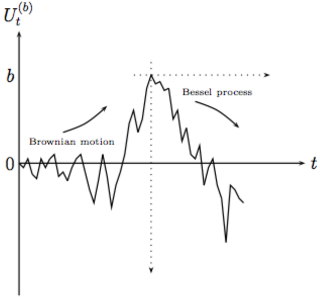

Take b random : $\mathbb{P}(b \in dx) = f(x)/c_1$ where

$$f(x) := \mathbb{E}\left[e^{-2\lambda \int_0^\infty G_v(\sigma U_v^{(x)}) dv}\right] \tag{1}$$

and $c_1 =$ constant.

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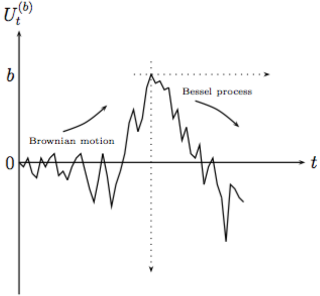


Take b random : $\mathbb{P}(b \in dx) = f(x)/c_1$ where

$$f(x) := \mathbb{E}\left[e^{-2\lambda \int_0^\infty G_v(\sigma U_v^{(x)})dv}\right] \quad (1)$$

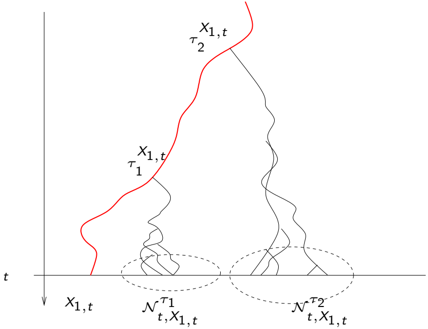
and $c_1 =$ constant.
Conditionally on b, Y has a density $/U^{(b)}$ given by

$$\frac{1}{f(b)} e^{-2\lambda \int_0^\infty G_v(\sigma U_v^{(b)})dv} \quad (2)$$

Structure of $\mathcal{Q}$

Theorem (Aidekon, B., Brunet, Shi '11)

Let $(Y(t), t \geq 0)$ be as above. Start independent BBMs $\mathcal{N}^*_{-Y(t)}(t)$ conditioned to finish to the right of $X_{1,t}$ along the path Y at rate $2\lambda(1 - \mathbb{P}(\min \mathcal{N}_{Y(t)}(t) \leq 0))dt$. Then $\cup_{t \in \pi} \mathcal{N}^*_{-Y(t)}(t)$ is distributed as $\mathcal{Q}$.



$D(\zeta, t) := \bigcup_{\tau_i^{X_{1,t}} \geq t-\zeta} \mathcal{N}^{(i)}_{t,X_{1,t}}$ particles born off $X_{1,t}$ less than ζ unit of time ago, then we have the following joint convergence in distribution

$$\lim_{\zeta \to \infty} \lim_{t \to \infty} \{(X_{1,t}(t-s) - X_{1,t}(t), s \geq 0), D(\zeta, t)\} = \{(Y(s), s \geq 0), \mathcal{Q}\}.$$

Structure of $\mathcal{Q}$

$$I_\zeta(t) = \mathbb{E}\Big\{ \exp\Big(-\sum_i \mathbf{1}_{t-\tau_i<\zeta} \sum_{j=1}^n \alpha_j \#[\mathcal{N}^{\tau_i}_{t,X_{1,t}} \cap (X_1(t)+A_j)]\Big)\Big\}$$

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Theorem (Aidekon, B., Brunet, Shi '11)

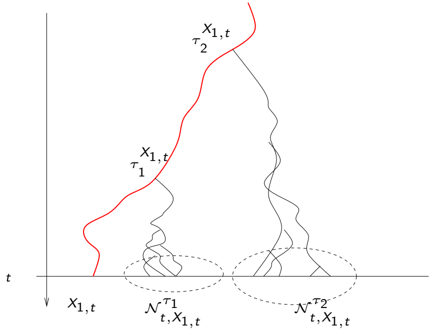
$\forall \alpha_j \geq 0$ *(for* $1 \leq j \leq n$*),*

$$\mathbb{E}\Big\{ e^{-\sum_{j=1}^n \alpha_j \mathcal{Q}(A_j)}\Big\} = \lim_{\zeta\to\infty} \lim_{t\to\infty} I_\zeta(t) = \frac{\int_0^\infty \mathbb{E}\big(e^{-2\lambda \int_0^\infty G_v^*(\sigma U_v^{(b)})dv}\big)db}{\int_0^\infty \mathbb{E}\big(e^{-2\lambda \int_0^\infty G_v(\sigma U_v^{(b)})dv}\big)db},$$

where $G_v(x) := \mathbb{P}\{\min \mathcal{N}(v) \leq x\}$,
$G_v^*(x) := 1 - \mathbb{E}\Big[e^{-\sum_{j=1}^n \alpha_j \#[\mathcal{N}(v)\cap(x+A_j)]}\mathbf{1}_{\{\min \mathcal{N}(v)\geq x\}}\Big]$.

Define

$$I(t) := \mathbb{E}\Big\{ F(X_{1,t}(s),\ s \in [0,\ t]) \\ \exp\Big(-\sum_i f(t-\tau_i^{X_{1,t}}) \sum_{j=1}^{n} \alpha_j \#[\mathcal{N}^{\tau_i}_{t,X_{1,t}} \cap (X_1(t)+A_j)] \Big) \Big\},$$



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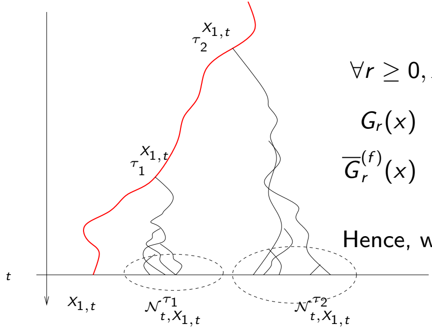

$\forall r \geq 0, x \in \mathbb{R}$ define

$$G_r(x) := \mathbb{P}\{\min \mathcal{N}(r) \leq x\},$$

$$\overline{G}_r^{(f)}(x) := \mathbb{E}\Big[e^{-f(r)\sum_{j=1}^{n}\alpha_j \#[\mathcal{N}(r)\cap(x+A_j)]}\mathbf{1}_{\{\min \mathcal{N}(r)\geq x\}}\Big].$$

Hence, when $f \equiv 0$ we have $\overline{G}_r^{(f)}(x) = 1 - G_r(x))$.

Define

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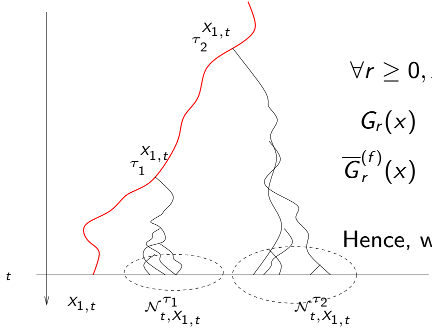

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Theorem (Aidekon, B., Brunet, Shi '11)

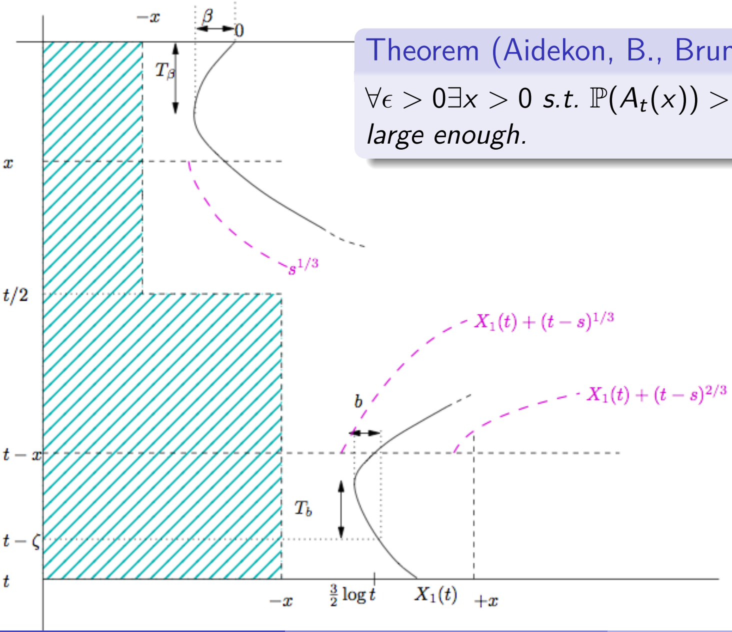
We have $I(t) = \mathbb{E}\Big[e^{\sigma B_t} F(\sigma B_s,\, s \in [0,\, t])\, e^{-2\lambda \int_0^t [1 - \overline{G}_{t-s}^{(f)}(\sigma B_t - \sigma B_s)] ds} \Big]$,
where B is BM.

Theorem (Aidekon, B., Brunet, Shi '11)

In particular, the path $(s \mapsto X_{1,t}(s), 0 \leq s \leq t)$ is a standard Brownian motion in a potential :

$$\mathbb{E}\Big[F(X_{1,t}(s),\ s \in [0,\ t])\Big] = \mathbb{E}\Big[e^{\sigma B_t} F(\sigma B_s,\ s \in [0,\ t])\, e^{-2\lambda \int_0^t G_{t-s}(\sigma B_t - \sigma B_s) ds}\Big]. \tag{3}$$

path of the leftmost particle



Theorem (Aidekon, B., Brunet, Shi '11)

$\forall \epsilon > 0 \exists x > 0$ s.t. $\mathbb{P}(A_t(x)) > 1 - \epsilon$ for all t large enough.

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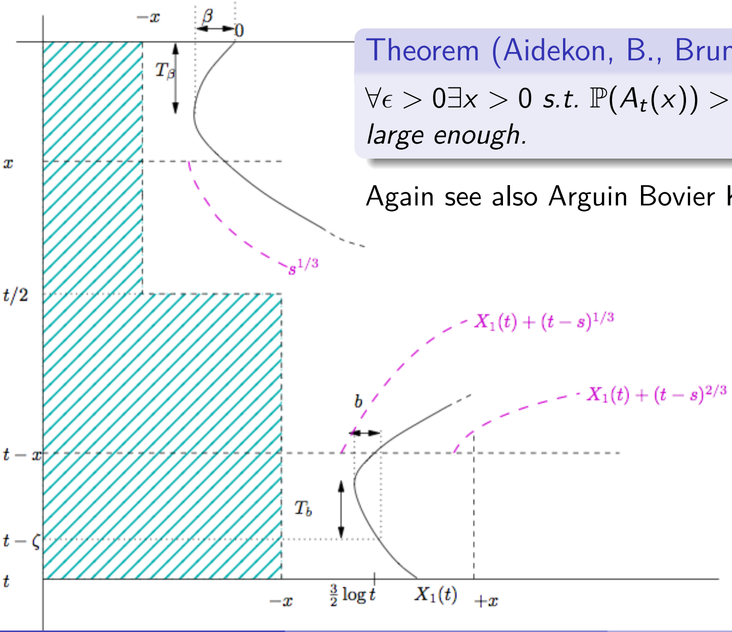


Theorem (Aidekon, B., Brunet, Shi '11)

$\forall \epsilon > 0 \exists x > 0$ *s.t.* $\mathbb{P}(A_t(x)) > 1 - \epsilon$ *for all* t *large enough.*

Again see also Arguin Bovier Kistler.

Theorem (Aidekon, B., Brunet, Shi '11)

As $t \to \infty$ the point process $(Y_i(t), 1 \le i \le N(t))$ converges in distribution to the point process $\mathcal{L}$ obtained as follows

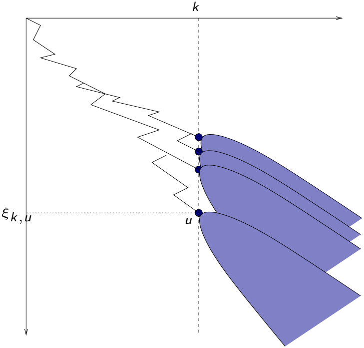
(i) *Define $\mathcal{P}$ a Poisson point process on $\mathbb{R}_+$, with intensity measure $\frac{\sqrt{2}}{\sigma} e^{\sqrt{2}x/\sigma} dx$ where a is a constant.*

(ii) *For each atom x of $\mathcal{P} \cup \{0\}$, we attach a point process $x + \mathcal{Q}^{(x)}$ where $\mathcal{Q}^{(x)}$ are i.i.d. copies of a certain point process $\mathcal{Q}$.*

(iii) *$\mathcal{L}$ is then the superposition of all the point processes $x + \mathcal{Q}^{(x)}$: $\mathcal{L} := \{x + y : \; x \in \mathcal{P} \cup \{0\}, \; y \in \mathcal{Q}^{(x)}\}$*

With extreme value theory.

Idea : Fix $K > 0$ large and stop particles when they hit k for the first time. (no escape).



$H_k = \#$ particles stoped at k

$$Z = \lim_{k \to \infty} 2^{-1/2} k e^{-k} H_k, \qquad (4)$$

exists a.s., is in $(0, \infty)$.

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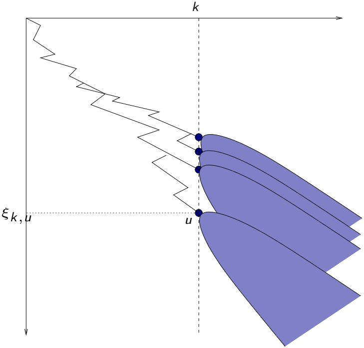

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exists a.s., is in $(0, \infty)$.
Start iid BBM so that
$\forall u \in \mathcal{H}_k, X_1^u(t) =_d k + X_1(t - \xi_{k,u})$, where
$\xi_{k,u} =$ time when u reaches k. By Bramson
$\forall k \geq 1, u \in \mathcal{H}_k$,

$$X_1^u(t) - m_t \overset{law}{\to} k + W_u, \qquad t \to \infty.$$

(see also Bovier Arguin and Kistler (2010) : extremal particles in BBM either branch at the very beginning or at the end)

Define

$$\mathcal{P}^*_{k,\infty} := \sum_{u \in \mathcal{H}_k} \delta_{k+W(u)+\log Z}.$$

Recall that $\mathcal{P} =$ PPP with intensity $ae^x dx$.

Proposition

$$\mathcal{P}^*_{k,\infty} \to \mathcal{P}$$

In the sense of convergence in distribution.

Take $(X_i, i \in \mathbb{N})$ a sequence of i.i.d. r.v. such that

$$\mathbb{P}(X_i \geq x) \sim Cxe^{-x}, \text{ as } x \to \infty.$$

Call $M_n = \max_{i=1,\dots,n} X_i$ the record. Define $b_n = \log n + \log\log n$. Then

$$\begin{aligned}
\mathbb{P}(M_n - b_n \leq y) &= (\mathbb{P}(X_i \leq y + b_n))^n \\
&= (1 - (1 + o(1))C(y + b_n)e^{-(y+b_n)})^n \\
&\sim \exp\left(-nC(y + b_n)\frac{1}{n\log n}e^{-y}\right) \\
&\sim \exp(-Ce^{-y})
\end{aligned}$$

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Therefore $\mathbb{P}(M_n - (b_n + \log C) \leq y) \sim \exp(-e^{-y})$. By classical results the point process

$$\zeta_n := \sum_{i=1}^{n} \delta_{X_i - b_n - \log C}$$

converges in distribution to a Poisson point process on $\mathbb{R}$ with intensity $e^{-x}dx$.

Recall that $\mathbb{P}(W \leq x) \sim Cxe^{-x}$ so apply to

$$\sum_{u \in \mathcal{H}_k} \delta_{W(u)+(\log H_k + \log\log H_k)+\log C}$$

which converges in dist. to a PPP on $\mathbb{R}$ with intensity $e^x dx$.

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Use $k2^{-1/2}e^{-k}H_k \to Z$ to obtain

$$H_k \sim \sigma k^{-1} e^k Z$$

and therefore

$$\log H_k + \log\log H_k \sim k + \log Z + c$$

and hence

$$\mathcal{P}^*_{k,\infty} = \sum_{u \in \mathcal{H}_k} \delta_{k+W(u)+\log Z}$$

also converges (as $k \to \infty$) towards a PPP on $\mathbb{R}$ with intensity $\frac{\sqrt{2}}{\sigma} e^{\sqrt{2}x/\sigma} dx$.

many-to-one

$$W_t := \sum_{u \in \mathscr{N}(t)} e^{-X(u)}, \qquad t \geq 0,$$

is the additive martingale. Because critical not UI and $\to 0$.

$$\mathbb{Q}|_{\mathscr{F}_t} = W_t \bullet \mathbb{P}|_{\mathscr{F}_t}.$$

Under $\mathbb{Q}$ spine = BM drift (0), branch at rate 2 into two particles.

$$\mathbb{Q}\{\Xi_t = u \,|\, \mathscr{F}_t\} = \frac{e^{-X(u)}}{W_t}.$$

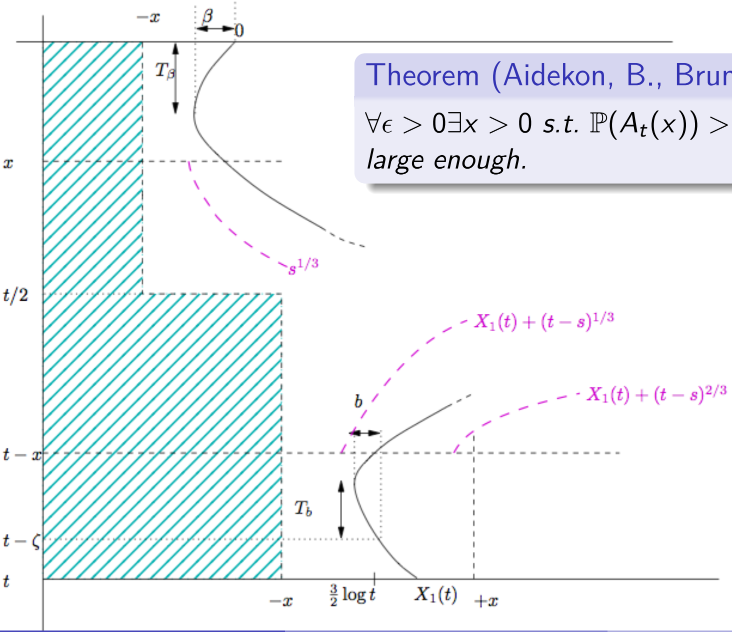
For each $u \in \mathscr{N}(t)$, G_u = a r.v. in $\mathcal{F}_t$.

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[\sum_{u \in \mathscr{N}(t)} G_u] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{W_t} \sum_{u \in \mathscr{N}(t)} G_u\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{X(\Xi_t)} G_{\Xi}\right]. \end{aligned}$$

Suppose that we want to check if $\exists$ a path $X^u(s)$ with some property A

$$\mathbb{P}(\exists |u| = t : (X^u(s), s \in [0,t]) \in A) = \mathbb{P}(e^{\sigma B_t}; (\sigma B_s, s \in [0,t]) \in A).$$

path of the leftmost particle



Theorem (Aidekon, B., Brunet, Shi '11)

$\forall \epsilon > 0 \exists x > 0$ s.t. $\mathbb{P}(A_t(x)) > 1 - \epsilon$ for all t large enough.

$$A_t(x) := E_1(x) \cap E_2(x) \cap E_3(x) \cap E_4(x)$$

where the events E_i are defined by

$$\begin{aligned}
E_1(x) &:= \left\{ \left| X_{1,t} - \frac{3}{2}\log t \right| \leq x \right\}, \\
E_2(x) &:= \left\{ \min_{[0,t]} X_{1,t}(s) \geq -x, \ \min_{[t/2,t]} X_{1,t}(s) \geq \frac{3}{2}\log t - x \right\} \\
E_3(x) &:= \left\{ \forall s \in [x, t/2], X_{1,t}(s) \geq s^{1/3} \right\} \\
E_4(x) &:= \left\{ \forall s \in [t/2, t-x], \ X_{1,t}(s) - X_{1,t} \in [(t-s)^{1/3}, (t-s)^{2/3}] \right\}.
\end{aligned}$$

the event E_3

Suppose we have $E_1(z)$ and $E_2(z)$ for z large enough. By the many-to-one lemma, we get

$$\begin{aligned}&\mathbb{P}(E_3(x)^c, E_1(z), E_2(z))\\ \leq\ & e^z t^{3/2}\mathbb{P}\Big\{\exists s\in[x,t/2] : B_s \leq s^{1/3}, \min_{[0,t/2]} B_s \geq -z,\\ &\qquad \min_{t/2,t} B_s \geq \frac{3}{2}\log t - z, B_t \leq \frac{3}{2}\log t + z\Big\}.\end{aligned}$$

Applying the Markov property at time $t/2$, it yields that

$$\begin{aligned}&\mathbb{P}\Big\{\exists s\in[x,t/2] : B_s \leq s^{1/3}, \min_{[0,t/2]} B_s \geq -z, \min_{t/2,t} B_s \geq \frac{3}{2}\log t - z, B_t \leq \frac{3}{2}\log t + z\Big\}\\ =\ & \mathbb{E}\Big[\mathbf{1}_{\{\exists s\in[x,t/2]:B_s\leq s^{1/3}\}}\mathbf{1}_{\{\min_{[0,t/2]} B_s\geq -z\}}\\ &\qquad \mathbb{P}_{B_{t/2}}\Big\{\min_{s\in[0,t/2]} B_s \geq \frac{3}{2}\log t - z, B_{t/2} \leq \frac{3}{2}\log t + z\Big\}\Big]\\ \leq\ & c2zt^{-3/2}\mathbb{E}\Big[\mathbf{1}_{\{\exists s\in[x,t/2]:B_s\leq s^{1/3}\}}\mathbf{1}_{\{\min_{[0,t/2]} B_s\geq -z\}}(B_{t/2} - \frac{3}{2}\log t + z)\Big]\\ \leq\ & c2zt^{-3/2}\mathbb{E}\Big[\mathbf{1}_{\{\exists s\in[x,t/2]:B_s\leq s^{1/3}\}}\mathbf{1}_{\{\min_{[0,t/2]} B_s\geq -z\}}(B_{t/2} + z)\Big]\end{aligned}$$

where the second inequality comes from bound on $\mathbb{P}\Big\{\min_{s\in[0,t]} B_s \geq -x,\ B_t \leq -x + y\Big\}$. We recognize the h-transform of the Bessel.

We end up with

$$\begin{aligned}\mathbb{P}(E_3(x)^c, E_1(z), E_2(z)) &\leq e^z c2z\mathbb{P}_z(\exists s \in [x, t/2] : R_s \leq s^{1/3}) \\ &\leq e^z c2z\mathbb{P}_z(\exists s \geq x : R_s \leq s^{1/3})\end{aligned}$$

which is less than ε for x large enough.

proof

Step 1 : The process $V^x(t) := \prod_{u \in N(t)} w(\sqrt{2}t - X_i(t) + x)$ is a $\mathcal{F}_t$-martingale

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Step 2 : $\sum_{u\in N(t)} e^{\sqrt{2}X_u(t)-2t}$ is a positive martingale, converges to a finite value, so $\min_u(\sqrt{2}t - X_u\ \ (t)) = +\infty$ a.s.

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Step 3 : $1 - w(y) = Cye^{-\sqrt{2}y}$.

$$\log V^x(t) = \sum_{u \in N(t)} \log w(\sqrt{2}t - X_i(t) + x)$$
$$\sim \sum_{u \in N(t)} -C(\sqrt{2}t - X_i(t) + x)e^{-2t + \sqrt{2}X_i(t) - \sqrt{2}x}$$
$$\sim -CZ(t)e^{-\sqrt{2}x} - CY(t)xe^{-\sqrt{2}x}$$

with $Y(t) = \sum_{u \in N(t)} e^{\sqrt{2}X_i(t) - 2t}$. Clearly $\lim Y/Z = 0$. $\lim Y(t) = Y \geq 0$ exists a.s. so $Z(t) \to \infty$ a.s. on the event $Y > 0$, this $\Rightarrow V^x = 0$. But since $\mathbb{E}(V^x) = w(x) \to 1$ when $x \to \infty, \mathbb{P}(Y > 0) = 0$.

proof 2

Step 4 : Thus $\lim Z(t) = -e^{\sqrt{2}x} C^{-1} \log V^x$. We conclude that $\lim Z(t)$ exists and > 0 a.s.

$$w(x) = \mathbb{E}[V^x(\infty)] = \mathbb{E}\left[\exp\left\{-CZe^{-\sqrt{2}x}\right\}\right].$$

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Step 5 :
$\mathbb{P}(M(t+s) \leq m(t+s) + x | \mathcal{F}_s) = \prod_{u \in N(t)} u(t, x + m(t+s) - X_u(s))$.

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$$w(x) = \mathbb{E}[V^x(\infty)] = \mathbb{E}\left[\exp\left\{-CZe^{-\sqrt{2}x}\right\}\right].$$

Step 5 :
$\mathbb{P}(M(t+s) \leq m(t+s) + x | \mathcal{F}_s) = \prod_{u \in N(t)} u(t, x + m(t+s) - X_u(s))$.Recall that $u(t, x + m(t)) = \mathbb{P}(M(t) \leq m(t) + x) \rightarrow w(x)$ and that $lim_t(m(t+s) - m(t) - \sqrt{2}s) = 0$

proof 2

Step 4 : Thus $\lim Z(t) = -e^{\sqrt{2}x} C^{-1} \log V^x$. We conclude that $\lim Z(t)$ exists and > 0 a.s.

$$w(x) = \mathbb{E}[V^x(\infty)] = \mathbb{E}\left[\exp\left\{-CZe^{-\sqrt{2}x}\right\}\right].$$

Step 5 :
$\mathbb{P}(M(t+s) \leq m(t+s) + x | \mathcal{F}_s) = \prod_{u \in N(t)} u(t, x + m(t+s) - X_u(s))$.Recall that $u(t, x + m(t)) = \mathbb{P}(M(t) \leq m(t) + x) \to w(x)$ and that $lim_t(m(t+s) - m(t) - \sqrt{2}s) = 0$ so that

$$\begin{aligned}\lim_t \mathbb{P}(M(t+s) \leq m(t+s) + x | \mathcal{F}_s) &= \prod_{u \in N(t)} w(x + m(t+s) - X_u(s) - m(t\\ &= \prod_{u \in N(t)} w(x + \sqrt{2}s - X_u(s))\\ &:= V^x(s)\end{aligned}$$