Fast and slow scales in a superprocess limit for interacting age and trait-structured particle system

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Particle system and difficulties

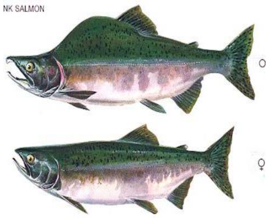
Superprocess limit

Sketch of the proof of Step 2

Motivation

★ Structured populations: individuals are characterized by variables that affect their reproducing and survival capacities. Here: trait $x \in \mathcal{X} \subset \mathbb{R}^d$ and age $a \in \mathbb{R}_+$

★ Continuous time discrete population, stochastic evolution based on individual dynamics.



Description

★ Population carrying capacity in $n \in \mathbb{N}^*$ $(n \to +\infty)$ and fixed resources: individuals biomass is in $1/n$:

$$X_t^n(dx, da) = \frac{1}{n} \sum_{i=1}^{N_t^n} \delta_{(X_i, A_i)}$$

★ Heredity of the trait x unless a mutation happens with proba $p \in [0, 1]$

★ Allometric demographies: lifetime and gestation length are proportional to individual biomass. Thus, birth and death rates are of order n, while preserving the demographic balance.

$$nr(x, a) + b(x, a)$$

$$nr(x, a) + d(x, a) + \int_{\mathcal{X} \times \mathbb{R}_+} U((x, a), (y, \alpha)) X_t^n(dy, d\alpha)$$

★ the right scale to observe age-structure is of order $1/n$. If the birth time is c: $a = n(t - c)$

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Short Bibliography

★ With trait, interaction, without age: Fournier-Méléard (2004), Champagnat-Ferrière-Méléard (2006)

★ With age, interaction but other rescaling: Oelschläger (1990), Tran (2008). Plus trait: Méléard-Tran (2009), Ferrière-Tran (2009)

★ With age, location but without interaction: Dynkin (1991), Bose-Kaj (1995), Kaj-Sagitov (1998), Bose-Kaj (2000), Dawson *et al.* (2002).

The limit may not be a superprocess. Laplace techniques are used: no generalization when interactions are added.

Difficulties:

- competition creates a dependence between age and trait-structure.
- Individualities are lost in the limit: how keep age ?
- Aging velocity is $n \to +\infty$.

Generator

★ For $F \in \mathcal{C}_b^1(\mathbb{R})$ and $\varphi \in \mathcal{C}_b^{0,1}(\mathcal{X} \times \mathbb{R}_+)$, and for $\mu \in \mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+)$, we define $F_\varphi(\mu) = F(\langle \mu, \varphi \rangle)$ with:

$$\langle \mu, \varphi \rangle = \int_{\mathcal{X} \times \mathbb{R}_+} \varphi(x, a) \mu(dx, da)$$

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★ The generator of X^n is given for these functions by:

$$\begin{aligned} L^n F_\varphi(\mu) &= n^2 \langle \mu, \partial_a \varphi(.) \rangle F'_\varphi(\mu) \\ &+ n \int_{\mathcal{X} \times \mathbb{R}_+} \Big[\big(nr(x,a) + d(x,a) + \mu U(x,a)\big) \Big(F_\varphi\big(\mu - \frac{1}{n}\delta_{(x,a)}\big) - F_\varphi(\mu) \Big) \\ &+ \big(nr(x,a) + b(x,a)\big) \int_{\mathcal{X}} \Big(F_\varphi\big(\mu + \frac{1}{n}\delta_{(x+h,0)}\big) - F_\varphi(\mu) \Big) K^n(x, dh) \Big] \mu(dx, da), \end{aligned}$$

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where

$$K^n(x, dh) = p \int_{\mathbb{R}^d} \delta_{h'}(dh) \pi^n(x, h') dh' + (1 - p)\ \delta_0(dh)$$

$\pi^n(x, h')$ is the Gaussian density with mean 0 and covariance $\Sigma(x)/n$.

Semi-martingale decomposition

★ There exists a càdlàg Markov process with generator L^n, solution of an SDE for which trajectorial uniqueness holds.

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★ Let $\varphi \in \mathcal{C}_b^{0,1}(\mathcal{X} \times \mathbb{R}_+)$ + moment assumptions on the initial conditions.
Then the process:

$$\begin{aligned} M_t^{n,\varphi} =& \langle X_t^n, \varphi\rangle - \langle X_0^n, \varphi\rangle - n \int_0^t \langle X_s^n, \partial_a \varphi(x,a)\rangle \, ds \\ & - \int_0^t \int_{\mathcal{X}\times\mathbb{R}_+} \Big(\big(nr(x,a) + b(x,a)\big) \int_{\mathbb{R}^d} \varphi(x+h, 0) K^n(x, dh) \\ & - \big(nr(x,a) + d(x,a) + X^n U(x,a)\big) \varphi(x,a) \Big) X_s^n(dx, dc)\, ds \end{aligned}$$

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is a square integrable martingale started at 0 with quadratic variation:

$$\begin{aligned} \langle M^{n,\varphi}\rangle_t =& \frac{1}{n} \int_0^t \int_{\mathcal{X}\times\mathbb{R}_+} \Big((nr(x,a) + b(x,a)) \int_{\mathbb{R}} \varphi^2(x+h, 0) K^n(x, dh) \\ & + (nr(x,a) + d(x,a) + X_s^n U(x,a)) \varphi^2(x,a) \Big) X_s^n(dy, dc) \, ds. \end{aligned}$$

Averaging phenomenon

★ The age is a "fast" evolving component. Mutation rate is of order n but since mutation steps are small, the trait will be viewed as a "slow" component.
We expect that the age distribution stabilizes in an equilibrium that depends on the trait.

★ Slow-fast phenomena are known in the literature: Freidlin-Ventzell (1984-1993), Kurtz (1992), Ball-Kurtz-Popovic-Rempala (2006).

★ Here, our "slow-fast" components are distributions. We introduce the trait-marginal $\bar{X}^n_t(dx)$ of $X^n_t(dx, da)$: $\forall f(x) \in \mathcal{B}_b(\mathcal{X}, \mathbb{R})$,

$$\int_{\mathcal{X}} f(x)\bar{X}^n_t(dx) = \int_{\mathcal{X}\times\mathbb{R}_+} f(x)X^n_t(dx, da).$$

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Main result

Th: Under sufficient assumption for the initial condition, $\bar{X}^n$ converge in distribution in $\mathbb{D}([0,T],\mathcal{M}_F(\mathcal{X}))$ to the unique continuous superprocess $\bar{X}$ such that

$$M_t^f = \langle \bar{X}_t, f\rangle - \langle \bar{X}_0, f\rangle - \int_0^t \int_{\mathcal{X}} \Big(\widehat{r}(x)\,\frac{p}{2}\sum_{i,j=1}^{d} \Sigma_{ij}(x)\partial_{ij}^2 f(x) + \big[\widehat{b}(x) - \big(\widehat{d}(x) + \bar{X}_s \widehat{U}(x)\big)\big] f(x)\Big)\bar{X}_s(dx)\,ds$$

is a square integrable martingale with quadratic variation:

$$\langle M^f\rangle_t = \int_0^t \int_{\mathcal{X}} 2\widehat{r}(x) f^2(x)\bar{X}_s(dx)\,ds.$$

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Notation: Here, any $\widehat{\psi}(x)$ is defined for a bounded function $\psi(x,a)$ by

$$\widehat{\psi}(x) = \int_{\mathbb{R}_+} \psi(x,a)\widehat{m}(x,a)da, \quad \text{with } \widehat{m}(x,a) = \frac{\exp\big(-\int_0^a r(x,\alpha)d\alpha\big)}{\int_0^{+\infty} \exp\big(-\int_0^a r(x,\alpha)d\alpha\big)\, da}$$

and

$$\bar{X}_t\widehat{U}(x) = \int_{\mathcal{X}} \Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} U((x,a),(y,\alpha))\widehat{m}(y,\alpha)d\alpha\ \widehat{m}(x,a)da \Big)\, \bar{X}_t(dy).$$

Idea of the proof

Step 1: Uniform tightness of $(\bar{X}^n)_{n\in\mathbb{N}^*}$ in $\mathbb{D}([0,T],\mathcal{M}_F(\mathcal{X}))$

We use Aldous-Rebolledo criterion + moment estimates +
Roelly-Méléard criterion with an argument due to Jourdain-Méléard.

Step 2: Averaging phenomenon

We consider a subsequence of $(\bar{X}^n)_{n\in\mathbb{N}^*}$ converging to $\bar{X}$. To identify the limiting values, we shall prove that for all $t\in[0,T]$, $(X^n_t(dx,da))_{n\in\mathbb{N}^*}$ converges in distribution to $\bar{X}_t(dx)\widehat{m}(x,a)da$.

Step 3: Identification of the limiting values

With Step 2 + moment estimates, we can characterize $\bar{X}$ as the solution of the martingale problem given in the theorem.

Step 4: Uniqueness of the solution of the martingale problem

Dawson-Girsanov transform + computation of the Laplace transform:

$$\mathbb{E}\big(\exp(\langle \bar{X}_t, f\rangle)\big) = \mathbb{E}\big(\exp(\langle \bar{X}_0, U_t(f)\rangle)\big)$$

where $U_t(f)$ solves:

$$\frac{\partial u}{\partial t}(t,x) = Au(t,x) - \widehat{r}(x)u^2(t,x).$$

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1) For every t, $(X^n_t)_{n\in\mathbb{N}^*}$ is uniformly tight in $\mathcal{M}_F(\mathcal{X}\times\mathbb{R}_+)$

★ **Assumption:** $r(x) > \underline{r}$. The lifelengths of particles are stochastically dominated by i.i.d. exponential r.v. E_i of parameter $n\underline{r}$.

★ For $A>0$ and $n_0\in\mathbb{N}^*$:

$$\sup_{n\geq n_0}\mathbb{P}\big(X^n_t((K\times[0,A])^c)>2\varepsilon\big)$$

$$\leq \sup_{n\geq n_0}\mathbb{P}\big(\bar{X}^n_t(K^c)>\varepsilon\big)+\sup_{n\geq n_0}\mathbb{P}\Big(\sum_{i=1}^{N^n_t}\mathbb{I}_{nA_i(t)>A}/n>\varepsilon\Big)$$

$$\leq \varepsilon+\sup_{n\geq n_0}\mathbb{P}\Big(\sum_{i=1}^{nN}\mathbb{I}_{E_i>A}>n\varepsilon\Big)+\sup_{n\geq n_0}\mathbb{P}\big(N^n_t>nN\big)$$

with N so that term 3$<\varepsilon$, A such that $\exp(-\underline{r}A)<\varepsilon/2N$ and since:

$$\mathbb{P}\Big(\sum_{i=1}^{nN}\mathbb{I}_{E_i>A}>n\varepsilon\Big)=\mathbb{P}\Big(\sum_{i=1}^{nN}(\mathbb{I}_{E_i>A}-e^{-\underline{r}A})>n(\varepsilon-Ne^{-\underline{r}A})\Big)$$

$$\leq\mathbb{P}\Big(\sum_{i=1}^{nN}(\mathbb{I}_{E_i>A}-e^{-\underline{r}A})>n\varepsilon/2\Big)\leq e^{-\frac{n\varepsilon^2}{8(Ne^{-\underline{r}A}(1-e^{-\underline{r}A})+\varepsilon/6)}}<\varepsilon$$

for n_0 large.

2) Kurtz' argument (1) - Occupation measure

★ Generalizing Kurtz, we define $\Gamma \in \mathcal{M}_F(\mathbb{R}_+ \times \mathcal{X} \times \mathbb{R}_+)$ for $B \subset \mathcal{X} \times \mathbb{R}_+$ and $t \in \mathbb{R}_+$ by:

$$\Gamma^n([0,t] \times B) = \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \mathbb{I}_B(x,a) X_s^n(dx, da)\, ds$$

★ A sufficient condition for the uniform tightness of $(\Gamma^n)_{n \in \mathbb{N}^*}$ is that

$$\sup_{n\in\mathbb{N}^*} \mathbb{E}\Big(\Gamma^n([0,t] \times (K \times [0,A])^c)\Big) = \sup_{n\in\mathbb{N}^*} \int_0^t \mathbb{E}\Big(X_s^n((K \times [0,A])^c)\Big) ds \leq C(t)\varepsilon.$$

★ And:

$$\begin{aligned}\mathbb{E}\Big(X_s^n((K \times [0,A])^c)\Big) \\ =& 2\varepsilon \mathbb{P}\Big(X_s^n((K \times [0,A])^c) \leq 2\varepsilon\Big) + \mathbb{E}\Big(\langle X_s^n, 1\rangle \mathbb{I}_{X_s^n\big((K\times[0,A])^c\big) > 2\varepsilon}\Big) \\ \leq& 2\varepsilon + \sqrt{\mathbb{E}\Big(\langle X_s^n, 1\rangle^2\Big)} \sqrt{\mathbb{P}\Big(X_s^n((K \times [0,A])^c) > 2\varepsilon\Big)} = C(s)(\varepsilon + \sqrt{\varepsilon}).\end{aligned}$$

Kurtz' argument (2) - Consequences

★ From the definition, the marginal measure of $\Gamma^n(ds, dx, da)$ on $\mathcal{M}_F(\mathcal{X}) \times \mathbb{R}_+$ is $\bar{X}^n_s(dx)ds$

★ The sequence $(\Gamma^n(ds, dx, da), \bar{X}^n_s(dx)ds)$ is uniformly tight and there is a subsequence converging to $(\Gamma(ds, dx, da), \bar{X}_s(dx)ds)$.

★ $\bar{X}_s(dx)ds$ is the marginal measure of $\Gamma(dx, ds, da)$.

★ There exists a (random) probability-valued process $\gamma_{s,x}(da)$, predictable in (ω, s) such that:

$$\Gamma(ds, dx, da) = \gamma_{s,x}(da)\bar{X}_s(dx)ds.$$

3) Averaging (1)

★ The following process is a martingale:

$$\frac{M_t^{n,\varphi}}{n} = \frac{\langle X_t^n, \varphi\rangle - \langle X_0^n, \varphi\rangle}{n} - \int_0^t \int_{\mathbb{D}\times\mathbb{R}_+} \Big[\partial_a \varphi(x,a) + \Big(r(x,a) + \frac{b(x,a)}{n}\Big) \int_{\mathbb{R}} \varphi(x,0) K^n(x,dh) - \Big(r(x,a) + \frac{d(x,a) + X_s^n U(x,a)}{n}\Big)\varphi(x,a)\Big] \Gamma^n(ds,dx,da)$$

★ For each t, the process:

$$\widetilde{M}_t^{n,\phi} := \int_0^t \int_{\mathcal{X}\times\mathbb{R}_+} \Big[\partial_a \varphi(x,a) + r(x,a)\Big(\varphi(x,0) - \varphi(x,a)\Big)\Big] \Gamma^n(ds,dx,da)$$

converges in distribution to:

$$\int_0^t \int_{\mathcal{X}\times\mathbb{R}_+} \Big[\partial_a \varphi(x,a) + r(x,a)\big(\varphi(x,0) - \varphi(x,a)\big)\Big] \gamma_{s,x}(da) \bar{X}_s(dx) ds$$

which is a martingale since $\lim_{n\to+\infty} \mathbb{E}\Big(\Big|\frac{M_t^{n,\phi}}{n} - \widetilde{M}_t^{n,\phi}\Big|\Big) = 0$.

As it is also continuous and of bounded variation, it must hence be zero.

Averaging (2)

★ Thus, dt-almost everywhere

$$\int_{\mathbb{D}\times\mathbb{R}_+} \Big[\partial_a\varphi(x,a) + r(x,a)\big(\varphi(x,0)-\varphi(x,a)\big)\Big]\gamma_{t,x}(da)\bar{X}_t(dx) = 0.$$

For $\varphi(x,a)=\phi(x)\psi(a)$, we have that a.s. and $\bar{X}_t(dx)$-almost everywhere,

$$\int_{\mathbb{R}_+} \Big[\partial_a\psi(a) + r(x,a)\big(\psi(0)-\psi(a)\big)\Big]\gamma_{t,x}(da) = 0$$

★ There exists a unique probability solution to this equation, which has a $\widehat{m}(x,a)$ density w.r.t. the Lebesgue measure

For $\psi(a)=f(0)+\int_0^a f(\alpha)d\alpha$, where $f\in\mathcal{C}^1(\mathbb{R}_+,\mathbb{R}_+)$ is positive:

$$\begin{aligned}\int_{\mathbb{R}_+} f(a)\gamma_{t,x}(da) &= \int_{\mathbb{R}_+}\Big[\Big(f(0)+\int_0^{+\infty}\mathbb{I}_{\alpha<a}f(\alpha)d\alpha\Big)-f(0)\Big]r(x,a)\gamma_{t,x}(da)\\ &= \int_{\mathbb{R}_+} f(\alpha)\int_\alpha^{+\infty} r(x,a)\gamma_{t,x}(da)\,d\alpha. \qquad (1)\end{aligned}$$

This entails that $\gamma_{t,x}(da)$ is absolutely continuous with respect to the Lebesgue measure with density $\widehat{m}(x,a)=\int_a^{+\infty} r(x,\alpha)\widehat{m}(x,\alpha)d\alpha$ (of class $\mathcal{C}^\infty$).

Averaging (3)

★ By an integration by part formula:

$$-\psi(0)\widehat{m}(x,0) - \int_{\mathbb{R}_+} \psi(a)\partial_a \widehat{m}(x,a)da = \int_{\mathbb{R}_+} \big(\psi(a) - \psi(0)\big) r(x,a)\widehat{m}(x,a)da.$$

By identification, $\widehat{m}(x,a)$ is a solution of:

$$\partial_a \widehat{m}(x,a) = -r(x,a)\widehat{m}(x,a)$$
$$\widehat{m}(x,0) = \int_{\mathbb{R}_+} r(x,a)\widehat{m}(x,a)da.$$

which is solved by:

$$\widehat{m}(x,a) = \widehat{m}(x,0)e^{-\int_0^a r(x,\alpha)d\alpha} = \frac{e^{-\int_0^a r(x,\alpha)d\alpha}}{\int_0^{+\infty} e^{-\int_0^a r(x,\alpha)d\alpha} da}$$

since $\gamma_{t,x}(da)$ is a probability measure.