

Continuous (convex) optimisation

M2 - PSL / Dauphine / S.U.

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Lecture 5: Saddle points, Primal-dual splitting

- 1 Optimisation for saddle-point problems, duality
 - Uzawa
 - ADMM
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 - Extensions
 - PDHG: acceleration

Constrained problems. Duality.

Assume we need to solve:

$$\min_x \{f(x) : g_i(x) \leq 0, i = 1, \dots, m\}$$

with f, g_i convex (KKT framework), and we assume in addition:

- f is strongly convex with some parameter $\gamma > 0$,
- $|g(x) - g(x')| \leq L|x - x'|$ ($g = (g_1, \dots, g_m)$ is L -Lipschitz).

We can introduce a Lagrange multiplier for the constraints as in the KKT's theorem:

$$\min_{g(x) \leq 0} f(x) = \min_x \sup_{\lambda \geq 0} f(x) + \langle \lambda, g(x) \rangle = (\geq) \sup_{\lambda \geq 0} \min_x f(x) + \langle \lambda, g(x) \rangle$$

and try to solve the *dual problem*

$$\max_{\lambda \geq 0} \mathcal{D}(\lambda) \quad \text{where} \quad \mathcal{D}(\lambda) = \min_x f(x) + \langle \lambda, g(x) \rangle.$$

Assume now we are able to solve for any $\lambda \geq 0$ the unconstrained problem

$$\min_x f(x) + \langle \lambda, g(x) \rangle$$

(for instance, using FISTA...)

Let $x(\lambda)$ be the (unique) solution. Then for any $\mu \geq 0$,

$$\begin{aligned} \mathcal{D}(\mu) &= f(x(\mu)) + \langle \mu, g(x(\mu)) \rangle = f(x(\mu)) + \langle \lambda, g(x(\mu)) \rangle + \langle \mu - \lambda, g(x(\mu)) \rangle \\ &\geq f(x(\lambda)) + \langle \lambda, g(x(\lambda)) \rangle + \frac{\gamma}{2} |x(\mu) - x(\lambda)|^2 + \langle \mu - \lambda, g(x(\mu)) \rangle, \end{aligned}$$

that is:

$$\mathcal{D}(\lambda) \leq \mathcal{D}(\mu) + \langle \lambda - \mu, g(x(\mu)) \rangle - \frac{\gamma}{2} |x(\mu) - x(\lambda)|^2$$

and it follows:

$$\mathcal{D}(\lambda) \leq \mathcal{D}(\mu) + \langle \lambda - \mu, g(x(\mu)) \rangle - \frac{\gamma}{2} |x(\mu) - x(\lambda)|^2$$

and it follows:

$$g(x(\mu)) \in \partial \mathcal{D}(\mu)$$

[here the *supergradient* of the concave function g] and

$$\gamma |x(\mu) - x(\lambda)|^2 \leq \langle \lambda - \mu, g(x(\mu)) - g(x(\lambda)) \rangle \leq |\lambda - \mu| |g(x(\mu)) - g(x(\lambda))|.$$

Now we have $|g(x(\mu)) - g(x(\lambda))| \leq L|x(\mu) - x(\lambda)|$ and we deduce

$$|g(x(\mu)) - g(x(\lambda))| \leq \frac{L^2}{\gamma} |\lambda - \mu|$$

that is, \mathcal{D} is concave with L^2/γ -Lipschitz gradient.

Then, it can be solved using “ISTA” or “FISTA”, for instance:

$$\lambda^{k+1} = \left(\lambda^k + \tau g(x(\lambda^k)) \right)^+$$

for $\tau = \gamma/L^2$, which will ensure that:

$$\mathcal{D}(\lambda^*) - \mathcal{D}(\lambda^N) \leq \frac{L^2}{2\gamma N} |\lambda^0 - \lambda^*|^2.$$

In addition (using $\mu = \lambda^*$ in the first inequality of the previous slide),

$$|x(\lambda^N) - x^*|^2 \leq \frac{2}{\gamma} \left(\mathcal{D}(\lambda^*) - \mathcal{D}(\lambda^N) \right) \leq \frac{L^2}{\gamma^2 N} |\lambda^0 - \lambda^*|^2.$$

(Of course, one should use acceleration, but for this we need to be able to solve the primal problems very precisely.)

The “ADMM”

or Alternating Directions Method of Multipliers

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The “ADMM” aims at solving a slightly more general form than $f(Kx) + g(x)$, namely:

$$\min_{Ax+By=b} f(x) + g(y) \quad (1)$$

for f, g convex, lsc., and A, B continuous, linear operators. [Of course, it is still of the form $f(Kx) + g(x)$ for some other functions f, g , which?]

It has the dual form:

$$\max_p \langle b, p \rangle - f^*(A^*p) - g^*(B^*p)$$

with strong duality if f, g are continuous at some x, y with $Ax + By = b$ (in finite dimension, x, y in the relative interiors of the domains, respectively, of f, g) or if f^* is continuous at some point A^*p and g^* at B^*p (in finite dimension, $A^*p \in \text{ri dom } f^*$, $B^*p \in \text{ri dom } g^*$ for some p). This seems not particularly easier to solve for generic f, g .

ADMM: Augmented Lagrangian

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An “augmented Lagrangian” approach consists in introducing the constraint in the form

$$\min_{x,y} \sup_z f(x) + g(y) - \langle z, Ax + By - b \rangle + \frac{\gamma}{2} |Ax + By - b|^2$$

for some $\gamma > 0$, which is equivalent (as the sup is $+\infty$ if $Ax + By \neq b$) to the original problem. Why use $\gamma > 0$? It makes the problem more regular.

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One considers the dual (concave) function:

$$\mathcal{D}(z) = \inf_{x,y} f(x) + g(y) - \langle z, Ax + By - b \rangle + \frac{\gamma}{2} \|Ax + By - b\|^2$$

Thanks to the quadratic term, it has $(1/\gamma)$ -Lipschitz gradient. This follows from the following result which we will prove next week in a slightly more general setting:

Lemma

Let f be convex, lsc: then f is γ -convex (strongly convex with parameter γ) if and only if f^ has $(1/\gamma)$ -Lipschitz gradient.*

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Hence, a natural method for maximizing the dual could be to implement an (accelerated) gradient ascent, using that (the supergradient)

$$\partial\mathcal{D}(z) = \{-(Ax + By - b)\}$$

where (x, y) minimizes the problem which defines $\mathcal{D}(z)$. (Same proof as for the Uzawa method, or simply Legendre-Fenchel identity.)

However, it means we are able to solve for (x, y) , which is not necessarily easy. Hence the “Alternating Directions Methods of Multipliers”.

ADMM: algorithm

[Proposed initially by Glowinski and Marroco 75 / Gabay and Mercier 76]

Choose $\gamma > 0$, y^0 , z^0 .

for all $k \geq 0$ **do**

Find x^{k+1} by minimising $x \mapsto f(x) - \langle z^k, Ax \rangle + \frac{\gamma}{2} |b - Ax - By^k|^2$,

Find y^{k+1} by minimising $y \mapsto g(y) - \langle z^k, By \rangle + \frac{\gamma}{2} |b - Ax^{k+1} - By|^2$,

Update $z^{k+1} = z^k + \gamma(b - Ax^{k+1} - By^{k+1})$.

end for

Convergence: for f, g convex, lsc. and provided there exists a saddle-point, the method converges.

Proof is omitted. In fact, it can be related to a Douglas-Rachford iteration on the dual problem. Or it is an “inexact” gradient ascent on the dual, with an error which needs to be controlled.

ADMM: difficulties

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In practice, it is not necessarily easy to solve

$$\min_x f(x) - \langle z^k, Ax \rangle + \frac{\gamma}{2} |b - Ax - By^k|^2$$

and one may revert to “proximal” ADMM: one introduces G, H symmetric positive-definite operators and considers rather the steps:

$$\begin{aligned} x^{k+1} &= \arg \min_x f(x) - \langle z^k, Ax \rangle + \frac{\gamma}{2} |b - Ax - By^k|^2 + \frac{1}{2} |x - x^k|_F^2, \\ y^{k+1} &= \arg \min_y g(y) - \langle z^k, By \rangle + \frac{\gamma}{2} |b - Ax^{k+1} - By|^2 + \frac{1}{2} |y - y^k|_G^2. \end{aligned}$$

In practice, choosing $F = I/\tau - \gamma A^*A$ and $G = I/\sigma - \gamma B^*B$ with τ, σ small enough allows to solve the problems if the “prox” of f, g can be computed. Then, again, the algorithm will converge.

“PDHG”

(primal dual hybrid gradient)

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One considers again:

$$\min_x f(Kx) + g(x) = \min_x \sup_y \langle Kx, y \rangle + g(x) - f^*(y).$$

A basic idea consists in performing a gradient descent in x and a gradient ascent in y (“Arrow-Hurwicz” method):

$$\begin{aligned}x^{k+1} &= (I + \tau \partial g)^{-1}(x^k - \tau K^* y^k), \\y^{k+1} &= (I + \sigma \partial f^*)^{-1}(y^k + \sigma K x^k)\end{aligned}$$

for some $\sigma, \tau > 0$, however in general this will not converge (case $f, g = 0$: this is similar to an explicit update for a monotone operator).

We observe though that in this specific case, one could use x^{k+1} in the second step (\rightarrow semi implicit). Does it help?

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Well, almost. For $f, g = 0$ one has:

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} I & -\tau K^* \\ \sigma K & I - \sigma\tau K K^* \end{pmatrix} \begin{pmatrix} x^k \\ y^k \end{pmatrix}$$

and the eigenvalues of this matrix have modulus equal to 1 for $\sigma\tau$ small enough.

Well, almost. For $f, g = 0$ one has:

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and the eigenvalues of this matrix have modulus equal to 1 for $\sigma\tau$ small enough.

We write, for $\lambda \in \mathbb{C}$,

$$\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} I & -\tau K^* \\ \sigma K & I - \sigma\tau KK^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} x = \frac{\tau}{1-\lambda} K^* y & \text{or } \lambda = 1, K^* y = 0 \\ \sigma Kx + y - \sigma\tau KK^* y = \lambda y \end{cases}$$

In case $\lambda = 1$ we also deduce that $Kx = 0$. So the eigenvalue 1 corresponds to $x \in \ker K$, $y \in \ker K^*$. If $K \neq 0$ there must be another eigenvalue $\lambda \neq 1$. Then, one has:

$$\frac{\sigma\tau}{1-\lambda} KK^* y - \sigma\tau KK^* y = (\lambda - 1)y \Leftrightarrow KK^* y = -\frac{(\lambda - 1)^2}{\sigma\tau\lambda} y.$$

unless $\lambda = 0$ but in this case $y = 0$, then $x = 0$, and it is not an eigenvalue.

We see that y is an eigenvector of KK^* , corresponding to an eigenvalue $\mu > 0$ (otherwise $\lambda = 1$). λ solves:

$$-\frac{(\lambda - 1)^2}{\sigma\tau\lambda} = \mu \Leftrightarrow \lambda^2 - 2\lambda + 1 = -\sigma\tau\mu\lambda \Leftrightarrow \lambda^2 - 2\left(1 - \frac{\sigma\tau\mu}{2}\right)\lambda + 1 = 0$$

If $\sigma\tau\|K^*K\| \leq 2$, letting $1 - \sigma\tau\mu/2 = \cos\theta$ we find that $\lambda = \cos\theta \pm i\sin\theta$.

Hence, in that case, the algorithm will not converge nor diverge (the iterates “rotate”). Of course, for $f, g \neq 0$, the method may actually converge, in practice.

The PDHG algorithm is a stable and converging variant of the previous case. Its simplest form is:

$$\begin{aligned}x^{k+1} &= (I + \tau \partial g)^{-1}(x^k - \tau K^* y^k), \\y^{k+1} &= (I + \sigma \partial f^*)^{-1}(y^k + \sigma K(2x^{k+1} - x^k))\end{aligned}\quad (PDHG)$$

Proposition (He-Yuan 2011)

If $\tau \sigma \|K^* K\| < 1$ then PDHG^a is a proximal-point algorithm.

^a“Primal-dual hybrid gradient”

To see this we write the iterates as follows:

$$\begin{cases} \frac{x^{k+1} - x^k}{\tau} + \partial g(x^{k+1}) \ni -K^* y^k = K^*(y^{k+1} - y^k) - K^* y^{k+1} \\ \frac{y^{k+1} - y^k}{\sigma} + \partial f^*(y^{k+1}) \ni K(x^{k+1} - x^k) + Kx^{k+1}, \end{cases}$$

that is

$$\begin{pmatrix} \frac{1}{\tau} I & -K^* \\ -K & \frac{1}{\sigma} I \end{pmatrix} \begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} - y^k \end{pmatrix} + \begin{pmatrix} \partial g(x^{k+1}) \\ \partial f^*(y^{k+1}) \end{pmatrix} + \begin{pmatrix} 0 & K^* \\ -K & 0 \end{pmatrix} \begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} \ni 0.$$

We remark that if S is symmetric, positive-definite (defines a metric/coercive in infinite dimension) then for A a maximal monotone operator:

$$S(z^{k+1} - z^k) + Az^{k+1} \ni 0$$

is the iteration of the proximal point algorithm for the maximal monotone operator $S^{-1}A$ in the metric defined by the scalar product $\langle z, z' \rangle_S := \langle Sz, z' \rangle$.

Hence here, one find that the algorithm is a PPA iff

$$M_{\tau,\sigma} := \begin{pmatrix} \frac{1}{\tau}I & -K^* \\ -K & \frac{1}{\sigma}I \end{pmatrix}$$

is symmetric, coercive.

One has:

$$\left\langle M_{\tau,\sigma} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle = \frac{1}{\tau}|\xi|^2 + \frac{1}{\sigma}|\eta|^2 - 2 \langle K\xi, \eta \rangle$$

is positive if and only if for any $X, Y \geq 0$

$$\sup_{|\xi| \leq X, |\eta| \leq Y} 2 \langle K\xi, \eta \rangle = 2\|K\|XY < \frac{X^2}{\tau} + \frac{Y^2}{\sigma}$$

if and only if

$$2\|K\| < \min_{X \geq 0, Y \geq 0} \frac{X}{\tau Y} + \frac{Y}{\sigma X} = \frac{2}{\sqrt{\tau\sigma}}$$

if and only if

$$\tau\sigma\|K\|^2 < 1,$$

hence the theorem.

PDHG: rate

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One inherits the rate of convergence for the iterates of a proximal-point algorithm.

Yet for this specific form (using the convexity of f^*, g) one can improve the rate.

We denote $z = (x, y)^T$ and take the scalar product of the algorithm and $z^{k+1} - z$:

$$\begin{aligned} \langle z^{k+1} - z^k, z^{k+1} - z \rangle_{M_{\tau, \sigma}} + \left\langle \begin{pmatrix} 0 & K^* \\ -K & 0 \end{pmatrix} \begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix}, \begin{pmatrix} x^{k+1} - x \\ y^{k+1} - y \end{pmatrix} \right\rangle \\ + g(x^{k+1}) + f^*(y^{k+1}) \leq g(x) + f^*(y) \end{aligned}$$

The scalar product is

$$- \langle K^* y^{k+1}, x \rangle + \langle K x^{k+1}, y \rangle$$

while

$$\langle z^{k+1} - z^k, z^{k+1} - z \rangle_{M_{\tau, \sigma}} = \frac{1}{2} |z^{k+1} - z^k|_{M_{\tau, \sigma}}^2 + \frac{1}{2} |z^{k+1} - z|_{M_{\tau, \sigma}}^2 - \frac{1}{2} |z^k - z|_{M_{\tau, \sigma}}^2.$$

Hence:

$$\begin{aligned} \frac{1}{2} |z^{k+1} - z^k|_{M_{\tau, \sigma}}^2 + \frac{1}{2} |z^{k+1} - z|_{M_{\tau, \sigma}}^2 - \frac{1}{2} |z^k - z|_{M_{\tau, \sigma}}^2 - \langle K^* y^{k+1}, x \rangle + \langle K x^{k+1}, y \rangle \\ + g(x^{k+1}) + f^*(y^{k+1}) \leq g(x) + f^*(y) \end{aligned}$$

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Therefore, introducing the Lagrangian $\mathcal{L}(x, y) = g(x) - f^*(y) + \langle Kx, y \rangle$ and using:

$$\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) = g(x^{k+1}) + \langle y, Kx^{k+1} \rangle - f^*(y) - g(x) - \langle y^{k+1}, Kx \rangle + f^*(y^{k+1})$$

we obtain for any $z = (x, y)^T$:

$$\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) + \frac{1}{2}|z^{k+1} - z^k|_{M_{\tau, \sigma}}^2 + \frac{1}{2}|z^{k+1} - z|_{M_{\tau, \sigma}}^2 \leq \frac{1}{2}|z^k - z|_{M_{\tau, \sigma}}^2.$$

so that, if $M_{\tau, \sigma} \geq 0$, for any $N \geq 1$,

$$\sum_{k=0}^{N-1} \mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) + \frac{1}{2}|z^N - z|_{M_{\tau, \sigma}}^2 \leq \frac{1}{2}|z^0 - z|_{M_{\tau, \sigma}}^2.$$

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By convexity, we obtain, denoting $Z^N = (X^N, Y^N)^T := \frac{1}{N} \sum_{k=1}^N z^k$:

$$\mathcal{L}(X^N, y) - \mathcal{L}(x, Y^N) \leq \frac{1}{2N} \|z^0 - z\|_{M_{\tau, \sigma}}^2.$$

If the domains of x, y are bounded we deduce:

$$\mathcal{P}(X^N) - \mathcal{D}(Y^N) \leq \frac{1}{N} \left(\frac{D_x^2}{\tau} + \frac{D_y^2}{\sigma} \right)$$

where D_\bullet are the diameters of the corresponding sets. (Similar to Nemirovsky, 2010, for an extragradient variant.)

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where D_\bullet are the diameters of the corresponding sets. (Similar to Nemirovsky, 2010, for an extragradient variant.)

Remark: we just used $\tau\sigma\|K\|^2 \leq 1$ (not $<$). If g, f^* provide additional information on the coerciveness of g, f^* it is enough (in finite dimension) to show convergence of the algorithm.

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- one can over-relax;
- one can add an “explicit” (co-coercive) term:

we obtain an extension due to L. Condat (in a generalized form to B.C. Vu, referred usually as Condat-Vu’s primal-dual algorithm). If h is a convex function with L_h -Lipschitz gradient one writes:

$$\begin{pmatrix} \frac{1}{\tau}I & -K^* \\ -K & \frac{1}{\sigma}I \end{pmatrix} \begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} - y^k \end{pmatrix} + \begin{pmatrix} \partial g(x^{k+1}) \\ \partial f^*(y^{k+1}) \end{pmatrix} + \begin{pmatrix} 0 & K^* \\ -K & 0 \end{pmatrix} \begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} \ni \begin{pmatrix} -\nabla h(x^k) \\ 0 \end{pmatrix}.$$

Then, this is exactly a forward-backward splitting for two operators and we know that it will converge provided, in the metric $M_{\tau,\sigma}$:

$$C = M_{\tau,\sigma}^{-1} \begin{pmatrix} -\nabla h(x) \\ 0 \end{pmatrix}$$

is μ -co-coercive for some $\mu > 1/2$.

Condat-Vu's variant

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That is, if for all z, z' :

$$\langle M_{\tau,\sigma}(z - z'), Cz - Cz' \rangle \geq \mu \|Cz - Cz'\|_{M_{\tau,\sigma}}^2.$$

Some algebra (see notes) show that μ can be estimated as $\mu \geq (1 - \sigma\tau\|K\|^2)/(\tau L_h)$ and one needs $\mu > 1/2$, hence:

$$\frac{1}{\sigma} \left(\frac{1}{\tau} - \frac{L_h}{2} \right) > \|K\|^2.$$

Condat-Vu's variant

In the end the method reads:

Input: initial pair of primal and dual points (x^0, y^0) , steps $\tau, \sigma > 0$.

for all $k \geq 0$ **do**

find (x^{k+1}, y^{k+1}) by solving

$$x^{k+1} = \text{prox}_{\tau g}(x^k - \tau(K^*y^k + \nabla h(x^k))) \quad (2)$$

$$y^{k+1} = \text{prox}_{\sigma f^*}(y^k + \sigma K(2x^{k+1} - x^k)). \quad (3)$$

end for

which will converge to a fixed point (if it exists) if $\tau < 2/L_h$ and $\sigma\|K\|^2 < 1/\tau - L_h/2$. [A rate can also be shown with a proof similar to the previous.]

Condat-Vu's variant

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end for

which will converge to a fixed point (if it exists) if $\tau < 2/L_h$ and $\sigma \|K\|^2 < 1/\tau - L_h/2$. [A rate can also be shown with a proof similar to the previous.]

(!) One should additionally check that a fixed point of these iterations solves:

$$\min_x f(Kx) + g(x) + h(x) = \min_x \sup_y \langle y, Kx \rangle - f^*(y) + g(x) + h(x).$$

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The previous method can be accelerated if g or f^* is strongly convex (and even further if **both** are strongly convex), similarly to the forward-backward splitting. We explain how it works, for instance if g is strongly convex. To make the computation a little bit easier we rather write the method as:

$$\begin{aligned}y^{k+1} &= (I + \sigma \partial f^*)^{-1}(y^k + \sigma K(x^k + \theta(x^k - x^{k-1}))) \\x^{k+1} &= (I + \tau \partial g)^{-1}(x^k - \tau K^* y^{k+1}).\end{aligned}$$

for some $\sigma, \tau > 0$, and some $\theta \in [0, 1]$ (we had $\theta = 1$ in the previous parts).

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Actually, the general form considers “old points” $(\bar{x}, \tilde{x}, \bar{y}, \tilde{y})$ and finds a “new point” (\hat{x}, \hat{y}) by solving:

$$\hat{y} = (I + \sigma \partial f^*)^{-1}(\bar{y} + \sigma K \tilde{x})$$

$$\hat{x} = (I + \tau \partial g)^{-1}(\bar{x} - \tau K^* \tilde{y}).$$

In particular, if g is μ_g -convex and/or f^* is μ_{f^*} -convex, then for all x, y , one has:

$$g(x) + \langle Kx, \tilde{y} \rangle + \frac{1}{2\tau} |x - \bar{x}|^2 \geq g(\hat{x}) + \langle K\hat{x}, \tilde{y} \rangle + \frac{1}{2\tau} |\hat{x} - \bar{x}|^2 + \frac{1 + \tau\mu_g}{2\tau} |x - \hat{x}|^2$$

$$f^*(y) - \langle K\tilde{x}, y \rangle + \frac{1}{2\sigma} |y - \bar{y}|^2 \geq f^*(\hat{y}) - \langle K\tilde{x}, \hat{y} \rangle + \frac{1}{2\sigma} |\hat{y} - \bar{y}|^2 + \frac{1 + \sigma\mu_{f^*}}{2\sigma} |y - \hat{y}|^2$$

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$$g(x) + \langle Kx, \tilde{y} \rangle + \frac{1}{2\tau} |x - \bar{x}|^2 \geq g(\hat{x}) + \langle K\hat{x}, \tilde{y} \rangle + \frac{1}{2\tau} |\hat{x} - \bar{x}|^2 + \frac{1 + \tau\mu_g}{2\tau} |x - \hat{x}|^2$$

$$f^*(y) - \langle K\tilde{x}, y \rangle + \frac{1}{2\sigma} |y - \bar{y}|^2 \geq f^*(\hat{y}) - \langle K\tilde{x}, \hat{y} \rangle + \frac{1}{2\sigma} |\hat{y} - \bar{y}|^2 + \frac{1 + \sigma\mu_{f^*}}{2\sigma} |y - \hat{y}|^2$$

as before we sum and see that:

$$\begin{aligned} \mathcal{L}(\hat{x}, y) - \mathcal{L}(x, \hat{y}) + \frac{1}{2\tau} |\hat{x} - \bar{x}|^2 + \frac{1 + \tau\mu_g}{2\tau} |x - \hat{x}|^2 + \frac{1}{2\sigma} |\hat{y} - \bar{y}|^2 + \frac{1 + \sigma\mu_{f^*}}{2\sigma} |y - \hat{y}|^2 \\ \leq \frac{1}{2\tau} |x - \bar{x}|^2 + \frac{1}{2\sigma} |y - \bar{y}|^2 \\ + \langle K\hat{x}, y \rangle - \langle Kx, \hat{y} \rangle + \langle K(x - \hat{x}), \tilde{y} \rangle - \langle K\tilde{x}, y - \hat{y} \rangle. \end{aligned}$$

Then, we add and remove $\langle K\hat{x}, \hat{y} \rangle$ to rewrite the last terms:

$$\langle K(x - \hat{x}), \tilde{y} - \hat{y} \rangle - \langle K(\tilde{x} - \hat{x}), y - \hat{y} \rangle.$$

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$$g(x) + \langle Kx, \tilde{y} \rangle + \frac{1}{2\tau} |x - \bar{x}|^2 \geq g(\hat{x}) + \langle K\hat{x}, \tilde{y} \rangle + \frac{1}{2\tau} |\hat{x} - \bar{x}|^2 + \frac{1 + \tau\mu_g}{2\tau} |x - \hat{x}|^2$$

$$f^*(y) - \langle K\tilde{x}, y \rangle + \frac{1}{2\sigma} |y - \bar{y}|^2 \geq f^*(\hat{y}) - \langle K\tilde{x}, \hat{y} \rangle + \frac{1}{2\sigma} |\hat{y} - \bar{y}|^2 + \frac{1 + \sigma\mu_{f^*}}{2\sigma} |y - \hat{y}|^2$$

as before we sum and see that:

$$\begin{aligned} \mathcal{L}(\hat{x}, y) - \mathcal{L}(x, \hat{y}) + \frac{1}{2\tau} |\hat{x} - \bar{x}|^2 + \frac{1 + \tau\mu_g}{2\tau} |x - \hat{x}|^2 + \frac{1}{2\sigma} |\hat{y} - \bar{y}|^2 + \frac{1 + \sigma\mu_{f^*}}{2\sigma} |y - \hat{y}|^2 \\ \leq \frac{1}{2\tau} |x - \bar{x}|^2 + \frac{1}{2\sigma} |y - \bar{y}|^2 \\ + \langle K\hat{x}, y \rangle - \langle Kx, \hat{y} \rangle + \langle K(x - \hat{x}), \tilde{y} \rangle - \langle K\tilde{x}, y - \hat{y} \rangle. \end{aligned}$$

Then, we add and remove $\langle K\hat{x}, \hat{y} \rangle$ to rewrite the last terms:

$$\langle K(x - \hat{x}), \tilde{y} - \hat{y} \rangle - \langle K(\tilde{x} - \hat{x}), y - \hat{y} \rangle.$$

→ the best would be to take $\tilde{x} = \hat{x}$ and $\tilde{y} = \hat{y}$ to get rid of these terms... (but then it is totally implicit).

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$$\begin{aligned}\mathcal{L}(\hat{x}, y) - \mathcal{L}(x, \hat{y}) + \frac{1}{2\tau} |\hat{x} - \bar{x}|^2 + \frac{1 + \tau\mu_g}{2\tau} |x - \hat{x}|^2 + \frac{1}{2\sigma} |\hat{y} - \bar{y}|^2 + \frac{1 + \sigma\mu_{f^*}}{2\sigma} |y - \hat{y}|^2 \\ \leq \frac{1}{2\tau} |x - \bar{x}|^2 + \frac{1}{2\sigma} |y - \bar{y}|^2 + \langle K(x - \hat{x}), \tilde{y} - \hat{y} \rangle - \langle K(\tilde{x} - \hat{x}), y - \hat{y} \rangle.\end{aligned}$$

reads in our case:

$$\begin{aligned}\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) + \frac{1}{2\tau} |x^{k+1} - x^k|^2 + \frac{1 + \tau\mu_g}{2\tau} |x - x^{k+1}|^2 + \frac{1}{2\sigma} |y^{k+1} - y^k|^2 + \frac{1 + \sigma\mu_{f^*}}{2\sigma} |y - y^{k+1}|^2 \\ \leq \frac{1}{2\tau} |x - x^k|^2 + \frac{1}{2\sigma} |y - y^k|^2 + \langle K(x - x^{k+1}), \tilde{y} - y^{k+1} \rangle - \langle K(\tilde{x} - x^{k+1}), y - y^{k+1} \rangle.\end{aligned}$$

and we can specialize in a semi-implicit form: $\tilde{y} = y^{k+1}$ and $\tilde{x} = x^k + \theta(x^k - x^{k-1})$ for some θ chosen later on, so that the last term becomes:

$$- \langle K(x^k + \theta(x^k - x^{k-1}) - x^{k+1}), y - y^{k+1} \rangle = \langle K(x^{k+1} - x^k), y - y^{k+1} \rangle - \theta \langle K(x^k - x^{k-1}), y - y^{k+1} \rangle$$

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We end up with:

$$\begin{aligned} \mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) &+ \frac{1}{2\tau} |x^{k+1} - x^k|^2 + \frac{1}{2\sigma} |y^{k+1} - y^k|^2 \\ &+ \frac{1 + \tau\mu g}{2\tau} |x - x^{k+1}|^2 + \frac{1 + \sigma\mu f^*}{2\sigma} |y - y^{k+1}|^2 - \langle K(x^{k+1} - x^k), y - y^{k+1} \rangle \\ &\leq \frac{1}{2\tau} |x - x^k|^2 + \frac{1}{2\sigma} |y - y^k|^2 - \theta \langle K(x^k - x^{k-1}), y - y^{k+1} \rangle \end{aligned}$$

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We end up with:

$$\begin{aligned} \mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) &+ \frac{1}{2\tau} |x^{k+1} - x^k|^2 + \frac{1}{2\sigma} |y^{k+1} - y^k|^2 \\ &+ \frac{1 + \tau\mu g}{2\tau} |x - x^{k+1}|^2 + \frac{1 + \sigma\mu f^*}{2\sigma} |y - y^{k+1}|^2 - \langle K(x^{k+1} - x^k), y - y^{k+1} \rangle \\ &+ \theta \langle K(x^k - x^{k-1}), y^k - y^{k+1} \rangle \leq \frac{1}{2\tau} |x - x^k|^2 + \frac{1}{2\sigma} |y - y^k|^2 - \theta \langle K(x^k - x^{k-1}), y - y^k \rangle \end{aligned}$$

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We end up with:

$$\begin{aligned} \mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) &+ \frac{1}{2\tau} |x^{k+1} - x^k|^2 + \frac{1}{2\sigma} |y^{k+1} - y^k|^2 \\ &+ \frac{1 + \tau\mu_g}{2\tau} |x - x^{k+1}|^2 + \frac{1 + \sigma\mu_{f^*}}{2\sigma} |y - y^{k+1}|^2 - \langle K(x^{k+1} - x^k), y - y^{k+1} \rangle \\ &+ \theta \langle K(x^k - x^{k-1}), y^k - y^{k+1} \rangle \leq \frac{1}{2\tau} |x - x^k|^2 + \frac{1}{2\sigma} |y - y^k|^2 - \theta \langle K(x^k - x^{k-1}), y - y^k \rangle \end{aligned}$$

Provided we can control the cross term $\langle K(x^k - x^{k-1}), y^k - y^{k+1} \rangle$ with the terms

$\frac{1}{2\tau} |x^{k+1} - x^k|^2 + \frac{1}{2\sigma} |y^{k+1} - y^k|^2$ we can hope to obtain a rate of convergence, even linear if $\mu_g > 0$ and $\mu_{f^*} > 0$. Let us consider the more difficult case $\mu_g > 0, \mu_{f^*} = 0$.

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In this case, we assume θ , σ , τ are varying and depend on k and we write:

$$\begin{aligned} \mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) &+ \frac{1}{2\tau_k} |x^{k+1} - x^k|^2 + \frac{1}{2\sigma_k} |y^{k+1} - y^k|^2 + \theta_k \langle K(x^k - x^{k-1}), y^k - y^{k+1} \rangle \\ &+ \frac{1 + \tau_k \mu g}{2\tau_k} |x - x^{k+1}|^2 + \frac{1}{2\sigma_k} |y - y^{k+1}|^2 - \langle K(x^{k+1} - x^k), y - y^{k+1} \rangle \\ &\leq \frac{1}{2\tau_k} |x - x^k|^2 + \frac{1}{2\sigma_k} |y - y^k|^2 - \theta_k \langle K(x^k - x^{k-1}), y - y^k \rangle \end{aligned}$$

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In this case, we assume θ, σ, τ are varying and depend on k and we write:

$$\begin{aligned} \mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) &+ \frac{1}{2\tau_k} |x^{k+1} - x^k|^2 + \frac{1}{2\sigma_k} |y^{k+1} - y^k|^2 + \theta_k \langle K(x^k - x^{k-1}), y^k - y^{k+1} \rangle \\ &+ \frac{\tau_{k+1}(1 + \tau_k \mu g)}{\tau_k} \frac{1}{2\tau_{k+1}} |x - x^{k+1}|^2 + \frac{\sigma_{k+1}}{\sigma_k} \frac{1}{2\sigma_{k+1}} |y - y^{k+1}|^2 - \frac{\theta_{k+1}}{\theta_{k+1}} \langle K(x^{k+1} - x^k), y - y^{k+1} \rangle \\ &\leq \frac{1}{2\tau_k} |x - x^k|^2 + \frac{1}{2\sigma_k} |y - y^k|^2 - \theta_k \langle K(x^k - x^{k-1}), y - y^k \rangle \end{aligned}$$

so that if we can choose

$$\frac{\tau_{k+1}(1 + \tau_k \mu g)}{\tau_k} = \frac{\sigma_{k+1}}{\sigma_k} = \frac{1}{\theta_{k+1}} > 1$$

and let $A_k := \frac{1}{2\tau_k} |x - x^k|^2 + \frac{1}{2\sigma_k} |y - y^k|^2 - \theta_k \langle K(x^k - x^{k-1}), y - y^k \rangle$ it reads:

$$\begin{aligned} \mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) &+ \frac{1}{2\tau_k} |x^{k+1} - x^k|^2 + \frac{1}{2\sigma_k} |y^{k+1} - y^k|^2 + \theta_k \langle K(x^k - x^{k-1}), y^k - y^{k+1} \rangle \\ &+ \frac{\sigma_{k+1}}{\sigma_k} A_{k+1} \leq A_k \end{aligned}$$

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$$\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) + \frac{1}{2\tau_k} |x^{k+1} - x^k|^2 + \frac{1}{2\sigma_k} |y^{k+1} - y^k|^2 + \theta_k \langle K(x^k - x^{k-1}), y^k - y^{k+1} \rangle + \frac{\sigma_{k+1}}{\sigma_k} A_{k+1} \leq A_k$$

Then, we use that (denoting to simplify $L := \|K\|$):

$$-\theta_k \langle K(x^k - x^{k-1}), y^k - y^{k+1} \rangle \leq \frac{\theta_k^2 L^2 \sigma_k}{2} |x^k - x^{k-1}|^2 + \frac{1}{2\sigma_k} |y^k - y^{k+1}|^2$$

to arrive at

$$\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}) + \frac{1}{2\tau_k} |x^{k+1} - x^k|^2 - \frac{\theta_k^2 L^2 \sigma_k}{2} |x^k - x^{k-1}|^2 + \frac{\sigma_{k+1}}{\sigma_k} A_{k+1} \leq A_k$$

or (after multiplication with σ_k):

$$\sigma_k (\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1})) + \frac{\sigma_k}{2\tau_k} |x^{k+1} - x^k|^2 - \frac{\theta_k^2 L^2 \sigma_k^2}{2} |x^k - x^{k-1}|^2 + \sigma_{k+1} A_{k+1} \leq \sigma_k A_k.$$

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We now sum from $k = 0$ to $N - 1$ the inequality (we use also $\theta_k \sigma_k = \sigma_{k-1}$):

$$\sigma_k(\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1})) + \frac{\sigma_k}{2\tau_k} |x^{k+1} - x^k|^2 - \frac{L^2 \sigma_{k-1}^2}{2} |x^k - x^{k-1}|^2 + \sigma_{k+1} A_{k+1} \leq \sigma_k A_k.$$

We let $T_N = \sum_{k=0}^{N-1} \sigma_k$, $X^N = \frac{1}{T_N} \sum_{k=0}^{N-1} \sigma_k x^{k+1}$, $Y^N = \frac{1}{T_N} \sum_{k=0}^{N-1} \sigma_k y^{k+1}$, so that:

$$T_N(\mathcal{L}(X^N, y) - \mathcal{L}(x, Y^N)) \leq \sum_{k=0}^{N-1} \sigma_k(\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1}))$$

thanks to the convexity of $\mathcal{L}(\cdot, y) - \mathcal{L}(x, \cdot)$ Then we get:

$$T_N(\mathcal{L}(X^N, y) - \mathcal{L}(x, Y^N)) + \sum_{k=1}^N \frac{\sigma_{k-1}}{2\tau_{k-1}} |x^k - x^{k-1}|^2 - \sum_{k=0}^{N-1} \frac{L^2 \sigma_{k-1}^2}{2} |x^k - x^{k-1}|^2 + \sigma_N A_N \leq \sigma_0 A_0$$

or, choosing $x^{-1} = x^0$:

$$T_N(\mathcal{L}(X^N, y) - \mathcal{L}(x, Y^N)) + \frac{\sigma_{N-1}}{2\tau_{N-1}} |x^N - x^{N-1}|^2 + \sum_{k=1}^{N-1} \left(\frac{\sigma_{k-1}}{\tau_{k-1}} (1 - L^2 \tau_{k-1} \sigma_{k-1}) \right) \frac{|x^k - x^{k-1}|^2}{2} + \sigma_N A_N \leq \sigma_0 A_0$$

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Hence: we choose in addition $L^2\sigma_k\tau_k \leq 1$ (or = in practice) and end up with:

$$T_N(\mathcal{L}(X^N, y) - \mathcal{L}(x, Y^N)) + \frac{\sigma_{N-1}}{2\tau_{N-1}}|x^N - x^{N-1}|^2 + \frac{\sigma_N}{2\tau_N}|x^N - x|^2 + \frac{1}{2}|y^N - y|^2 \\ - \sigma_N\theta_N \langle K(x^N - x^{N-1}), y - y^N \rangle \leq \frac{\sigma_0}{2\tau_0}|x^0 - x|^2 + \frac{1}{2}|y^0 - y|^2$$

(using again $x^{-1} = x^0$ and recalling $A_k := \frac{1}{2\tau_k}|x - x^k|^2 + \frac{1}{2\sigma_k}|y - y^k|^2 - \theta_k \langle K(x^k - x^{k-1}), y - y^k \rangle$). We end up estimating again:

$$\sigma_N\theta_N \langle K(x^N - x^{N-1}), y - y^N \rangle \leq \frac{L^2\sigma_{N-1}^2}{2}|x^N - x^{N-1}|^2 + \frac{1}{2}|y - y^N|^2$$

and using $L^2\sigma_{N-1}^2 \leq \sigma_{N-1}/\tau_{N-1}$ to find:

$$T_N(\mathcal{L}(X^N, y) - \mathcal{L}(x, Y^N)) + \frac{\sigma_N}{2\tau_N}|x^N - x|^2 \leq \frac{\sigma_0}{2\tau_0}|x^0 - x|^2 + \frac{1}{2}|y^0 - y|^2$$

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Now we specify the parameters... In order to keep $L^2\sigma_k\tau_k = 1$ (to simplify) we keep $\sigma_k\tau_k = \sigma_0\tau_0 = 1/L^2$. Then, we should have:

$$\frac{\tau_{k+1}(1 + \tau_k\mu_g)}{\tau_k} = \frac{\sigma_{k+1}}{\sigma_k} = \frac{1}{\theta_{k+1}} > 1$$

and in particular, $\sigma_{k+1} = \sigma_k/\theta_{k+1}$ and $\tau_{k+1} = \theta_{k+1}\tau_k$, so that:

$$\theta_{k+1} = \frac{1}{\sqrt{1 + \mu_g\tau_k}}, \quad \tau_{k+1} = \frac{\tau_k}{\sqrt{1 + \mu_g\tau_k}}, \quad \sigma_{k+1} = \sigma_k\sqrt{1 + \mu_g\tau_k}.$$

In particular $\sigma_{k+1}^2 = \sigma_k^2 + \kappa\sigma_k$ where $\kappa = \mu_g/L^2$ is an (inverse) condition number.

One can then show that: if τ_0 is large, then after very few iterations, $\tau_k \leq 1$: we use

$$\mu_g\tau_{k+1} = \frac{\mu_g\tau_k}{\sqrt{1 + \mu_g\tau_k}} \leq \sqrt{\mu_g\tau_k} \Rightarrow \dots$$

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$$\mu_g \tau_{k+1} \leq \sqrt{\mu_g \tau_k} \Rightarrow \log \mu_g \tau_k \leq \frac{1}{2^k} \log \mu_g \tau_0$$

so that, for instance, $\mu_g \tau_k \leq 2$ as soon as $k \geq \log_2 \log_2(\mu_g \tau_0)$ (or $k = 0$), which is always very small. Then, for larger k s, one has $\sigma_k = 1/(L^2 \tau_k) \geq \kappa/2$ and:

$$\sigma_{k+1}^2 = \sigma_k^2 + \kappa \sigma_k \geq \sigma_k^2 + \alpha \kappa \sigma_k + (1 - \alpha) \frac{\kappa^2}{2} = \left(\sigma_k + \frac{\alpha}{2} \kappa\right)^2 + \left((1 - \alpha) - \frac{\alpha^2}{2}\right) \frac{\kappa^2}{2} = \left(\sigma_k + \frac{\alpha}{2} \kappa\right)^2$$

if we choose $\alpha = \sqrt{3} - 1 \approx 0.73$. Then it follows $\sigma_k \gtrsim (.73\kappa/2)k$ and in particular, $T_N \gtrsim (.73\kappa/4)N(N-1)$ (in fact, one can show $\sigma_k \sim (\kappa/2)k$ and $T_N \sim \kappa N^2/4$).

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Getting back to:

$$T_N(\mathcal{L}(X^N, y) - \mathcal{L}(x, Y^N)) + \frac{\sigma_N}{2\tau_N} |x^N - x|^2 \leq \frac{\sigma_0}{2\tau_0} |x^0 - x|^2 + \frac{1}{2} |y^0 - y|^2$$

we see that, taking $(x, y) = (x^*, y^*)$ (for which one can show: $\mathcal{L}(X^N, y) - \mathcal{L}(x, Y^N) \geq \mu_g |X^N - x^*|^2$) one has

$$|x^N - x^*|^2 + |X^N - x^*|^2 \lesssim \frac{CL^2}{\mu_g^2 N^2}$$

and

$$\mathcal{L}(X^N, y) - \mathcal{L}(x, Y^N) = O(\kappa^{-1} N^{-2})(|x - x^0|^2 + |y - y^0|^2)$$

(for $\sigma_0 < \tau_0$).

It shows an improvement over the non-accelerated method provided $N \gtrsim 1/\kappa$.

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A similar (easier) proof shows an accelerated rate in case both functions are strongly convex.