Communication trade-offs for synchronized distributed SGD with large step size

Aymeric DIEULEVEUT

EPLF, MLO

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Joint work with Kumar Kshitij Patel.

Outline

1. **Stochastic gradient descent - supervised machine learning - setting, assumptions and proof techniques**
2. **Synchronized distributed SGD - from mini-batch averaging to model averaging**
3. **Optimality of Local-SGD.**

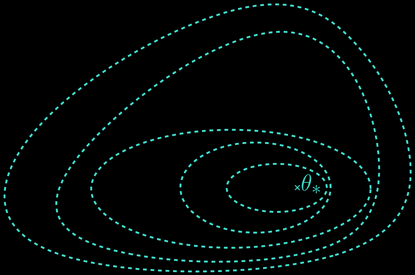
Stochastic Gradient Descent

- **Goal:**

$$\min_{\theta \in \mathbb{R}^d} F(\theta)$$

  given unbiased gradient estimates g_n

- $\theta^\star := \operatorname{argmin}_{\mathbb{R}^d} F(\theta)$.



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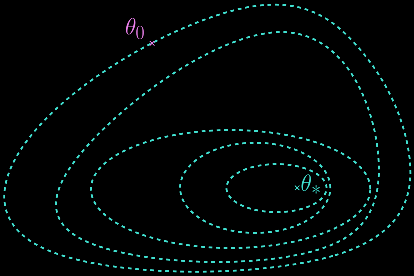

- Key algorithm: Stochastic Gradient Descent (SGD) (Robbins and Monro, 1951):

$$\boxed{\theta_k = \theta_{k-1} - \eta_k \, g_k(\theta_{k-1})}$$

- $\mathbb{E}[g_k(\theta_{k-1}) | \mathcal{F}_{k-1}] = F'(\theta_{k-1})$ for a filtration $(\mathcal{F}_k)_{k \geq 0}$, θ_k is $\mathcal{F}_k$ measurable.

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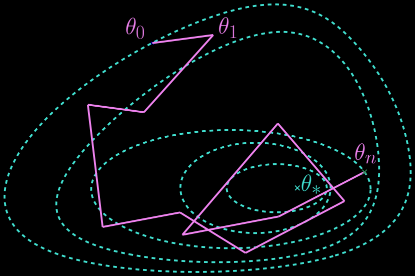

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Supervised Machine Learning

- We define the risk (generalization error) as

$$\mathcal{R}(\theta) := \mathbb{E}_{\rho}\left[\ell(Y, \langle \theta, \Phi(X) \rangle)\right].$$

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- For example, least-squares regression:

$$\min_{\theta\in\mathbb{R}^d} \quad \frac{1}{2n}\sum_{i=1}^{n} \left(y_i - \langle\theta, \Phi(x_i)\rangle\right)^2 \quad + \quad \mu\Omega(\theta),$$

- and logistic regression:

$$\min_{\theta\in\mathbb{R}^d} \quad \frac{1}{n}\sum_{i=1}^{n} \log\left(1 + \exp(-y_i\langle\theta, \Phi(x_i)\rangle)\right) \quad + \quad \mu\Omega(\theta).$$

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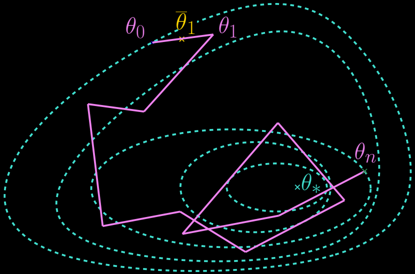

Introduced by Polyak and Juditsky (1992) and Ruppert (1988):

$$\bar{\theta}_n = \frac{1}{n+1} \sum_{k=0}^{n} \theta_k .$$

- **off line averaging reduces the noise effect.**

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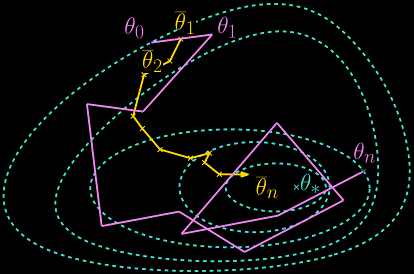

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- **off line averaging reduces the noise effect.**
- **on line computing:** $\bar{\theta}_{n+1} = \frac{1}{n+1}\theta_{n+1} + \frac{n}{n+1}\bar{\theta}_n$.

Assumptions

Goal: $\boxed{\min_{\theta} F(\theta)}$. **Recursion:** $\boxed{\theta_k = \theta_{k-1} - \eta_k\, g_k(\theta_{k-1})}$

A1 [Strong convexity] The function F is strongly-convex with convexity constant $\mu > 0$.

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A1 [Strong convexity] The function F is strongly-convex with convexity constant $\mu > 0$.

A2 [Smoothness and regularity] The function F is three times continuously differentiable with second and third uniformly bounded derivatives: $\sup_{\theta \in \mathbb{R}^d} \left|\left|\left| F^{(2)}(\theta) \right|\right|\right| < L$, and $\sup_{\theta \in \mathbb{R}^d} \left|\left|\left| F^{(3)}(\theta) \right|\right|\right| < M$. Especially F is L-smooth.

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Or:

Q1 [Quadratic function] There exists a positive definite matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{d \times d}$, such that the function F is the quadratic function $\theta \mapsto \|\boldsymbol{\Sigma}^{1/2}(\theta - \theta^\star)\|^2/2$,

Which step size would you use?

Smooth functions.

	$\eta_k \equiv \eta_0$	$\eta_k = 1/\sqrt{k}$	$\eta_k = 1/(\mu k)$
Convex			
Strongly Convex			
Quadratic			

Classical bound: Lyapunov approach

$$
\begin{aligned}
\mathbb{E}\left[||\theta_{k+1} - \theta^\star||^2 | \mathcal{F}_k\right] &\leq \mathbb{E}\left[||\theta_k - \theta^\star||^2\right] - 2\eta_k \left\langle F'(\theta_k), \theta_k - \theta^\star \right\rangle \\
&\quad + \eta_k^2 ||g_k(\theta_k)||^2 \\
&\leq \mathbb{E}\left[||\theta_k - \theta^\star||^2\right] - 2\eta_k(1 - \eta_k L) \left\langle F'(\theta_k), \theta_k - \theta^\star \right\rangle \\
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Conclusion: with $\eta_k = \frac{1}{\mu k}$, telescopic sum + Jensen:

$$
\mathbb{E}\left[F(\bar{\theta}_k) - F(\theta^\star)\right] \leq O(1/\mu k).
$$

Trivial case: decaying step sizes are not that great !

Consider least squares: $y_i = {\theta^\star}^\top x_i + \varepsilon_i$, $\varepsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$.

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Start with $\theta_0 = \theta^\star$:

Then:

$$\bar{\theta}_k - \theta^\star = \frac{1}{k} \sum_{i=1}^{k} M_i^k \eta_i^2 \varepsilon_i.$$

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Tight control is much easier on the stochastic process $\theta_k - \theta^\star$ than through the "Lyapunov approach".

Other proof: introduce decomposition

Original proof of averaging in Polyak and Juditsky (1992).

$$\begin{aligned}\eta_k F''(\theta^\star)(\theta_{k-1}-\theta^\star) = {} & \theta_{k-1}-\theta_k \\ & -\eta_k\left[g_k(\theta_{k-1}) - F'(\theta_{k-1})\right] \\ & +\eta_k\left[F'(\theta_{k-1}) - F''(\theta^\star)(\theta_{k-1}-\theta^\star)\right].\end{aligned}$$

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Thus, for $\eta_k \equiv \eta$

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Correct control of the noise for smooth and strongly convex

All step sizes $\eta_n = Cn^{-\alpha}$ with $\alpha \in (1/2, 1)$ lead to $O(n^{-1})$.

LMS algorithm: constant step-size $\to$ statistical optimality.

Problem: dependence in μ

Possible to recover convergence in function values:

$$F(\bar{\theta}_K) - F(\theta^\star) \leq \frac{L}{2}||\theta_K - \theta^\star||^2 \leq \frac{L}{2\mu^2}||F''(\theta^\star)\left(\bar{\theta}_K - \theta^\star\right)||^2$$

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However:

- Ok for least squares regression (with some more work (Défossez and Bach, 2015; Dieuleveut et al., 2016; Jain et al., 2016))
- Possible to recover tight convergence with self concordance (Bach 2013).

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For any $p \in [P], t \in [C], k \in [N^t]$:

- $\theta^t_{p,k}$ the model proposed by worker p, at phase t, after k local iterations.
- $\theta^1_{p,0} = \theta_0$.
- $$\boxed{\theta^t_{p,k} = \theta^t_{p,k-1} - \eta^t_k g^t_{p,k}(\theta^t_{p,k-1}).}$$

Link with classical algorithms.

Algo.	Work.	Com.	Phases	T
Local	P	C	$(N^1 \ldots N^C)$	$P\sum_{t=1}^{C} N^t$
Serial	1	-	(N)	N
P-MBA	P	C	$(1, \ldots, 1)$	PC
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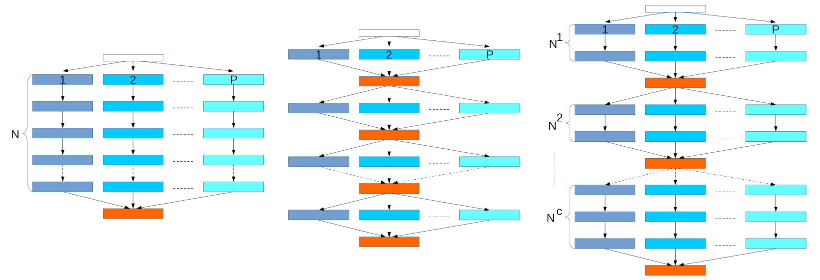

One Shot Averaging – Mini-Batch Averaging –Local SGD

Aggregation steps: $\hat{\theta}^t = \frac{1}{P}\sum_{p=1}^{P} \theta_{p,N^t}^t$.

At phase $t+1$, every worker $p \in [P]$ restarts from the averaged model: $\theta_{p,0}^{t+1} := \hat{\theta}^t$.

Goal: Risk of the Polyak-Ruppert averaged iterate:

$$\overline{\overline{\theta}}^C = \frac{1}{P\sum_{t=1}^{C} N^t} \sum_{t=1}^{C}\sum_{p=1}^{P}\sum_{k=1}^{N^t} \theta_{p,k}^t,$$

Assumptions

A3 [Oracle on the gradient] Filtration $(\mathcal{H}_k^t)_{(t,k)\in[C]\times[N^t]}$ such that for any $(t,k)\in[C]\times[N^t]$ and $\theta\in\mathbb{R}^d$, $g_{p,k+1}^t(\theta)$ is a $\mathcal{H}_{k+1}^t$-measurable random variable and $\mathbb{E}\left[g_{p,k+1}^t(\theta)|\mathcal{H}_k^t\right] = F'(\theta)$.

A4 [Uniformly bounded variance] $\mathbb{E}[\|g_{p,k}^t(\theta_{p,k}^t) - F'(\theta_{p,k}^t)\|^2] \leq \sigma_\infty^2$.

A5 [Cocoercivity of the random gradients] For any $t\in[C]$, $k\in[N^t]$, $p\in[P]$, $g_{p,k}^t$ is almost surely L-co-coercive

A6 [Finite variance at the optimal point] There exists $\sigma\geq 0$, such that for any $t\in[C], k\in[N^t], p\in[P]$, $\mathbb{E}[\|g_{p,k}^t(\theta^\star)\|^4]\leq\sigma^4$.

We assume A4 OR A5 + A6

Error decomposition

$$\begin{aligned}\eta_k^t F''(\theta^\star)(\theta_{p,k-1}^t - \theta^\star) = \theta_{p,k-1}^t - \theta_{p,k}^t \\ - \eta_k^t \left[g_{p,k}^t(\theta_{p,k-1}^t) - F'(\theta_{p,k-1}^t) \right] \\ + \eta_k^t \left[F'(\theta_{p,k-1}^t) - F''(\theta^\star)(\theta_{p,k-1}^t - \theta^\star) \right].\end{aligned}$$

Thus:

$$\begin{aligned}F''(\theta^\star)\left(\overline{\overline{\theta}}^C - \theta^\star\right) = \frac{1}{P\sum_{t=1}^C N^t} \sum_{t=1}^C \sum_{p=1}^P \sum_{k=1}^{N^t} \Bigg(\frac{\theta_{p,k-1}^t - \theta_{p,k}^t}{\eta_k^t} \\ - \left[g_{p,k}^t(\theta_{p,k-1}^t) - F'(\theta_{p,k-1}^t) \right] \\ + \left[F'(\theta_{p,k-1}^t) - F''(\theta^\star)(\theta_{p,k-1}^t - \theta^\star) \right] \Bigg).\end{aligned}$$

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Noise: **Additive + (Multiplicative** $\propto ||\theta_{p,k}^t - \theta^\star||^2$)
Residual: $\propto ||\theta_{p,k}^t - \theta^\star||^2$

Results MBA - OSA

Assume A1,2,3,5,6, and $\eta_k^t \equiv \eta$ for any $(t, k) \in [C] \times [N^t]$.

Proposition (Mini-batch Averaging)

For any $t \in [C]$,

$$\mathbb{E}\left[\left\|\hat{\theta}^t - \theta^\star\right\|^2\right] \leq (1 - \eta\mu)^t \left\|\theta_0 - \theta^\star\right\|^2 + \frac{2\sigma^2\eta}{P}\frac{1 - (1 - \eta\mu)^t}{\mu},$$

$$\mathbb{E}\left[\left\|\overline{\overline{\theta}}^C - \theta^\star\right\|^2_{F''(\theta^\star)}\right] \precsim \frac{\left\|\theta^0 - \theta^\star\right\|^2}{\eta^2 C^2} Q_{bias} + \frac{\sigma^2}{T}\left(1 + \frac{Q_{1,var}(C)}{P} + \frac{Q_{2,var}(C)}{P^2}\right).$$

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Proposition (One-shot Averaging)

For any $p \in [P]$, $t = 1$, $k \in [N]$,

$$\mathbb{E}\left[\left\|\theta^1_{p,k} - \theta^\star\right\|^2\right] \leq (1-\eta\mu)^k \left\|\theta_0 - \theta^\star\right\|^2 + 2\sigma^2\eta\frac{1-(1-\eta\mu)^k}{\mu},$$

$$\mathbb{E}\left[\left\|(\overline{\overline{\theta}}^C - \theta^\star)\right\|^2_{F''(\theta^\star)}\right] \precsim \frac{\left\|\theta^0 - \theta^\star\right\|^2}{\eta^2 N^2} Q_{bias} + \frac{\sigma^2}{T}\left(1 + Q_{1,var}(N) + Q_{2,var}(N)\right)$$

With

$$Q_{bias} = 1 + \frac{M^2\eta}{\mu}\left\|\theta^0 - \theta^\star\right\|^2 + \frac{L^2\eta}{\mu P},$$

$$Q_{1,var}(X) = \frac{L^2\eta}{\mu} + \frac{P}{X\eta\mu}, \quad Q_{2,var}(X) = \frac{M^2 X P \eta^2 \sigma^2}{\mu^2}.$$

With

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- Asymptotically equivalent for P constant.
- Non asymptotic result (vs Godichon and Saadane (2017))
- Proposition 1 corrects Bach 2011, with Needel 2014 remark (see also Dieuleveut Durmus 2017).
- "the noise is the noise and SGD doesn't care" (for asynchronous SGD, (Duchi et al., 2015))
- Extension to the on-line setting possible

Bridging the gap: convergence of Local-SGD: simple case

Assume Q1, A3, A4. For $p \in [P], t \in [C], k \in [N^t]$,

$$\mathbb{E}\left[\left\|\hat{\theta}^{t-1} - \theta^\star\right\|^2\right] \leq (1-\eta\mu)^{N_1^{t-1}} \|\theta_0 - \theta^\star\|^2 + \frac{\sigma_\infty^2 \eta}{P} \frac{1-(1-\eta\mu)^{N_1^{t-1}}}{\mu}$$

$$\mathbb{E}\left[\left\|\theta_{p,k}^t - \theta^\star\right\|^2\right] \leq (1-\eta\mu)^{N_1^{t-1}+k} \|\theta_0 - \theta^\star\|^2$$
$$+ \sigma_\infty^2 \eta \left(\underbrace{\frac{1-(1-\eta\mu)^{N_1^{t-1}}}{P\mu}}_{\text{long term reduced variance}} + \underbrace{\frac{1-(1-\eta\mu)^k}{\mu}}_{\text{local iteration variance}} \right).$$

Corollary: If for all $t \in [C]$, $N^t \leq \frac{1}{\mu\eta P}$, then the second order moment of $\theta_{p,k}^t$ admits the same upper bound as the mini-batch iterate $\hat{\theta}_{MB}^{N_1^{t-1}+k}$ up to a factor of 2. As a consequence, Local-SGD performs optimally.

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Example

With constant number of local steps $N^t = N$, and learning rate $\eta = \frac{c}{\sqrt{NC}}$ in order to obtain an optimal $O(\frac{\sigma^2}{T})$ parallel convergence rate, local-SGD can communicate $O(\frac{\sqrt{NC}}{P\mu})$ times less as compared to mini-batch averaging.

Quadratic + additive noise $\leftrightarrow$ too simple and un-realistic

- Least square regression: quadratic + multiplicative noise (Q1, A3, A5, A6)
- Logistic regression: non quadratic + uniformly bounded variance (A1, A2, A3, A4)

Key lemmas: control how the restart point of each phase differs from its mini-batch equivalent.

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Theorem

Under either of the following sets of assumptions, the convergence of the Polyak Ruppert iterate $\overline{\overline{\theta}}^{C}$ is as good as in the mini-batch case, up to a constant:

1. Assume Q1, A3, A5, A6, and for any $t \in [C]$, $N^t \leq \frac{1}{\mu\eta P}$ and $\mu\eta^2 N_1^t = O(1)$.

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1. Assume Q1, A3, A5, A6, and for any $t \in [C]$, $N^t \leq \frac{1}{\mu\eta P}$ and $\mu\eta^2 N_1^t = O(1)$.
2. Assume A1, A2, A3, A4, and for any $t \in [C]$, $N^t \leq \inf\left(\frac{1}{\eta P M \mathbb{E}\left[\left\|\hat{\theta}^t - \theta^\star\right\|\right]}, \frac{1}{\mu\eta P}\right)$.

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- Non asymptotic analysis of Local-SGD
- With “large” step sizes.
- better understanding of communication trade-offs $\rightarrow$ lower bounds on communication frequency
- Similar results for the on-line case (a bit faster, and much more painful for the eyes).

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Directions:

- Improve to optimal rates in terms of μ with self concordance
- Proving that those bounds are tight (dangerous to compare upper bounds!!)

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