Practical Session

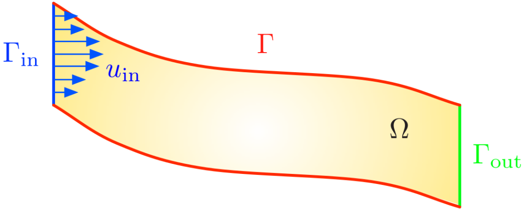
- Navier-Stokes Equations

Fluids

- potential flow: velocity comes from a gradient
  - advantage: simplicity - Laplace equation
  - disadvantage: not too accurate (no lift, no drag, no vorticity possible)
- Stokes equations: more accurate modeling
  - advantage: linear system
  - disadvantage: it is only good in the viscous case
  - it usually shows laminar flows
- Navier-Stokes equations: even more accurate modeling
  - advantage: it gives more realistic results
  - disadvantage: the system is non-linear and its resolution is more costly
  - the parameters are included in the Reynolds number
  - high Reynolds: turbulent regime giving numerical difficulties

Navier-Stokes Equations

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)u + \nabla p & = & 0 & \text{in } \Omega \\ \mathrm{div}(\mathbf{u}) & = & 0 & \text{in } \Omega \\ \mathbf{u} & = & \mathbf{u}_{in} & \text{on } \Gamma_{in} \\ \mathbf{u} & = & 0 & \text{on } \Gamma \\ \sigma(\mathbf{u},p)\mathbf{n} & = & 0 & \text{on } \Gamma_{out} \end{cases}$$



stress tensor: $\sigma(\mathbf{u},p) = 2\nu e(\mathbf{u}) - p\,\mathrm{Id}$ where $e(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$

Difficulty

- using the stress tensor and the incompressibility, the first equation becomes

$$-\operatorname{div}(\sigma(\mathbf{u}, p)) + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0$$

- the complicated non-linear term is $(\mathbf{u} \cdot \nabla)\mathbf{u}$:

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \left(u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2}, u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} \right)$$

- use a fixed point algorithm: linearize the system and solve it multiple times
- All information is contained in the Reynolds number $Re = 1/\nu$. Unfortunately, interesting situations come when $Re \gg 1$, the case which is difficult from a numerical point of view.

Algorithm

1. write a non-linear variational formulation for the system in the form

$$a(u,v)+b(p,v)+b(u,q)+c(u,u,v)=\ell(v)$$

2. the solution (u,p) to the NS system could be achieved using the following:
   - (u^0,p^0) is the solution to the Stokes system (ignoring the non-linear term above)
   - for $n=0,\dots$ till convergence

     $$(u^{n+1},p^{n+1})=(u^n,p^n)+(\delta u^n,\delta p^n)$$

     where the increment $(\delta u^n,\delta p^n)$ is obtained by solving the following linearized variational formulation

     $$a(\delta u^n,v)+b(\delta p^n,v)+b(\delta u^n,q)+c(\delta u^n,u^n,v)+c(u^n,\delta u^n,v)=\ell(v).$$

This algorithm is implemented in `FreeFem++` and it works for moderately large Reynolds number ($\nu\geq 0.005,\ Re\leq 200$)

Reynolds number

The Reynolds number is defined as[3]

$$\mathrm{Re} = \frac{\rho u L}{\mu} = \frac{u L}{\nu}$$

where:

- ρ is the density of the fluid (SI units: kg/m^3)
- u is the velocity of the fluid with respect to the object (m/s)
- L is a characteristic linear dimension (m)
- μ is the dynamic viscosity of the fluid (Pa·s or N·s/m^2 or kg/m·s)
- ν is the kinematic viscosity of the fluid (m^2/s).

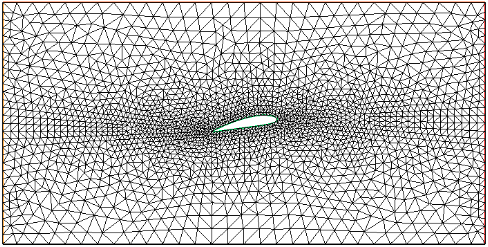
Online tools: Compute Reynolds Number

Conclusion: The `FreeFem++` code is valid for the air at $20^\circ C$, a speed of about 40 km/h and an object of diameter of roughly 30cm

Not bad... In the `FreeFem++` manual a more complex example using a $k-\varepsilon$ model is shown. Search for: *A Large Fluid Problem*

Usage

- For simplicity a method to construct the mesh using a photo is given in the `FreeFem++` code with the possibility to tilt the image with a given angle



- The image should contain one black shape in pgm format
- Use linux command `convert` or some online tools to convert images to this format
- Multiple components are also possible, but needs more work
- See the `FreeFem++` manual for details concerning the mesh generation

Compute the drag and lift

For an object Ω immersed in a fluid moving with speed having direction $\mathbf{u}_\infty$ at infinity the drag is the scalar product $F \cdot \mathbf{u}_\infty$ where F is the force the fluid exerts on the object Ω given by

$$F = \int_{\partial\Omega} \sigma(\mathbf{u}, p) n$$

The lift is equal to the component of F orthogonal to $\mathbf{u}_\infty$.

Recall that the stress tensor is given by

$$\sigma(\mathbf{u}, p) = 2\nu e(\mathbf{u}) - p\,\mathrm{Id}$$

where

$$e(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$$

is the symmetrized gradient

Shape optimization for fluids

References:

1. [Mohammadi, Pironneau, *Applied Shape Optimization for Fluids*]

this goes in depth to show the methods and challenges in this domain. A special emphasis is made on aeronautics applications.

2. [Dapogny, Frey, Omnés, Privat, *Geometrical shape optimization in fluid mechanics using `FreeFem++`*]

this paper is good for introductory purposes. It goes straight to the point showing the Navier-Stokes system and shape derivatives for some given functionals. The `FreeFem++` codes used in the paper can be found on GitHub.

Some ideas

- we saw previously that finding shape derivatives involves the construction of a Lagrangian
- often the shape derivatives involve the use of the adjoint state
- positive aspect (for encouragement): even if the state equation is non-linear, the adjoint state will be linear
- in complex applications (in aeronautics) in order to avoid possible instability issues, the derivatives are computed directly using automatic differentiation (coding challenge...)

Practical session

Investigate the design of the CATIA object for Uber Elevate

Questions:

1. What is fixed and what is optimizable in your design?
2. Produce 2D slices of your design along the vertical and displacement directions as filled contours and convert the images to the pgm format
3. Use `FreeFem++` to simulate the 2D flow around the slices. Investigate the drag with respect to various angles of attack. Validate your results by changing the mesh parameters and the size of the computational domain.
4. Make various modifications of your design and observe the changes in drag and lift forces with respect to different attack angles.