OPTIMAL SHAPES MAXIMIZING THE STEKLOV EIGENVALUES

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Abstract. In this paper we consider the problem of maximizing the k-th Steklov eigenvalue of the Laplacian (or a more general spectral functional), among all sets of $\mathbb{R}^d$ of prescribed volume. We prove existence of an optimal set and get some qualitative properties of the solutions in a relaxed setting. In particular, in $\mathbb{R}^2$, we prove that the optimal set consists in the union of at most k disjoint Jordan domains with finite perimeter. A key point of our analysis is played by an isodiametric control of the Stelkov spectrum. We also perform some numerical experiments and exhibit the optimal shapes maximizing the k-th eigenvalues under area constraint in $\mathbb{R}^2$, for $k = 1, \dots, 10$.

Keywords: Steklov eigenvalues, shape optimization, existence of solutions, numerical simulations

Contents

1. Introduction

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz set. A number $\sigma \in \mathbb{R}^+$ is an eigenvalue of the Steklov problem for the Laplace operator provided there exists a non-zero function $u \in H^1(\Omega)$ which satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Omega, \end{cases}$$

in the weak sense

$$\forall v \in H^1(\Omega) \, : \, \int_\Omega \nabla u \nabla v dx = \sigma \int_{\partial\Omega} uv dx.$$

All the eigenvalues of the Steklov problem can be computed by the usual min-max formula

$$\forall k \in \mathbb{N} \, : \, \sigma_k(\Omega) = \min_{S \in \mathcal{S}_{k+1}} \max_{u \in S \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 dx},$$

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where $\mathcal{S}_{k+1}$ denotes the family of all subspaces of dimension $k+1$ in $H^1(\Omega)$. Then

$$0 = \sigma_0(\Omega) \le \sigma_1(\Omega) \le \sigma_2(\Omega) \le \ldots \to +\infty.$$

The problem we discuss in this paper is the following

$$\max\{F(\sigma_1(\Omega), \ldots, \sigma_k(\Omega)) : \Omega \subseteq \mathbb{R}^d, \Omega \text{ bounded and Lipschitz}, |\Omega| = m\}, \tag{1.1}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ is a functional non decreasing in each variable and upper semi-continuous. By $|\Omega|$ we denoted the Lebesgue measure of Ω. Typical examples are

$$F(\sigma_1, \ldots, \sigma_k) = \sigma_k, \quad F(\sigma_1, \ldots, \sigma_k) = \Big(\frac{1}{\sigma_1} + \cdots + \frac{1}{\sigma_k}\Big)^{-1}.$$

Following the results of Weinstock [22] and Brock [8] (see also [17]), the ball maximizes the first Steklov eigenvalue $\sigma_1(\Omega)$ and, even more, the functional

$$\Big(\frac{1}{\sigma_1(\Omega)} + \cdots + \frac{1}{\sigma_d(\Omega)}\Big)^{-1}$$

among all sets of given volume of $\mathbb{R}^d$. We also refer to the pioneering paper of Hersch, Payne and Schiffer [18] for a series of results in the family of simply connected sets of the plane under a perimeter constraint, and to the paper of Girouard and Polterovich [15] for a recent and complete overview of this topic. It is important to notice that in many spectral inequalities associated to the Steklov spectrum the natural constraint is the surface area of the boundary, which is not considered in this paper. From a different perspective, we also refer to the recent result of Petrides [20] in which the maximization of the Steklov eigenvalues is studied in the class of Riemanian metrics on a prescribed smooth manifold with boundary, under a constraint on the length of the boundary.

Our first objective is to analyze the existence of a solution for problem (1.1), i.e. we search to prove that there exists some set Ω for which the maximum is attained in (1.1). In general, proving the existence of a solution for a shape optimization problem of spectral type is not an easy task. There exists only one general result, which involves the Dirichlet eigenvalues of the Laplacian, and was proved by Buttazzo and Dal Maso in 1993 (see [9]). For that purpose, Buttazzo and Dal Maso extended the class of competing sets to a larger one. Precisely, instead on looking for an optimal shape in the class of smooth open sets, they searched it in the class of quasi-open sets, which (by a monotonicity argument) is equivalent to search it in the class of measurable sets. Once existence is achieved, the question, which turns out to be classical in shape optimization, is to prove the smoothness of the solution, and return back in this way to the original problem. The regularity question is quite difficult as soon as the spectral functional involves higher eigenvalues, and for the moment is still unsolved even for more classical situations (e.g. the Dirichlet Laplacian).

Even in the absence of a regularity result, the existence of an optimal shape in a weak setting is still of interest. In this paper, we extend the variational definition of the Steklov eigenvalues to a measurable set in $\mathbb{R}^d$, which has a finite perimeter in the sense of De Giorgi, calling them *relaxed eigenvalues*. As soon as the measurable set is smooth, the classical definition of the Steklov eigenvalues is recovered. In Theorem 5.6 we prove the existence of a solution which maximizes the general functional (1.1) of the *relaxed* eigenvalues among all measurable sets of $\mathbb{R}^d$ with finite perimeter. Moreover, we prove that an optimal set has necessarily to have both perimeter and diameter below a certain threshold (depending on the functional, volume and the dimension of the space). A key result in our analysis is due to Colbois, El Soufi and Girouard [11], which gives a control on the Steklov spectrum in terms of the isoperimetric ratio. Roughly speaking, a maximizing sequence of measurable sets should have a uniformly bounded perimeter. A second key argument that we developed for proving existence of a solution, is a *local* control of the spectrum with respect to the mass. Precisely, we prove that if the mass is too small in some region, then either the set has very low eigenvalues or it has to be disconnected. In this way, we obtain also an isodiametric control of the Steklov spectrum, which is new, up to the authors' knowledge. Precisely, we prove that (Proposition 4.3)

$$\sigma_k(\Omega)\,\mathrm{diam}(\Omega) \le C(d) k^{\frac{2}{d}+1},$$

for every $k \in \mathbb{N}$ and every smooth (or not) connected set $\Omega \subseteq \mathbb{R}^d$, with a constant $C(d)$ depending only on the dimension of the space.

In Section 6 we consider the two dimensional case. In this case we can work with open sets by introducing a different relaxation framework which takes into account the full topological boundary and not only the reduced boundary. We prove (see Theorem 6.4) the existence of an optimal open set with a topological boundary of finite Hausdorff measure, which is a finite union of at most k Jordan domains, whose closures intersect pairwise in at most one point.

In general, when relaxing the shape optimization problem in a larger class of domains, the risk is that the value of the shape functional on the new class is strictly larger than the original one. In order to prove equality, one has either to prove a density argument in a topology for which the spectrum is continuous, or to prove a regularity result for the boundary of the optimal relaxed shape. For the relaxed formulations we give in this paper, we are not able to prove such result in the full generality, but we do not have a counterexample as well. In some particular cases, we can prove indeed that our relaxed formulation leads to the same optimal shape, as in the usual class of Lipschitz sets.

In Section 7 we give some numerical approximations of the sets maximizing $\sigma_k(\Omega)$ for k from 2 to 10, and for some other functionals of eigenvalues. We observe numerical evidence that the optimal shapes have the symmetry of the regular k-gons. With respect to the previous numerical simulations (e.g. [6, 1]), our method avoids imposing star shapedness of the competing domains and is applied to more general spectral functionals, satisfying the monotonicity assumption.

A technical tool which is connected to the upper semicontinuity properties of the relaxed eigenvalues but which may be of independent interest, is a lower semi-continuity result for the L^2-norms of the traces of a strongly convergent sequence in $H^1(\mathbb{R}^d)$ on boundaries of moving sets (Propositions 2.3 and 2.6).

The paper is organized as follows. In Section 2 we recall the notation and the basic facts concerning sets of finite perimeter and Hausdorff convergence employed throughout the paper. In particular we prove the two lower semicontinuity results mentioned above (Proposition 2.3 and Proposition 2.6). Section 3 collects some basic properties of the Steklov spectrum of Lipschitz domains, which yield indications on how a relaxation on larger classes of domains can be carried over. In Section 4 we prove a fundamental lemma (see Lemma 4.1) which is pivotal for the whole analysis of the paper. We show that this lemma readily implies the isodiametric control of the Steklov eigenvalues. Section 5 contains the relaxation to the class of sets of finite perimeter, the case of open planar domains is studied in Section 6, and Section 7 contains the numerical computations. As mentioned above, the existence of optimal shapes in both cases is based on the isoperimetric control of the spectrum analogous to that proved in [11]: we show how to adapt the arguments to the relaxed spectrum in the Appendix.

2. Notation and preliminaries

In this section we fix the basic notation employed throughout the paper, and recall some notions concerning sets of finite perimeter and Hausdorff convergence of compact sets. Moreover, we will prove two lower semicontinuity results (see Proposition 2.3 and Proposition 2.6) which will be important to deal with the shape optimization problems of Section 5 and Section 6.

2.1. **Basic notation.** Given $E \subseteq \mathbb{R}^d$, we will denote by $|E|$ its Lebesgue measure, by E^c its complement, by 1_E its characteristic function, and we set $tE := \{tx : x \in E\}$ for every $t \in \mathbb{R}$. $\mathcal{H}^{d-1}(E)$ will stand for the Hausdorff $(d-1)$-dimensional measure of E (see [13, Chapter 2]), which coincides with the usual area measure if E is a piecewise regular hypersurface. Two measurable sets $E_1, E_2 \subseteq \mathbb{R}^d$ are said to be "well separated" if there exist two open sets A_1, A_2 with $|E_1 \setminus A_1| = 0$ and $|E_2 \setminus A_2| = 0$, and $\mathrm{dist}(A_1, A_2) > 0$.

For $x \in \mathbb{R}^d$ and $r > 0$, $B_r(x)$ stands for the ball of center x and radius r, while $Q_r(x)$ denotes the cube centered at x, with sides parallel to the axis of length r.

We will denote by $\mathcal{M}_b(\mathbb{R}^d)$ the space of bounded Radon measures on $\mathbb{R}^d$. If μ is a Borel measure on $\mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$ is Borel regular, we will denote by $\mu\lfloor A$ the restriction of μ to A.

Given $\Omega \subseteq \mathbb{R}^d$ open and $1 \le p \le +\infty$, $L^p(\Omega;\mathbb{R}^k)$ stands for the usual space of (classes of) p-summable $\mathbb{R}^k$-valued functions on Ω, while $H^1(\Omega)$ will denote the Sobolev space of square summable functions whose gradient in the sense of distributions is also square-summable.

2.2. **Sets of finite perimeter.** For the general theory of sets of finite perimeter, we refer the reader to [3, Section 3.3]. Here we recall some basic facts in a form which is suitable to our analysis.

Given $E \subseteq \mathbb{R}^d$ measurable and $A \subseteq \mathbb{R}^d$ open, the *perimeter* of E in A is defined as

$$P(E,A) := \sup\left\{ \int_E \operatorname{div}(\varphi)\,dx \,:\, \varphi \in C^\infty_c(A;\mathbb{R}^d), \|\varphi\|_\infty \le 1 \right\},$$

and E is said to have *finite perimeter* in A if $P(E,A) < +\infty$. When $A = \mathbb{R}^d$, we write simply $P(E)$.

If $E \subseteq \mathbb{R}^d$ has finite perimeter, then there exists $\partial^* E \subseteq \partial E$, called the *reduced boundary of* E, such that for every $A \subseteq \mathbb{R}^d$ open we have

$$P(E,A) = \mathcal{H}^{d-1}(\partial^* E \cap A).$$

It turns out that the set $\partial^* E$ is countably $\mathcal{H}^{d-1}$-rectifiable, i.e., there exists a sequence $(M_n)_{n\in\mathbb{N}}$ of C^1-submanifold in $\mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\partial^* E \setminus \cup_n M_n) = 0$.

The following compactness result holds true.

Proposition 2.1 (**Compactness).** *Let $A \subseteq \mathbb{R}^d$ be open and bounded, and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of measurable subsets of A such that $\sup_n P(E_n,A) < +\infty$. Then there exists $E \subseteq A$ with finite perimeter in A such that up to a subsequence*

$$1_{E_n} \to 1_E \qquad \text{strongly in } L^1(A)$$

and

$$P(E,A) \le \liminf_n P(E_n,A).$$

In order to establish a fundamental lemma in Section 4, we will use a suitable isoperimetric inequality in annuli, which is uniform with respect to their width. In order to formulate the statement, we use the notation

$$A_{r_1,r_2}(0) := \{x \in \mathbb{R}^d \,:\, r_1 < |x| < r_2\}.$$

Lemma 2.2 (**Uniform relative isoperimetric inequality in annuli).** *Let $m > 0$ be given. Then there exist two constants $c = c(d)$ and $w = w(m,d)$ such that for every $r \ge 0$, $l \ge w$ and every measurable set $E \subseteq A_{r,r+l}(0)$ with $|E| \le m$ we have*

$$|E|^{\frac{d-1}{d}} \le cP(E, A_{r,r+l}(0)).$$

Proof. In the proof we will use some basic facts concerning the space $BV(\mathbb{R}^d)$ of functions of bounded variation: we refer the reader to [3] for a comprehensive treatment of the subject. Recall that $E \subseteq \mathbb{R}^d$ with $|E| < +\infty$ has finite perimeter if and only if $1_E \in BV(\mathbb{R}^d)$.

We divide the proof in several steps.

Step 1. We will make use of the following two relative isoperimetric inequalities.

(a) There exists $c_1 = c_1(d) > 0$ such that for every $r > 0$ and $E \subseteq B_r(0)$ measurable

$$\min\{|E|^{\frac{d-1}{d}}, |B_r(0) \setminus E|^{\frac{d-1}{d}}\} \le c_1 P(E, B_r(0)).$$

(b) There exists $c_2 = c_2(d) > 0$ such that for every $r > 0$ and $E \subseteq \bar{B}^c_r(0)$ measurable

$$|E|^{\frac{d-1}{d}} \le c_2 P(E, \bar{B}^c_r(0)). \tag{2.1}$$

The proof of (a) can be found for example in [3, Remark 3.50]. For the proof of (b) we can reason as follows. Let $u \in BV(\bar{B}^c_r(0))$. Then by Sobolev imbedding applied to $u1_{\bar{B}^c_r(0)} \in BV(\mathbb{R}^d)$ we obtain

$$\left(\int_{\bar{B}^c_r(0)} |u|^{\frac{d}{d-1}} \right)^{\frac{d-1}{d}} \le C(d)|D(u1_{\bar{B}^c_r(0)})|(\mathbb{R}^d) = C(d)\left(|Du|(\bar{B}^c_r(0)) + \int_{\partial B_r(0)} |u|\,d\mathcal{H}^{d-1} \right),$$

where $C(d)$ is the Sobolev imbedding constant, while the integration on $\partial B_r(0)$ involves the trace of u. Reasoning on smooth functions vanishing outside a ball, by density we obtain that for every $u \in BV(\bar{B}_r^c(0))$

$$\int_{\partial B_r(0)} |u| \, d\mathcal{H}^{d-1} \leq |Du|(\bar{B}_r^c(0)).$$

We thus obtain

$$\left(\int_{\bar{B}_r^c(0)} |u|^{\frac{d}{d-1}} \right)^{\frac{d-1}{d}} \leq 2C(d)|Du|(\bar{B}_r^c(0)),$$

so that the isoperimetric inequality (2.1) follows now by considering $u = 1_E$.

Step 2. Let us prove that there exist $\varepsilon = \varepsilon(d) > 0$ and $c = c(d)$ with the following property: for every $r \geq 0$, $l \geq 1$ and every measurable set E with $E \subseteq A_{r,r+l}(0)$ and $|E| < \varepsilon$ we have

$$|E|^{\frac{d-1}{d}} \leq cP(E, A_{r,r+l}(0)) \tag{2.2}$$

Indeed from the equality

$$|E| = \int_r^{r+l} \mathcal{H}^{d-1} \left(E \cap \partial B_{r+s}(0) \right) \, ds,$$

we can find $l_0 \in]0, l[$ such that

$$\mathcal{H}^{d-1} \left(E \cap \partial B_{r+l_0}(0) \right) \leq |E|.$$

Let us consider the sets

$$E_1 := E \cap B_{r+l_0}(0) \qquad \text{and} \qquad E_2 := E \setminus B_{r+l_0}(0).$$

By the relative isoperimetric inequality in a ball applied to E_2 in $B_{r+l}(0)$, we get

$$\begin{aligned} |E_2|^{\frac{d-1}{d}} \leq c_1 P(E_2, B_{r+l}(0)) &\leq c_1 \left(P(E, A_{r,r+l}(0)) + \mathcal{H}^{d-1} \left(E \cap \partial B_{r+l_0}(0) \right) \right) \\ &\leq c_1 \left(P(E, A_{r,r+l}(0)) + |E| \right). \end{aligned} \tag{2.3}$$

On the other hand, by the isoperimetric inequality outside a ball applied to E_1 in $\bar{B}_r^c$ we obtain

$$\begin{aligned} |E_1|^{\frac{d-1}{d}} \leq c_2 P(E_1, \bar{B}_r^c) &\leq c_2 \left(P(E, A_{r,r+l}(0)) + \mathcal{H}^{d-1} \left(E \cap \partial B_{r+l_0}(0) \right) \right) \\ &\leq c_2 \left(P(E, A_{r,r+l}(0)) + |E| \right). \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) we obtain for $c := \max\{c_1, c_2\}$

$$\left(\frac{|E|}{2} \right)^{\frac{d-1}{d}} \leq c \left(P(E, A_{r,r+l}(0)) + |E| \right),$$

so that (2.2) follows if ε is sufficiently small.

Step 3. Let $w = w(m, d) > 0$ be such that $w^d \varepsilon = m$, where ε is given in Step 2. If $E \subseteq A_{r,r+l}(0)$ with $l \geq w$ and $|E| \leq m$, then for $E_1 := \frac{1}{w} E$ we have

$$E_1 \subseteq \frac{1}{w} A_{r,r+l}(0) = A_{\frac{r}{w}, \frac{r}{w} + \frac{l}{w}}(0), \qquad |E_1| \leq \varepsilon, \qquad \frac{l}{w} \geq 1,$$

so that thanks to (2.2)

$$\frac{1}{w^{d-1}} |E|^{\frac{d-1}{d}} = |E_1|^{\frac{d-1}{d}} \leq cP(E_1, A_{\frac{r}{w}, \frac{r}{w} + \frac{l}{w}}(0)) = \frac{c}{w^{d-1}} P(E, A_{r,r+l}(0)),$$

i.e., the conclusion follows. $\square$

The following lower semicontinuity result will be essential in Section 5 to infer suitable upper semicontinuity properties of the relaxed Steklov eigenvalues.

Proposition 2.3. *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets of finite perimeter of $\mathbb{R}^d$ such that*

$$\limsup_{n\to\infty} \mathcal{H}^{d-1}(\partial^* E_n) < +\infty \qquad \text{and} \qquad 1_{E_n} \stackrel{L^1(\mathbb{R}^d)}{\longrightarrow} 1_E \tag{2.5}$$

for some set E of finite perimeter in $\mathbb{R}^d$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\mathbb{R}^d)$ such that

$$u_n \to u \qquad \text{strongly in } H^1(\mathbb{R}^d)$$

for some $u \in H^1(\mathbb{R}^d)$.

Then

$$\int_{\partial^* E} u^2 \, d\mathcal{H}^{d-1} \le \liminf_n \int_{\partial^* E_n} u_n^2 \, d\mathcal{H}^{d-1}. \tag{2.6}$$

Proof. In the proof we will use some basic facts concerning the space $BV(\mathbb{R}^d)$ of functions of bounded variation: we refer the reader to [3] for a comprehensive treatment of the subject.

It is not restrictive to assume

$$\sup_n \int_{\partial^* E_n} u_n^2 d\mathcal{H}^{d-1} < +\infty.$$

It turns out that (see [3, Theorem 3.84])

$$v_n := u_n^2 1_{E_n} \in BV(\mathbb{R}^d),$$

with distributional derivative given by

$$Dv_n = 2u_n \nabla u_n 1_{E_n} \, dx + u_n^2 \nu_n \mathcal{H}^{d-1} \lfloor \partial^* E_n,$$

where ν_n denotes the inner normal to E_n. Since

$$v_n \to v := u^2 1_E \qquad \text{strongly in } L^1(\mathbb{R}^d)$$

and

$$u_n \nabla u_n 1_{E_n} \to u \nabla u 1_E \qquad \text{strongly in } L^1(\mathbb{R}^d; \mathbb{R}^d),$$

the lower semicontinuity of the total variation (see [3, Remark 3.5]) entails precisely (2.6). □

2.3. **Hausdorff convergence of compact sets.** The family $\mathcal{K}(\mathbb{R}^d)$ of closed sets in $\mathbb{R}^d$ can be endowed with the Hausdorff metric d_H defined by

$$d_H(K_1, K_2) := \max \left\{ \sup_{x\in K_1} \operatorname{dist}(x, K_2), \sup_{y\in K_2} \operatorname{dist}(y, K_1) \right\} \tag{2.7}$$

with the conventions $\operatorname{dist}(x, \emptyset) = +\infty$ and $\sup \emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$ and $d_H(\emptyset, K) = +\infty$ if $K \neq \emptyset$.

The Hausdorff metric has good compactness properties (see [4, Theorem 4.4.15]).

Proposition 2.4 (Compactness). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets contained in a fixed compact set of $\mathbb{R}^d$. Then there exists a compact set $K \subseteq \mathbb{R}^d$ such that up to a subsequence*

$$K_n \to K \qquad \text{in the Hausdorff metric.}$$

For our analysis we will need the following property due to Gołąb: for the proof we refer the reader to [14, Theorem 3.18] or [4, Theorem 4.4.17].

Theorem 2.5 (Gołąb). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact connected sets in $\mathbb{R}^d$ such that*

$$K_n \to K \qquad \text{in the Hausdorff metric.}$$

Then K is connected and

$$\mathcal{H}^1(K) \le \liminf_n \mathcal{H}^1(K_n).$$

The following lower semicontinuity result will be essential in Section 6 to deduce suitable upper semicontinuity properties of the relaxed Steklov eigenvalues for planar domains.

Proposition 2.6. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets in $\mathbb{R}^2$ with at most k connected components, such that $\mathcal{H}^1(K_n) < +\infty$ and*

$$K_n \to K \qquad \text{in the Hausdorff metric,}$$

where K is compact with $\mathcal{H}^1(K) < +\infty$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\mathbb{R}^2)$ such that

$$u_n \to u \qquad \text{strongly in } H^1(\mathbb{R}^2),$$

for some $u \in H^1(\mathbb{R}^2)$. Then

$$\int_K u^2\, d\mathcal{H}^1 \le \liminf_n \int_{K_n} u_n^2\, d\mathcal{H}^1.$$

Proof. Let us divide the proof in several steps.

Step 1: A measure theoretic approach. Without loss of generality, we can assume that K_n and K are connected,

$$\sup_n \int_{K_n} u_n^2\, d\mathcal{H}^1 < +\infty,$$

and that the positive Radon measures on $\mathbb{R}^2$

$$\mu_n(A) := \int_{A\cap K_n} u_n^2\, d\mathcal{H}^1.$$

are such that

$$\mu_n \overset{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\mathbb{R}^2), \tag{2.8}$$

for some positive measure $\mu \in \mathcal{M}_b(\mathbb{R}^2)$. The result follows if we prove that

$$\frac{d\mu}{d\nu}(x) \ge u^2(x) \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in K, \tag{2.9}$$

where $\frac{d\mu}{d\nu}$ denotes the Radon-Nikodym derivative of μ with respect to $\nu := \mathcal{H}^1\lfloor K$. Indeed, if this is the case, is view of the fact that

$$\mu(A) = \int_{A\cap K} \frac{d\mu}{d\nu}\, d\mathcal{H}^1 + \mu^s(A),$$

where μ^s is singular with respect to ν, we can write

$$\liminf_n \int_{K_n} u_n^2\, d\mathcal{H}^1 = \liminf_n \mu_n(\mathbb{R}^2) \ge \mu(\mathbb{R}^2) \ge \mu^a(\mathbb{R}^2) = \int_K \frac{d\mu}{d\nu}\, d\mathcal{H}^1 \ge \int_K u^2\, d\mathcal{H}^1,$$

and the result follows.

Recall that K is $\mathcal{H}^1$-countably rectifiable, being compact and connected in $\mathbb{R}^2$ with $\mathcal{H}^1(K) < +\infty$ (see [14, Theorem 3.14]). So it suffices to prove inequality (2.9) for every $x \in K$ in which K admits an approximate tangent line l_x, which is a Lebesgue point for u, and for which

$$\frac{d\mu}{d\nu}(x) = \lim_{\rho\to 0^+} \frac{\mu(\bar{Q}_{2\rho}(x))}{\nu(\bar{Q}_{2\rho}(x))}.$$

Indeed $\mathcal{H}^1$-a.e. point in K satisfies these properties.

Step 2: Some geometric properties of K. Up to a translation, we may assume $x = 0$ and that the approximate tangent line l is horizontal. Then by definition of approximate tangent line, as $\varepsilon \to 0^+$ we have

$$\mathcal{H}^1\lfloor K_\varepsilon \overset{*}{\rightharpoonup} \mathcal{H}^1\lfloor l \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\mathbb{R}^2), \tag{2.10}$$

where $K_\varepsilon := \frac{1}{\varepsilon}K$.

We claim that for every $r > 0$

$$K_\varepsilon \cap \overline{Q}_{2r}(0) \to l \cap \overline{Q}_{2r}(0) \qquad \text{in the Hausdorff metric.} \tag{2.11}$$

Indeed, given any sequence $\varepsilon_n \to 0$, by the compactness of Hausdorff convergence and using a diagonal argument, we can find a subsequence $(\varepsilon_{n_h})_{h\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$, $m \geq 1$

$$K_{\varepsilon_{n_h}} \cap \bar{Q}_{2m}(0) \to K_0^m \qquad \text{in the Hausdorff metric.}$$

It is readily checked that for every $m \geq 1$

$$K_0^m \subseteq K_0^{m+1} \qquad \text{and} \qquad K_0^m \cap Q_{2m}(0) = K_0^{m+1} \cap Q_{2m}(0). \tag{2.12}$$

Let us set $K_0 := \bigcup_{m=1}^{\infty} K_0^m$. We claim that

$$K_0 = l \tag{2.13}$$

(a) We have $K_0 \subseteq l$. Indeed, assume by contradiction that $\xi \in K_0 \setminus l$ with $\overline{B}_\eta(\xi) \cap l = \emptyset$. Using the measure convergence (2.10), we obtain that

$$\mathcal{H}^1(K_{\varepsilon_{n_h}} \cap B_\eta(\xi)) \to 0. \tag{2.14}$$

But $K_{\varepsilon_{n_h}}$ is connected by arcs (see [14, Lemma 3.12]), so that the points $\xi_{n_h} \in K_{\varepsilon_{n_h}}$ such that $\xi_{n_h} \to \xi$ are connected to 0 through an arc contained in $K_{\varepsilon_{n_h}}$, against (2.14).

(b) We have on the contrary $l \subseteq K_0$. Indeed, assume by contradiction that $\xi \in l \setminus K_0$. Then there exists $\eta > 0$ such that $K_{\varepsilon_{n_h}} \cap B_\eta(\xi) = \emptyset$ for h large, against (2.10).

In view of (2.12) and (2.13) we deduce that for $\varepsilon \to 0$ and for every $r > 0$

$$K_\varepsilon \cap \bar{Q}_{2r}(0) \to l \cap \bar{Q}_{2r}(0) \qquad \text{in the Hausdorff metric,}$$

i.e., convergence (2.11) holds true.

Step 3: Blow up. Let now $\varepsilon := \varepsilon_m \to 0$. Notice that thanks to (2.10) we have

$$\frac{\mathcal{H}^1(\bar{Q}_{2\varepsilon_m}(0) \cap K)}{2\varepsilon_m} \to 1$$

so that

$$\frac{d\mu}{d\nu}(0) = \lim_m \frac{\mu(\bar{Q}_{2\varepsilon_m}(0))}{\nu(\bar{Q}_{2\varepsilon_m}(0))} = \lim_m \frac{\mu(\bar{Q}_{2\varepsilon_m}(0))}{2\varepsilon_m}. \tag{2.15}$$

Since for (2.8)

$$\mu(\bar{Q}_{\varepsilon_m}(0)) \geq \limsup_n \mu_n(\bar{Q}_{\varepsilon_m}(0)),$$

and in view of (2.11), using the Hausdorff convergence of K_n to K we can find a sequence $(n_m)_{m\in\mathbb{N}}$ with

$$\varepsilon_m^2 + \mu(\bar{Q}_{2\varepsilon_m}(0)) \geq \mu_{n_m}(\bar{Q}_{2\varepsilon_m}(0)),$$

such that setting

$$\hat{K}_m := \frac{1}{\varepsilon_m} K_{n_m} \cap \bar{Q}_2(0)$$

we have

$$\hat{K}_m \to l \cap \bar{Q}_2(0) \qquad \text{in the Hausdorff metric.} \tag{2.16}$$

We deduce that

$$\begin{aligned} \liminf_m \frac{\mu(\bar{Q}_{2\varepsilon_m}(0))}{2\varepsilon_m} \geq \liminf_m \frac{\mu_{n_m}(\bar{Q}_{2\varepsilon_m}(0))}{2\varepsilon_m} &= \liminf_m \frac{1}{2\varepsilon_m} \int_{K_{n_m} \cap \bar{Q}_{2\varepsilon_m}(0)} u_{n_m}^2 \, d\mathcal{H}^1 \\ &= \frac{1}{2} \liminf_m \int_{\hat{K}_m} v_m^2 \, d\mathcal{H}^1, \end{aligned} \tag{2.17}$$

where

$$v_m(y) := u_{n_m}(\varepsilon_m y) \in H^1(Q_2(0))$$

is such that

$$v_m \to u(0) \qquad \text{strongly in } H^1(Q_2(0)). \tag{2.18}$$

This last requirement is achieved taking into account that 0 is a Lebesgue point for u so that

$$\int_{Q_2(0)} |u(\varepsilon_m y) - u(0)|^2\, dy = \frac{1}{\varepsilon_m^2} \int_{Q_{2\varepsilon_m}(0)} |u(x) - u(0)|^2\, dx \to 0$$

and

$$\int_{Q_2(0)} |\nabla(u(\varepsilon_m y))|^2\, dy = \int_{Q_{2\varepsilon_m}(0)} |\nabla u(x)|^2\, dx \to 0.$$

Adding $\partial Q_2(0)$ to $\hat{K}_m$ (in the case $\hat{K}_m$ is not connected), we deduce by Gołąb theorem (see Theorem 2.5) that for every open set $A \subseteq Q_2(0)$

$$\mathcal{H}^1(l \cap A) \le \liminf_m \mathcal{H}^1(\hat{K}_m \cap A). \tag{2.19}$$

Collecting (2.17) and (2.15), in order to prove (2.9) it suffices to check that

$$\frac{1}{2} \liminf_m \int_{\hat{K}_m} v_m^2\, d\mathcal{H}^1 \ge u^2(0). \tag{2.20}$$

Step 4: Slicing and conclusion. Being $\hat{K}_m$ countably $\mathcal{H}^1$-rectifiable, using the area formula we have

$$\liminf_m \int_{\hat{K}_m} v_m^2\, d\mathcal{H}^1 \ge \liminf_m \int_{-1}^1 \int_{(\hat{K}_m)_{x_1}} v_m^2(x_1, s)\, d\mathcal{H}^0(s)\, dx_1, \tag{2.21}$$

where

$$(\hat{K}_m)_{x_1} := \{s \in \mathbb{R} \,:\, (x_1, s) \in \hat{K}_m\}.$$

In view of (2.18), for a.e. $x_1 \in [-1, 1]$ we have

$$v_m(x_1, \cdot) \to u(0) \qquad \text{strongly in } H^1(-1, 1). \tag{2.22}$$

Let $(m_k)_{k\in\mathbb{N}}$ be such that

$$\liminf_m \int_{-1}^1 \int_{(\hat{K}_m)_{x_1}} v_m^2(x_1, s)\, d\mathcal{H}^0(s)\, dx_1 = \lim_k \int_{-1}^1 \int_{(\hat{K}_{m_k})_{x_1}} v_{m_k}^2(x_1, s)\, d\mathcal{H}^0(s)\, dx_1. \tag{2.23}$$

Thanks to the Hausdorff convergence (2.16), notice that if $(\hat{K}_{m_k})_{x_1} \neq \emptyset$ we have

$$(\hat{K}_{m_k})_{x_1} \to (x_1, 0) \qquad \text{in the Hausdorff metric.} \tag{2.24}$$

Assume that for some $a \in]-1, 1[$ we have along a further subsequence $(m_{k_h})_{h\in\mathbb{N}}$

$$(\hat{K}_{m_{k_h}})_a = \emptyset. \tag{2.25}$$

Then for every $x_1 \neq a$ and $h \in \mathbb{N}$ we have $(\hat{K}_{m_{k_h}})_{x_1} \neq \emptyset$: indeed, if this is not the case, taking into account that in view of (2.19) we get

$$\hat{K}_{m_{k_h}} \cap (]a, x_1[\times\mathbb{R}) \neq \emptyset \qquad \text{(assuming for example } x_1 > a),$$

we deduce that the original compact set $K_{m_{k_h}}$ cannot be connected, against the assumption.

We can thus write using (2.24) and (2.22), and recalling that the H^1-convergence in dimension one entails also uniform convergence,

$$\begin{aligned}\liminf_m \int_{-1}^1 \int_{(\hat{K}_m)_{x_1}} v_m^2(x_1, s)\, d\mathcal{H}^0(s)\, dx_1 &= \lim_h \int_{-1}^1 \int_{(\hat{K}_{m_{k_h}})_{x_1}} v_{m_{k_h}}^2(x_1, s)\, d\mathcal{H}^0(s)\, dx_1 \\ &\ge \int_{-1}^1 \liminf_h \int_{(\hat{K}_{m_{k_h}})_{x_1}} v_{m_{k_h}}^2(x_1, s)\, d\mathcal{H}^0(s)\, dx_1 \ge \int_{-1}^1 u^2(0)\, dx_1 = 2u^2(0).\end{aligned}$$

If (2.25) does not occur, the same calculation starting from (2.23) leads again to the previous inequality. In view of (2.21), we infer that inequality (2.20) holds true, so that the proof is concluded. □

3. Some basic properties of the Steklov spectrum

In this section, we collect some basic properties of the Steklov eigenvalues of Lipschitz domains, which yield some hints on how to find a suitable extension to a larger class of sets in which the optimization problem (1.1) is well posed.

Definition of the Steklov spectrum and first properties. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz open set. As mentioned in the introduction, σ is an eigenvalue of the Steklov problem for the Laplace operator provided there exists a non-zero function $u \in H^1(\Omega)$ which satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Omega, \end{cases}$$

The Steklov eigenvalues are given by the min-max formula

$$\forall k \in \mathbb{N} \,:\, \sigma_k(\Omega) = \min_{S \in \mathcal{S}_{k+1}} \max_{u \in S\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 dx},$$

where $\mathcal{S}_{k+1}$ denotes the family of all subspaces of dimension $k+1$ in $H^1(\Omega)$. Then

$$0 = \sigma_0(\Omega) \leq \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \dots$$

and $\sigma_k(\Omega) \to +\infty$. As usual, in this formulation an eigenvalue can appear several times, according to its multiplicity.

For every $t > 0$ we have the following rescaling property:

$$\sigma_k(t\Omega) = \frac{1}{t}\sigma_k(\Omega). \tag{3.1}$$

The min-max formula entails also the following property for the spectrum of disconnected domains: if $\Omega_1, \Omega_2 \subseteq \mathbb{R}^d$ are bounded Lipschitz disjoint domains, then $\sigma_k(\Omega_1 \cup \Omega_2)$ is given by the value of the $(k+1)$-th term of the ordered non decreasing rearrangement of the numbers

$$\sigma_0(\Omega_1), \dots, \sigma_k(\Omega_1), \sigma_0(\Omega_2), \dots, \sigma_k(\Omega_2).$$

Isoperimetric control of the eigenvalues. Following the result of [11, Theorem 2.2], one has a control on the k-th Steklov eigenvalue via the perimeter of the boundary of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$:

$$\sigma_k(\Omega) \leq C_d \frac{k^{\frac{2}{d}}}{\mathcal{H}^{d-1}(\partial\Omega)^{\frac{1}{d-1}}}. \tag{3.2}$$

Monotonicity with respect to inclusions. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded connected Lipschitz open set. Assume that K is the closure of an open Lipschitz subset of Ω, such that $K \subset \Omega$. We claim that for every $k \in \mathbb{N}$,

$$\sigma_k(\Omega \setminus K) \leq \sigma_k(\Omega). \tag{3.3}$$

Indeed, let $S_{k+1} = \text{span}\{u_0, \dots, u_k\}$ be a system of $L^2(\partial\Omega)$-orthogonal eigenfunctions corresponding to $\sigma_0(\Omega), \dots, \sigma_k(\Omega)$. Then, they generate a subspace of dimension $k+1$ in $H^1(\Omega)$. Moreover, the same functions restricted to $\Omega \setminus K$ are independent as well. This is a consequence of the fact that they are harmonic, via the unique continuation principle.

Consequently, for every $u \in S_{k+1}$ we have

$$\frac{\int_{\Omega\setminus K} |\nabla u|^2 dx}{\int_{\partial(\Omega\setminus K)} u^2 dx} \leq \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 dx},$$

and hence

$$\max_{u \in S_{k+1}\setminus\{0\}} \frac{\int_{\Omega\setminus K} |\nabla u|^2 dx}{\int_{\partial(\Omega\setminus K)} u^2 dx} \leq \max_{u \in S_{k+1}\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 dx} = \sigma_k(\Omega).$$

Taking the infimum on the left hand side, among subspaces of dimension $k+1$ in $H^1(\Omega \setminus K)$, we get inequality (3.3).

The previous items lead to the following considerations concerning the shape optimization problem (1.1).

(a) The isoperimetric inequality (3.2) plays a crucial role to prove the existence of optimal domains: it implies that for a maximizing sequence we have a uniform bound on the perimeters, the bound depending essentially on the functional F, the volume m and the dimension of the space. This remark yields that some compactness could be available in the class of sets of finite perimeter.

(b) In view of (3.3), one understands that optimal domains have to be searched in the class of Lipschitz sets which do not have "inner holes". Indeed, removing the hole as above increases the eigenvalues and the measure of the domain. If one rescales the set without the hole in order to have the volume equal to m, the eigenvalues will increase again, thanks to (3.1).

If a hole shrinks to an inner crack, the previous arguments assert that a Lipschitz set from which we remove a Lipschitz crack can not be optimal (provided that we define suitably the Steklov eigenvalues of this *non admissible* domain). In dimension two this implies that necessarily the maximum has to be searched in the class of open sets whose complement is connected, i.e., union of open disjoint simply connected sets.

4. A fundamental lemma and a new isodiametric inequality

In this section we establish Lemma 4.1 on which a large part of the analysis of the present paper is based. Then we show how this lemma implies an isodiametric control of the Steklov spectrum, which is new up to the authors' knowledge.

In order to formulate the statement of our fundamental lemma, we use the notation

$$A_{r_1,r_2}(0) := \{x \in \mathbb{R}^d \,:\, r_1 < |x| < r_2\}.$$

Lemma 4.1. *Let $m, \lambda > 0$, and let $w = w(m,d) > 0$ be the constant of the isoperimetric inequality given by Lemma 2.2. There exists $L = L(m,\lambda,d) > w$ such that for every $r \geq 0$, $l \geq L$ and every measurable set $E \subseteq A_{r,r+l}(0)$ with finite perimeter and with $|E| = m$, at least one of the following two possibilities occur.*

(a) *There exists a function $\varphi \in H^1_0(A_{r,r+l}(0))$ with $\int_{\partial^* E} \varphi^2 d\mathcal{H}^{d-1} > 0$ and*

$$\frac{\int_E |\nabla \varphi|^2 dx}{\int_{\partial^* E} \varphi^2 d\mathcal{H}^{d-1}} \leq \lambda.$$

(b) *We have*

$$|E \cap A_{r+\frac{l-w}{2}, r+\frac{l+w}{2}}(0)| = 0.$$

Proof. Let L be a number such that

$$\frac{L-w}{2} > \frac{m^{\frac{1}{2d}}}{(\lambda c)^{\frac{1}{2}}} \sum_{k=1}^{\infty} \frac{1}{(2^{\frac{1}{2d}})^k}, \tag{4.1}$$

where c is the isoperimetric constant of Lemma 2.2.

Let $l \geq L$ and let $E \subseteq A_{r,r+l}(0)$ be a measurable set with finite perimeter such that $|E| = m$. Let us assume that conclusion (a) does not hold, and let us infer that situation (b) takes place.

For every $t \in \left[0, \frac{l-w}{2}\right]$ we introduce the quantities

$$m(t) := |E \cap (A_{r,r+t}(0) \cup A_{r+l-t,r+l}(0))| \qquad \text{and} \qquad p(t) := P(E, A_{r+t,r+l-t}(0)).$$

Notice that we can assume $p(t) \neq 0$ since otherwise the second possibility occurs trivially. Considering the test function

$$\varphi_1(x) := \frac{1}{t} \mathrm{dist}(x, \mathbb{R}^d \setminus A_{r,r+l}(0)) \wedge 1,$$

we have

$$\frac{\int_E |\nabla\varphi_1|^2 dx}{\int_{\partial^* E} \varphi_1^2 d\mathcal{H}^{d-1}} \le \frac{\frac{m(t)}{t^2}}{p(t)}.$$

so that

$$\lambda t^2 p(t) \le m(t). \tag{4.2}$$

Since the width of the annulus $A_{r+t,r+l-t}(0)$ is greater than w, using the relative isoperimetric inequality given by Lemma 2.2 we deduce

$$p(t) \ge c(m - m(t))^{\frac{d-1}{d}}.$$

so that

$$m(t) \ge \lambda c t^2 (m - m(t))^{\frac{d-1}{d}}. \tag{4.3}$$

Notice that there exists $t_1 \in]0, \frac{l-w}{2}[$ such that $m(t_1) = \frac{m}{2}$. For otherwise we would get

$$\frac{m}{2} \ge \lambda c \left(\frac{l-w}{2}\right)^2 \left(\frac{m}{2}\right)^{\frac{d-1}{d}},$$

which is against (4.1). We can estimate t_1 using again (4.3), and we obtain

$$t_1 \le \frac{m^{\frac{1}{2d}}}{(\lambda c)^{\frac{1}{2}}} \frac{1}{2^{\frac{1}{2d}}} < \frac{l-w}{2}.$$

We now repeat the same argument on the annulus $A_{r+t_1,r+l-t_1}(0)$ with the set $E \cap A_{r+t_1,r+l-t_1}(0)$ which has a measure equal to $\frac{m}{2}$. For every $t \in [0, \frac{l-w}{2} - t_1]$ we set

$$m_2(t) := |E \cap (A_{r+t_1,r+t_1+t}(0) \cup A_{r+l-t_1-t,r+l-t_1}(0))| \quad \text{and} \quad p_2(t) := P(E, A_{r+t_1+t,r+l-t_1-t}(0)),$$

and consider the test function

$$\varphi_2(x) := \frac{1}{t}\mathrm{dist}(x, \mathbb{R}^d \setminus A_{r+t_1,r+l-t_1}(0)) \wedge 1.$$

Proceeding as before, in view of our choice (4.1) for the constant L, we obtain the existence of $t_2 \in]0, \frac{l}{2} - w - t_1[$ such that $|E \cap A_{r+t_1+t_2,r+l-t_1-t_2}(0)| = \frac{m}{2^2}$, with the explicit estimate

$$t_2 \le \frac{m^{\frac{1}{2d}}}{(\lambda c)^{\frac{1}{2}}} \frac{1}{(2^{\frac{1}{2d}})^2}.$$

Thanks to (4.1), the argument can be carried out an infinite number of times exhausting in this way the entire measure of E. This means that

$$|E \cap A_{r+\frac{l-w}{2},r+\frac{l+w}{2}}(0)| = 0,$$

so that situation (b) occurs, and the proof is thus completed. □

Remark 4.2. Lemma 4.1 is clearly valid if we replace the reduced boundary $\partial^* E$ with the full topological boundary ∂E, being $\partial^* E \subseteq \partial E$.

Lemma 4.1 entails the following isodiametric control of the Steklov spectrum on connected domains.

Proposition 4.3 (Isodiametric control of the spectrum). *There exists a constant $C(d)$ depending only on the dimension of the space, such that for every $k \in \mathbb{N}$ and for every bounded connected Lipschitz open set $\Omega \subset \mathbb{R}^d$*

$$\sigma_k(\Omega)\,\mathrm{diam}(\Omega) \le C(d) k^{\frac{2}{d}+1}. \tag{4.4}$$

Proof. Notice that inequality (4.4) is scale invariant thanks to (3.1). We can thus assume that $|\Omega| = 1$ and that $0 \in \Omega$. Let us apply Lemma 4.1 with $m = \lambda = 1$, and associated constant $L = L(d)$. Then two possibilities can occur: either

(a) $\mathrm{diam}(\Omega) \le (k+1)L$,

or

(b) $\operatorname{diam}(\Omega) > (k+1)L$.

Assume that point (a) holds true: the we can write thanks to the isoperimetric inequality (3.2)

$$\begin{aligned}\sigma_k(\Omega)\operatorname{diam}(\Omega) &\le \sigma_k(\Omega)(k+1)L = \sigma_k(\Omega)|\Omega|^{\frac{1}{d}}(k+1)L \\ &\le c_d^{\frac{1}{d-1}}\sigma_k(\Omega)\mathcal{H}^{d-1}(\partial\Omega)^{\frac{1}{d-1}}(k+1)L \le c_d^{\frac{1}{d-1}}C_d k^{\frac{2}{d}}(k+1)L,\end{aligned} \tag{4.5}$$

where c_d is the isoperimetric constant such that $|\Omega|^{\frac{d-1}{d}} \le c_d\mathcal{H}^{d-1}(\partial\Omega)$.

Assume point (b) holds true. Let us pick $0 < t < 1$ so that $\operatorname{diam}(t\Omega) = (k+1)L$. Then, since $t\Omega$ is connected, we are in the first alternative of Lemma 4.1 relative to any annulus of the form $A_{iL,(i+1)L}(0)$ for $i = 0, \dots, k$. We find thus $(k+1)$ functions $\varphi_0, \dots, \varphi_k \in H^1(\mathbb{R}^d)$ with disjoint supports such that for every $i = 0, \dots, k$

$$\frac{\int_{t\Omega}|\nabla\varphi_i|^2\,dx}{\int_{\partial(t\Omega)}\varphi_i^2\,d\mathcal{H}^{d-1}} \le 1.$$

If we set $S := \operatorname{span}\{\varphi_0, \dots, \varphi_k\}$, then we get

$$\sigma_k(t\Omega) \le \max_{\varphi\in S\setminus\{0\}} \frac{\int_{t\Omega}|\nabla\varphi|^2\,dx}{\int_{\partial(t\Omega)}\varphi^2\,d\mathcal{H}^{d-1}} \le 1.$$

Then the rescaling property of σ_k together with the choice of t yields

$$\sigma_k(\Omega) = t\sigma_k(t\Omega) \le t = \frac{(k+1)L}{\operatorname{diam}(\Omega)}. \tag{4.6}$$

The conclusion follows gathering (4.5) and (4.6), by choosing

$$C(d) = 2L\max\{c_d^{\frac{1}{d-1}}C_d, 1\}.$$

□

We point out that the power $\frac{2}{d}+1$ in (4.4) is probably not sharp, but it is sufficient for the purposes of our paper.

5. Relaxed formulation for the Steklov eigenvalues: existence of optimal shapes

In this section we extend the notion of the Steklov eigenvalues to arbitrary sets of finite perimeter, and reformulate the associated shape optimization problem (1.1) in such a way to get existence of optimal domains.

The choice of the class of sets of finite perimeter is motivated by the remarks contained in Section 3, in particular by the isoperimetric control (3.2). A natural candidate to replace the topological boundary in the variational definition of the eigenvalues is given by the reduced boundary.

Let $\Omega \subseteq \mathbb{R}^d$ have finite perimeter, and let $u \in H^1(\mathbb{R}^d)$. Since $\mathcal{H}^{d-1}$-a.e. point in $\mathbb{R}^d$ is a Lebesgue point for u (indeed the set of Lebesgue points is full in capacity, see [13, Section 4.8]), the term $\int_{\partial^*\Omega} u^2 d\mathcal{H}^{d-1}$ is well defined, possibly taking a value equal to $+\infty$.

The definition of the relaxed eigenvalues is the following.

Definition 5.1 (Relaxed Steklov eigenvalues). *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter. For every $k \in \mathbb{N}$ we set*

$$\tilde\sigma_k(\Omega) := \inf_{S\in\mathcal{S}_{k+1}} \max_{u\in S\setminus\{0\}} \frac{\int_\Omega|\nabla u|^2dx}{\int_{\partial^*\Omega}u^2d\mathcal{H}^{d-1}},$$

where $\mathcal{S}_{k+1}$ denotes the family of all subspaces of dimension $k+1$ in $H^1(\mathbb{R}^d)$ which are $(k+1)$-dimensional also as subspaces of $L^2(\Omega)$ (we assume that the Rayleigh quotient is zero if the denominator is $+\infty$).

The following lemma contains some basic properties of the relaxed eigenvalues.

Lemma 5.2. *Let $\Omega \subseteq \mathbb{R}^d$ have finite perimeter. Then the following items hold true.*

(a) Classical setting. *If $\Omega \subseteq \mathbb{R}^d$ is bounded and Lipschitz, then $\tilde\sigma_k(\Omega) = \sigma_k(\Omega)$.*

(b) Rescaling. *For every $t > 0$*

$$\tilde{\sigma}_k(t\Omega) = \frac{1}{t}\tilde{\sigma}_k(\Omega).$$

(c) Isoperimetric control. *If $|\Omega| < +\infty$,*

$$\tilde{\sigma}_k(\Omega) \le C_d \frac{k^{\frac{2}{d}}}{\mathcal{H}^{d-1}(\partial^*\Omega)^{\frac{1}{d-1}}}, \tag{5.1}$$

where C_d is the constant of the isoperimetric inequality (3.2).

(d) "Disconnected" sets. *If $\Omega_1, \Omega_2 \subseteq \mathbb{R}^d$ are well separated with Ω_1 bounded, then $\tilde{\sigma}_k(\Omega_1 \cup \Omega_2)$ is given by the value of the $(k+1)$-th term of the ordered non decreasing rearrangement of the numbers*

$$\tilde{\sigma}_0(\Omega_1), \dots, \tilde{\sigma}_k(\Omega_1), \tilde{\sigma}_0(\Omega_2), \dots, \tilde{\sigma}_k(\Omega_2).$$

In particular, if $\Omega \subset \mathbb{R}^d$ is given by the union of $k+1$ well separated sets of finite perimeter, the first k sets being bounded, then $\tilde{\sigma}_k(\Omega) = 0$.

Proof. The proof of (a), (b), (d) is completely analogous to that of the classical setting. The proof of (5.1) can be obtained by adapting the arguments of [11, Theorem 2.2] to sets of finite perimeter, replacing the topological boundary with the reduced boundary: for the sake of the reader, we reproduce the proof in the Appendix. □

Remark 5.3. If the boundary of Ω is not smooth, it is possible that $\tilde{\sigma}_k(\Omega) = 0$ for every $k \in \mathbb{N}$. This is not in contradiction with our objective, which is to maximize the relaxed eigenvalues and to expect optimal sets to be smooth.

The following result will be important to get compactness in shape optimization problems.

Proposition 5.4 (**A priori bound on the diameter**). *Let $\Omega \subset \mathbb{R}^d$ be a set of finite perimeter such that*

$$|\Omega| = m \qquad \text{and} \qquad \tilde{\sigma}_k(\Omega) > \lambda,$$

where $m, \lambda > 0$. Then, up to negligible sets, we can split Ω in at most k parts

$$\Omega = \Omega^1 \cup \dots \cup \Omega^h, \qquad h \le k$$

where the sets Ω^i are bounded, with finite perimeter and well separated. Moreover the bound on the diameter depends only on m and λ.

Proof. We apply Lemma 4.1 relative to m and λ: let $L = L(m, \lambda, d) > 0$ be the associated constant.

Let us construct Ω^1. Up to a translation, we can assume that the origin is a point of density one for Ω. Let us consider the annuli

$$A_i := A_{iL,(i+1)L} := \{x \in \mathbb{R}^d \,:\, iL < |x| < (i+1)L\}, \qquad i = 0, \dots, k.$$

Notice that it cannot happen that all the sets $\Omega \cap A_i$ for $i = 0, \dots, k$ satisfy the first alternative of Lemma 4.1. Indeed, if this was the case, we could build $k+1$ functions $\varphi_0, \dots, \varphi_k$ with disjoint supports and such that for every $i = 0, \dots, k$

$$\frac{\int_\Omega |\nabla \varphi_i|^2 dx}{\int_{\partial^*\Omega} \varphi_i^2 d\mathcal{H}^{d-1}} \le \lambda.$$

This would imply $\tilde{\sigma}_k(\Omega) \le \lambda$, in view of the variational definition, against the assumption.

We have thus two alternatives.

(a) One of the annuli, say A_j, has negligible intersection with Ω: in this case we set

$$\Omega^1 := \Omega \cap B_{jL}(0).$$

(b) If all the annuli have an intersection with positive measure with Ω, then in one of them, say A_j, the second situation of Lemma 4.1 occurs. In this case we set

$$\Omega^1 := \Omega \cap B_{jL+\frac{L}{2}}(0).$$

Notice that

$$\mathrm{diam}(\Omega^1) \leq (k+1)L.$$

We proceed to construct Ω^2 following the previous arguments reasoning on the set $\Omega \setminus \Omega^1$, if this is not negligible: indeed it is such that $|\Omega \setminus \Omega^1| < m$ and still $\tilde{\sigma}_k(\Omega \setminus \Omega^1) > \lambda$ thanks to point (d) of Lemma 5.2. According to Lemma 4.1, in this case we have by construction that up to negligible sets

$$\mathrm{dist}(\Omega^1, \Omega \setminus \Omega^1) \geq w,$$

where w depends only on m and d. The procedure can be repeated to build at most k subsets of Ω which satisfy the conclusion, the bound on the diameter being given by $(k+1)L$: the existence of $k+1$ components would readily imply $\tilde{\sigma}_k(\Omega) = 0$ in view of point (d) in Lemma 5.2, against the assumption. □

Remark 5.5. The previous proof is based only on the alternatives given by Lemma 4.1.

Let now $F : \mathbb{R}^k \to \mathbb{R}$ be such that

$$F \text{ is non decreasing in each variable and upper semi-continuous.} \tag{5.2}$$

We relax the original shape optimization problem (1.1) to

$$\max\{F(\tilde{\sigma}_1(\Omega), \dots, \tilde{\sigma}_k(\Omega)) : \Omega \subseteq \mathbb{R}^d \text{ has finite perimeter and } |\Omega| = m\}. \tag{5.3}$$

In order to avoid trivial situations, we assume that F is not constant on the family of admissible sets, i.e., there exists Ω_0 with

$$F(\tilde{\sigma}_1(\Omega_0), \dots, \tilde{\sigma}_k(\Omega_0)) > F(0, \dots, 0). \tag{5.4}$$

The main result of the section is the following.

Theorem 5.6 (**Existence of optimal domains).** *Assume* (5.2) *and* (5.4)*. Then problem* (5.3) *has at least one solution. Moreover, up to negligible sets, any optimal set is bounded and can be written as the union of at most k subsets of finite perimeter, pairwise disjoint and lying at positive distance.*

Proof. In view of the assumptions on F, there exists some value $\lambda > 0$ such that for every $0 \leq \lambda_i \leq \lambda$, $i = 1, \dots, k$, we have

$$F(\tilde{\sigma}_1(\Omega_0), \dots, \tilde{\sigma}_k(\Omega_0)) > F(\lambda_1, \dots, \lambda_k).$$

This means that every domain with $\tilde{\sigma}_k(\Omega) \leq \lambda$ can not be optimal.

Let now $(\Omega_n)_{n\in\mathbb{N}}$ be a maximizing sequence. We have $\tilde{\sigma}_k(\Omega_n) > \lambda$ for every $n \in \mathbb{N}$. By the isoperimetric control (5.1) we infer that

$$\sup_n \mathcal{H}^{d-1}(\partial^*\Omega_n) < +\infty. \tag{5.5}$$

In view of Proposition 5.4 we can write up to negligible sets

$$\Omega_n = \Omega_n^1 \cup \dots \cup \Omega_n^{h_n}, \qquad h_n \leq k,$$

where the sets are equibounded and well separated. Up to a translation of these subsets, we can thus assume that the Ω_n are contained in a fixed ball of $\mathbb{R}^d$. Thanks to (5.5), we deduce that, up to a subsequence,

$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^d), \tag{5.6}$$

where $\Omega \subseteq \mathbb{R}^d$ has finite perimeter and $|\Omega| = m$.

Notice that for every $h \in \mathbb{N}$

$$\limsup_{n\to\infty} \tilde{\sigma}_h(\Omega_n) \leq \tilde{\sigma}_h(\Omega). \tag{5.7}$$

Indeed, let $\varepsilon > 0$ and let $S_{h+1} = \mathrm{span}\{u_0, \dots, u_h\} \subseteq H^1(\mathbb{R}^d)$ be an admissible subspace for the computation of $\tilde{\sigma}_h(\Omega)$ such that

$$\tilde{\sigma}_h(\Omega) \geq \max_{u\in S_{h+1}\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial^*\Omega} u^2 d\mathcal{H}^{d-1}} - \varepsilon.$$

For each index n, assume that

$$u_n := \sum_{i=0}^{h} \alpha_i^n u_i$$

attains the maximum

$$\max_{u \in S_{h+1}\setminus\{0\}} \frac{\int_{\Omega_n} |\nabla u|^2 dx}{\int_{\partial^* \Omega_n} u^2 d\mathcal{H}^{d-1}}.$$

Without restricting the generality, we may assume that

$$\sum_{i=0}^{h} (\alpha_i^n)^2 = 1, \qquad \alpha_i^n \to \alpha_i.$$

Denoting $u := \sum_{i=0}^{h} \alpha_i u_i$ we have

$$u_n \to u \qquad \text{strongly in } H^1(\mathbb{R}^d).$$

We deduce

$$\lim_{n\to\infty} \int_{\Omega_n} |\nabla u_n|^2 dx = \int_{\Omega} |\nabla u|^2 dx \qquad \text{and} \qquad \liminf_{n\to\infty} \int_{\partial^*\Omega_n} u_n^2 d\mathcal{H}^{d-1} \geq \int_{\partial^*\Omega} u^2 d\mathcal{H}^{d-1}, \tag{5.8}$$

the first equality being a consequence of the convergence (5.6) of the Ω_n, the second inequality following by Proposition 2.3.

Notice that S_{h+1} is admissible for the computation of $\tilde{\sigma}_h(\Omega_n)$ for n large enough.We obtain

$$\limsup_{n\to\infty} \tilde{\sigma}_h(\Omega_n) \leq \limsup_{n\to\infty} \frac{\int_{\Omega_n} |\nabla u_n|^2 dx}{\int_{\partial^*\Omega_n} u_n^2 d\mathcal{H}^{d-1}} \leq \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\partial^*\Omega} u^2 d\mathcal{H}^{d-1}} \leq \tilde{\sigma}_h(\Omega) + \varepsilon.$$

Letting $\varepsilon \to 0$, inequality (5.7) follows.

Thanks to assumption (5.2) on F we deduce that

$$F(\tilde{\sigma}_1(\Omega), \ldots, \tilde{\sigma}_k(\Omega)) \geq \limsup_{n\to\infty} F(\tilde{\sigma}_1(\Omega_n), \ldots, \tilde{\sigma}_k(\Omega_n)),$$

so that Ω is the optimum we are looking for.

If Ω is an optimal domain, then $\tilde{\sigma}_k(\Omega) > \lambda$ as noticed above: we can thus apply Proposition 5.4 to get that, up to negligible sets, Ω is bounded and can be written as the union of at most k subsets of finite perimeter, pairwise disjoint and lying at positive distance. The proof is now concluded. $\square$

6. Dimension two: existence of optimal open sets

In this section we consider the two-dimensional setting and propose a different relaxation of the notion of Steklov eigenvalues which permits us to work within the class of open sets. The key point is that the two-dimensional setting together with some geometric lower semicontinuity properties for the $\mathcal{H}^1$-measure on connected compact sets (Gołąb Theorem 2.5), permit us to deal successfully with the full topological boundary of the domain, instead only with the reduced boundary as in the previous section.

Let $\Omega \subseteq \mathbb{R}^2$ be open. The arguments leading to Definition 5.1 of the relaxed Steklov eigenvalues are still valid for Ω: as mentioned above, we replace the reduced boundary with the full topological boundary.

Definition 6.1. *Let $\Omega \subseteq \mathbb{R}^2$ be open. For every $k \in \mathbb{N}$ we set*

$$\tilde{\sigma}_k(\Omega) := \inf_{S\in\mathcal{S}_{k+1}} \max_{u\in S\setminus\{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\mathcal{H}^1}, \tag{6.1}$$

where $\mathcal{S}_{k+1}$ denotes the family of all subspaces of dimension $k+1$ in $H^1(\mathbb{R}^2)$ which are $(k+1)$-dimensional also as subspaces of $L^2(\Omega)$ (we assume that the Rayleigh quotient is zero if the denominator is $+\infty$).

The following lemma collects some basic properties of the relaxed eigenvalues.

Lemma 6.2. *Let $\Omega \subseteq \mathbb{R}^2$ be open. The following items hold true for every $k \in \mathbb{N}$.*

(a) Classical setting. *If Ω is bounded and Lipschitz, then $\tilde{\sigma}_k(\Omega) = \sigma_k(\Omega)$.*

(b) Rescaling*: for every $t > 0$*

$$\tilde{\sigma}_k(t\Omega) = \frac{1}{t}\tilde{\sigma}_k(\Omega).$$

(c) Isoperimetric inequality*. If $|\Omega| < +\infty$,*

$$\tilde{\sigma}_k(\Omega) \leq C_2 \frac{k}{\mathcal{H}^1(\partial\Omega)}, \tag{6.2}$$

where C_2 is a universal constant.

(d) Disconnected domains*. If $\Omega_1, \Omega_2 \subseteq \mathbb{R}^2$ are well separated with Ω_1 bounded, then $\tilde{\sigma}_k(\Omega_1 \cup \Omega_2)$ is given by the value of the $(k+1)$-th term of the ordered non decreasing rearrangement of the numbers*

$$\tilde{\sigma}_0(\Omega_1), \dots, \tilde{\sigma}_k(\Omega_1), \tilde{\sigma}_0(\Omega_2), \dots, \tilde{\sigma}_k(\Omega_2).$$

In particular, if $\Omega \subset \mathbb{R}^2$ is given by the union of $k+1$ well separated open subsets, the first k subsets being bounded, then $\tilde{\sigma}_k(\Omega) = 0$.

Proof. The proof of items (a), (b), (d) is straightforward. The isoperimetric control (6.2) is the analogue of the isoperimetric inequality (3.2) in the new setting. The proof is obtained by adapting the arguments of [11, Theorem 2.2]: for the sake of the reader, we reproduce the proof in the Appendix. □

The following result will be important to get compactness in shape optimization problems: the proof is precisely that of Proposition 5.4 taking into account Remark 5.5 and Remark 4.2.

Proposition 6.3 (A priori bound on the diameter). *Let $\Omega \subset \mathbb{R}^2$ be such that*

$$|\Omega| = m \qquad \text{and} \qquad \tilde{\sigma}_k(\Omega) > \lambda,$$

where $m, \lambda > 0$. Then we can split Ω in at most k open subsets

$$\Omega = \Omega^1 \cup \dots \cup \Omega^h, \qquad h \leq k$$

where the Ω^i are bounded, with $\mathcal{H}^1(\partial\Omega^i) < +\infty$ and well separated. The bound on the diameter depends only on m and λ.

Proof. □

As in the preceding section, let $F : \mathbb{R}^k \to \mathbb{R}$ be such that

$$F \text{ is non decreasing in each variable and upper semi-continuous.} \tag{6.3}$$

We relax the original shape optimization problem (1.1) to

$$\max\{F(\tilde{\sigma}_1(\Omega), \dots, \tilde{\sigma}_k(\Omega)) : \Omega \subseteq \mathbb{R}^2 \text{ open}, |\Omega| = m, \mathcal{H}^1(\partial\Omega) < +\infty\}. \tag{6.4}$$

The requirement $\mathcal{H}^1(\partial\Omega) < +\infty$ is motivated by item (c) in Lemma 6.2. Indeed, the isoperimetric control (6.2) entails that only sets with $\mathcal{H}^1(\partial\Omega)$ below a certain threshold (depending on F and m) are interesting for problem (6.4). In order to avoid trivial situations, we assume that F is not constant on the family of admissible sets, i.e., there exists Ω_0 with

$$F(\tilde{\sigma}_1(\Omega_0), \dots, \tilde{\sigma}_k(\Omega_0)) > F(0, \dots, 0). \tag{6.5}$$

The main result of the section is the following.

Theorem 6.4 (Existence of optimal domains). *Assume* (6.3) *and* (6.5)*. Then problem* (6.4) *has at least one solution which is bounded and given by the union of at most k disjoint Jordan domains whose closures intersect pairwise in at most one point. Moreover, if in addition F is strictly increasing in its arguments, every optimal set Ω_{opt} is bounded and it is contained in an optimal domain $\hat{\Omega}_{opt}$ (so $|\hat{\Omega}_{opt} \setminus \Omega_{opt}| = 0$) satisfying the previous properties.*

In order to prove Theorem 6.4, we need some preliminary work in order to construct a suitable maximizing sequence for problem (6.4). We start with the following observation which heavily depends on our two-dimensional setting.

Lemma 6.5. *Let $\Omega \subset \mathbb{R}^2$ be an admissible domain for problem* (6.4) *such that $\Omega \subseteq B_R(0)$. Then there exists $\hat{\Omega} \subseteq B_R(0)$ with $\Omega \subseteq \hat{\Omega}$ and $\partial\hat{\Omega} \subseteq \partial\Omega$, such that the following items hold true.*

(a) *$\hat{\Omega}$ is union of open simply connected sets.*
(b) $\hat{\Omega}^c = \overline{\mathrm{int}(\hat{\Omega}^c)}$.
(c) *For every $h \in \mathbb{N}$*
$$\tilde{\sigma}_h(\hat{\Omega}) \geq \tilde{\sigma}_h(\Omega).$$
(d) *If $\tilde{\sigma}_k(\Omega) > 0$, then $\partial\hat{\Omega}$ has at most k connected components.*

Proof. Let $U := \mathrm{int}(\Omega^c)$, and let us denote by V the unbounded connected component of U. Then $\partial V \subseteq \partial\Omega$. We define
$$\hat{\Omega} := \mathbb{R}^2 \setminus \overline{V}.$$
By construction we have
$$\Omega \subseteq \hat{\Omega} \subseteq B_R(0), \qquad \partial\hat{\Omega} \subseteq \partial\Omega \qquad \text{and} \qquad \hat{\Omega}^c = \overline{\mathrm{int}(\hat{\Omega}^c)}, \tag{6.6}$$
so that point (b) is proved, and in particular $\mathcal{H}^1(\partial\hat{\Omega}) < +\infty$. Moreover the connected components of $\hat{\Omega}$ are open sets which are simply connected, as their complement is readily seen to be connected, so that point (a) follows.

Thanks to (6.6) we have for every $h \in \mathbb{N}$
$$\tilde{\sigma}_h(\Omega) = \inf_{S \in \mathcal{S}_{h+1}} \max_{u \in S \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\mathcal{H}^{d-1}} \leq \inf_{S \in \mathcal{S}_{h+1}} \max_{u \in S \setminus \{0\}} \frac{\int_{\hat{\Omega}} |\nabla u|^2 dx}{\int_{\partial\hat{\Omega}} u^2 d\mathcal{H}^{d-1}} = \tilde{\sigma}_h(\hat{\Omega}). \tag{6.7}$$

In order to prove point (d), let us proceed as follows. Since $\partial\hat{\Omega}$ is compact, for every $n \geq 1$ we can find a finite number of open balls such that
$$\partial\hat{\Omega} \subseteq \bigcup_{j=1}^{k_n} B(x_j^n, r_j^n)$$
with $x_j^n \in \partial\hat{\Omega}_n$ and $r_j^n \leq \frac{1}{n}$. Let us set $U_n := \bigcup_{j=1}^{k_n} B(x_j^n, r_j^n)$. We claim that for n large enough
$$U_n := A_n^1 \cup A_n^2 \cup \cdots \cup A_n^k \tag{6.8}$$
where A_n^i is open and connected (possibly empty), i.e., U_n has at most k connected components. As a consequence, if we set
$$K_n := \bigcup_{j=1}^{k_n} \bar{B}(x_j^n, r_j^n),$$
we have that K_n has at most k connected components with
$$d_H(\partial\hat{\Omega}, K_n) \leq \frac{1}{n}, \tag{6.9}$$
where d_H is defined in (2.7). Since $K_n \to \partial\hat{\Omega}$ in the Hausdorff metric, we infer that $\partial\hat{\Omega}$ has at most k connected components, and point (d) follows.

In order to conclude the proof, we need only to check claim (6.8). This is a consequence of the fact that $\tilde{\sigma}_k(\hat{\Omega}) \geq \tilde{\sigma}_k(\Omega) > 0$. If indeed by contradiction U_n has more than k connected components, that is
$$U_n := A_n^1 \cup A_n^2 \cup \cdots \cup A_n^{m_n}$$
with $m_n > k$ (notice that the number of connected components is finite as U_n is given by the union of a finite number of balls), then up to reducing the radii we can assume that they are also well separated in $\mathbb{R}^2$. Then we can divide the connected components of $\hat{\Omega}$ in $m_n > k$ well separated groups : the i-th group is defined by collecting the connected components whose boundary is contained in A_n^i. Notice that the definition of the groups is well posed since the connected components of $\hat{\Omega}$ are simply connected by point (a): as a consequence, their boundaries are connected, and thus contained in at most one of the A_n^i. From point (d) of Lemma 6.2, we get $\tilde{\sigma}_k(\Omega) = 0$, a contradiction. The proof is now concluded. □

In the following lemma we exhibit a suitable maximizing sequence for problem (6.4) with additional geometric properties.

Lemma 6.6. *Assume* (6.3) *and* (6.5). *Then there exists a maximizing sequence* $(\Omega_n)_{n\in\mathbb{N}}$ *for problem* (6.4) *such that the following items hold true.*

(a) *There exists $\lambda > 0$ such that $\tilde{\sigma}_i(\Omega_n) > \lambda$ for every $n \in \mathbb{N}$ and $i = 1, \dots, k$. In particular*
$$\sup_n \mathcal{H}^1(\partial\Omega_n) < +\infty.$$
(b) *There exists $R > 0$ such that for every $n \in \mathbb{N}$*
$$\Omega_n \subset B_R(0).$$
(c) *For every $n \in \mathbb{N}$ the domain Ω_n satisfies points (a), (b) of Lemma 6.5, and $\partial\Omega_n$ has at most k connected components.*

Proof. In view of the assumptions on F, there exists some value $\lambda > 0$ such that for every $0 \le \lambda_i \le \lambda$, $i = 1, \dots, k$, we have
$$F(\tilde{\sigma}_1(\Omega_0), \dots, \tilde{\sigma}_k(\Omega_0)) > F(\lambda_1, \dots, \lambda_k).$$
If $(\Omega_n)_{n\in\mathbb{N}}$ is a maximizing sequence, we can thus assume $\tilde{\sigma}_i(\Omega_n) > \lambda$ for every $n \in \mathbb{N}$ and $i = 1, \dots, k$. The bound on $\mathcal{H}^1(\partial\Omega_n)$ is then a consequence of the isoperimetric inequality (6.2).

By Proposition 6.3, up to translating some connected components, the sequence $(\Omega_n)_{n\in\mathbb{N}}$ satisfies also point (b).

Let us apply Lemma 6.5 to the sequence $(\Omega_n)_{n\in\mathbb{N}}$, and rescale in order to recover the measure constraint: thanks to the assumptions on F, and since the rescaling operation increases the eigenvalues (see point (b) of Lemma 6.2), we obtain a new maximizing sequence of domains which satisfy in addition points (a) and (b) of Lemma 6.5. Finally, the bound on the number of connected components follows since the k-th eigenvalue is greater than $\lambda > 0$, so that the proof is concluded. □

We are now in a position to prove the main existence result of the section.

Proof of Theorem 6.4. Recall that if $(\Omega_n)_{n\in\mathbb{N}}$ is a sequence of open sets contained in a fixed bounded set $D \subset \mathbb{R}^2$, we say that
$$\Omega_n \to \Omega \qquad \text{in the Hausdorff complementary topology}$$
for some open set $\Omega \subseteq \mathbb{R}^2$ provided that $\Omega_n^c \to \Omega^c$ in the Hausdorff metric. Let us divide the proof in several steps.

Step 1: Compactness. Let $(\Omega_n)_{n\in\mathbb{N}}$ be the maximizing sequence given by Lemma 6.6. Then there exists an admissible domain such that up to a subsequence
$$\Omega_n \to \Omega \qquad \text{in the Hausdorff complementary topology,}$$
$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^2), \tag{6.10}$$
and
$$\partial\Omega_n \to K \qquad \text{in the Hausdorff metric,}$$
where the compact set K has at most k connected components with $\partial\Omega \subseteq K$ and $\mathcal{H}^1(K) < +\infty$. Indeed, up to a subsequence we have
$$\Omega_n \to \Omega \qquad \text{in the Hausdorff complementary topology}$$
and
$$\partial\Omega_n \to K \qquad \text{in the Hausdorff metric,}$$
where $\Omega \subset \mathbb{R}^2$ is open and bounded, and $K \subset \mathbb{R}^2$ is compact with at most k connected components. By the definition of Hausdorff complementary convergence we get easily
$$\partial\Omega \subseteq K \qquad \text{and} \qquad \overline{\Omega}_n \to \Omega \cup K \text{ in the Hausdorff metric.}$$

By Gołąb's theorem (see Theorem 2.5) we infer that

$$\mathcal{H}^1(K) \le \liminf_n \mathcal{H}^1(\partial\Omega_n)$$

so that $\mathcal{H}^1(K) < +\infty$, and in particular $\mathcal{H}^1(\partial\Omega) < +\infty$. Moreover, since by the Hausdorff complementary convergence

$$1_{\Omega_n} \to 1_\Omega \qquad \text{pointwise on } \mathbb{R}^2 \setminus K,$$

being $|K| = 0$ we infer also that

$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^2).$$

We deduce $|\Omega| = m$, so that Ω is an admissible domain for problem (6.4), and the conclusion follows.

Step 2: Upper semicontinuity and existence of an optimal domain. For every $h \in \mathbb{N}$ we have

$$\limsup_n \tilde{\sigma}_h(\Omega_n) \le \tilde{\sigma}_h(\Omega). \tag{6.11}$$

Indeed let $\varepsilon > 0$ and let $S_{h+1} = \text{span}\{u_0, \dots, u_h\} \subseteq H^1(\mathbb{R}^2)$ be an admissible subspace for the computation of $\tilde{\sigma}_h(\Omega)$ such that

$$\tilde{\sigma}_h(\Omega) \ge \max_{u \in S_{h+1} \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\mathcal{H}^1} - \varepsilon.$$

For each index n, assume that

$$u_n := \sum_{i=0}^h \alpha_i^n u_i$$

attains the maximum

$$\max_{u \in S_{h+1} \setminus \{0\}} \frac{\int_{\Omega_n} |\nabla u|^2 dx}{\int_{\partial\Omega_n} u^2 d\mathcal{H}^1}.$$

Without restricting the generality, we may assume that

$$\sum_{i=0}^h (\alpha_i^n)^2 = 1, \qquad \alpha_i^n \to \alpha_i.$$

Denoting $u := \sum_{i=0}^h \alpha_i u_i$ we have

$$u_n \to u \qquad \text{strongly in } H^1(\mathbb{R}^2).$$

In view of (6.10) we deduce

$$\lim_{n\to\infty} \int_{\Omega_n} |\nabla u_n|^2 dx = \int_\Omega |\nabla u|^2 dx.$$

Moreover, thanks to Proposition 2.6 and since $\partial\Omega \subseteq K$ we have

$$\liminf_{n\to\infty} \int_{\partial\Omega_n} u_n^2 d\mathcal{H}^1 \ge \int_K u^2 d\mathcal{H}^1 \ge \int_{\partial\Omega} u^2 d\mathcal{H}^1.$$

Notice that S_{h+1} is admissible for the computation of $\tilde{\sigma}_h(\Omega_n)$ for n large enough.We obtain

$$\limsup_{n\to\infty} \tilde{\sigma}_h(\Omega_n) \le \limsup_{n\to\infty} \frac{\int_{\Omega_n} |\nabla u_n|^2 dx}{\int_{\partial\Omega_n} u_n^2 d\mathcal{H}^{d-1}} \le \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\mathcal{H}^{d-1}} \le \tilde{\sigma}_h(\Omega) + \varepsilon.$$

Letting $\varepsilon \to 0$, inequality (6.11) follows.

Thanks to the assumptions on F we get

$$\limsup_{n\to\infty} F(\tilde{\sigma}_1(\Omega_n), \dots, \tilde{\sigma}_k(\Omega_n)) \le F(\tilde{\sigma}_1(\Omega), \dots, \tilde{\sigma}_k(\Omega)),$$

so that Ω is an optimal domain for problem (6.4).

We can apply Lemma 6.5 to Ω and rescale to get a new optimal bounded domain, still denoted by Ω, satisfying $\mathrm{int}(\Omega^c)$ connected, unbounded, with $\Omega^c = \overline{\mathrm{int}(\Omega^c)}$, and such that

$$\Omega = \bigcup_{n\in\mathbb{N}} A_n,$$

where A_n is simply connected. Moreover, $\partial\Omega$ has at most k-connected components.

Step 3: First properties of the optimal domain. Let Ω be the optimal domain given by Step 2.

Since ∂A_n is connected and with $\mathcal{H}^1(\partial A_n) < +\infty$, we deduce that ∂A_n is locally connected (see [10]). By [21, Theorem 2.1], we infer that there exists a conformal mapping

$$f : B_1(0) \to A_n \tag{6.12}$$

which admits a continuous extension to $\overline{B}_1(0)$ (with of course $f(\partial B_1(0)) \subseteq \partial A_n$).

We claim that the following items hold true.

(a) A_n is a Jordan domain, i.e., ∂A_n is a Jordan curve. In particular the function f in (6.12) admits an extension

$$f : \bar{B}_1(0) \to \bar{A}_n$$

which is a homeomorphism.

(b) $\partial A_i \cap \partial A_j$ consists of at most one point.

In order to prove point (a), let us check that ∂A_n has no *cut points*, i.e., points a such that $\partial A_n \setminus \{a\}$ is not connected: the property then follows by [21, Theorem 2.6]. By contradiction, let a be a cut point for ∂A_n. Then, thanks to [21, Proposition 2.5], $f^{-1}(a)$ contains at least two points x_1, x_2. Let $[x_1, x_2]$ be the segment in $\bar{B}_1(0)$ connecting x_1, x_2. Then $[x_1, x_2]$ divides the disk in two parts: let C be the one such that $f(C)$ is inside the loop $f([x_1, x_2])$ which intersects ∂A_n in a. We have that if l is the arc on $\partial B_1(0)$ bounding C with $[x_1, x_2]$, then $f(l) = \{a\}$. For, if $f(x) \neq a$ for some $x \in l$, then $f(x)$ would be a point of $\partial\Omega$ inside the loop $f([x_1, x_2])$. Being $\mathrm{int}(\Omega^c)$ connected, unbounded and with $\Omega^c = \overline{\mathrm{int}(\Omega^c)}$, there would be a curve in $\mathrm{int}(\Omega^c)$ with one extreme inside the loop (so inside A_n) and one extreme outside A_n. This curve should then cross the loop, which is impossible as its points are not in $\mathrm{int}(\Omega^c)$. We reach a contradiction since it is known that $f^{-1}(a)$ has zero measure (even zero capacity, see the comment after [21, Proposition 2.5]).

In order to prove point (b), let us assume by contradiction that $\partial A_i \cap \partial A_j$ contains two points a_1 and a_2. Let

$$f_i : \overline{B}_1(0) \to \overline{A}_i, \qquad f_j : \overline{B}_1(0) \to \overline{A}_j \tag{6.13}$$

be the associated homeomorphisms. Let $x_1, x_2 \in \partial B_1(0)$ and $y_1, y_2 \in \partial B_1(0)$ be the points associated to a_1, a_2 through f_i and f_j respectively. Reasoning as above, the loop

$$f_i([x_1, x_2]) \cup f_j([y_1, y_2])$$

contains points in $\mathrm{int}(\Omega^c)$. Again there would be a curve in $\mathrm{int}(\Omega^c)$ with one extreme inside the loop and one extreme outside: this curve should then cross the loop, which is impossible as its points are not in $\mathrm{int}(\Omega^c)$.

Step 4: Some properties of the boundary. Let us prove some topological properties concerning $\partial\Omega$, more precisely in connection with the boundaries of its connected components. We claim that the following property holds true.

Let A_i and A_j be two connected components of Ω. Let us consider the open set

$$N_\delta(\partial\Omega) := \bigcup_{n\in\mathbb{N}} B_{\delta_n}(x_n)$$

containing $\partial\Omega$, where $x_n \in \partial\Omega$, $\delta_n < \delta$ and

$$2\pi \sum_{n\in\mathbb{N}} \delta_n < 2(\mathcal{H}^1(\partial\Omega) + 1). \tag{6.14}$$

The existence of such a covering follows directly from the definition of Hausdorff measure. We claim that there exists a point $a \in \partial A_i$ such that for any given $\varepsilon > 0$ the following property holds true: for δ small enough, a connected component of $N_\delta(\partial\Omega) \setminus \overline{B}_\varepsilon(a)$ cannot touch both ∂A_i and ∂A_j.

Let us assume firstly that $\partial A_i \cap \partial A_j = \{a\}$ according to Step 2. Let us proceed by contradiction, assuming that for every $\delta > 0$ small enough, ∂A_i and ∂A_j both touch a connected component of $N_\delta(\partial\Omega) \setminus \overline{B}_\varepsilon(a)$. Thanks to (6.14), there exists a curve γ_δ joining ∂A_i and ∂A_j with length less than $2(\mathcal{H}^1(\partial\Omega) + 1)$ and contained in $\overline{N_\delta(\partial\Omega) \setminus \overline{B}_\varepsilon(a)}$. As $\delta \to 0$, we conclude for the existence of a curve γ with finite length in $\partial\Omega \setminus B_\varepsilon(a)$ joining ∂A_i and ∂A_j. We replace γ by a geodesic (still denoted by γ), so that we can assume that that γ intersects ∂A_i in a unique point a_i and ∂A_j in a unique point a_j, with $a_i, a_j \neq a$. We can now connect a_i to a inside A_i (through the image a of a cord via the conformal mapping (6.13)) and a to a_j inside A_j, creating with these two arcs and γ a Jordan curve in $\overline{\Omega}$. This curve cannot contain points of Ω^c, since this would be against the fact that $\Omega^c = \overline{\mathrm{int}(\Omega^c)}$ and $\mathrm{int}(\Omega^c)$ is connected. Then the curve is the boundary of an open connected set in Ω which intersects A_i and A_j, a contradiction.

Let us assume that $\partial A_i \cap \partial A_j = \emptyset$, and that they belong to the same connected component of $\partial\Omega$ (otherwise the result is trivial, without the need to choose a point a). Since $\partial\Omega$ has finite $\mathcal{H}^1$-measure, we get that its connected components are arcwise connected (see [14, Lemma 3.12]). This implies that there exists a curve γ in $\partial\Omega$ joining ∂A_i and ∂A_j, which we may assume to be a geodesic. Then we have that γ intersects ∂A_i in a unique point a. Let us assume by contradiction that for every $\delta > 0$, a connected component of $N_\delta(\partial\Omega) \setminus \overline{B}_\varepsilon(a)$ intersects both ∂A_i and ∂A_j. Reasoning as above, there exists a curve $\hat{\gamma}$ of finite length in $\partial\Omega \setminus B_\varepsilon(a)$ joining ∂A_i and ∂A_j, which again we may assume to be geodesic. Let $\hat{a}$ be the unique point of intersection of $\hat{\gamma}$ and ∂A_i. As before, we form a Jordan curve out of γ, $\hat{\gamma}$, ∂A_j and a simple arc in A_i connecting a and a_i. Again this curve encloses an open connected set contained in Ω, different from A_i and intersecting A_i, a contradiction.

Step 5: Bound on the number of connected components. Let us prove that $\Omega = \cup_{h\in\mathbb{N}} A_h$ admits at most k connected components, completing thus the proof of the properties of the optimal domain given by Step 2.

We proceed again by contradiction assuming that there are $A_1, \dots, A_{k+1}$ different connected components of Ω. For every $\eta > 0$ we will construct $\varphi_{i,\eta} \in H^1(\mathbb{R}^2)$ with $i = 1, \dots, k+1$ such that, setting $S^\eta_{k+1} := \mathrm{span}\{\varphi_{1,\eta}, \dots, \varphi_{k+1,\eta}\}$,

$$\lim_{\eta\to 0} \max_{\varphi \in S^\eta_{k+1}\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2\,dx}{\int_{\partial\Omega} \varphi^2\,d\mathcal{H}^1} = 0, \tag{6.15}$$

yielding $\tilde{\sigma}_k(\Omega) = 0$, against the fact that $\tilde{\sigma}_k(\Omega) > 0$ (see Step 3).

According to Step 4, there exists a finite number of points $a_1, \dots, a_m$ such that for every given fixed $\varepsilon > 0$, by choosing $\delta > 0$ small enough a connected component of the open set

$$N_\delta(\partial\Omega) \setminus (\overline{B}_\varepsilon(a_1) \cup \dots \cup \overline{B}_\varepsilon(a_m)) \tag{6.16}$$

cannot intersect both ∂A_i and ∂A_j for $i \neq j \in \{1, \dots, k+1\}$. As a consequence, a connected component of the open set

$$(\Omega \cup N_\delta(\partial\Omega)) \setminus (\overline{B}_\varepsilon(a_1) \cup \dots \cup \overline{B}_\varepsilon(a_m)) \tag{6.17}$$

cannot intersect both A_i and A_j for $i \neq j \in \{1, \dots, k+1\}$.

Let us now fix $\eta > 0$ small enough so that the balls $\{B_\eta(a_i) : i = 1, \dots, k+1\}$ are disjoint. Let us consider $\psi_\eta \in C^\infty(\mathbb{R}^2)$ such that

$$\psi_\eta = \begin{cases} 1 & \text{outside } B_\eta(a_1) \cup \dots \cup B_\eta(a_m), \\ 0 & \text{in } B_{\eta_1}(a_1) \cup \dots \cup B_{\eta_1}(a_m), \end{cases}$$

where $\eta_1 < \eta$ and

$$\lim_{\eta\to 0} \int_{\mathbb{R}^2} |\nabla\psi_\eta|^2\,dx = 0.$$

The function ψ_η is easily constructed through capacity arguments.

Let now choose $\varepsilon < \eta_1$, and let $\delta > 0$ be small enough. We consider the smooth function

$$\phi_{i,\eta} \in C^\infty((\Omega \cup N_\delta(\partial\Omega)) \setminus (\overline{B}_\varepsilon(a_1) \cup \cdots \cup \overline{B}_\varepsilon(a_m)))$$

which is equal to ψ_η on the connected components which intersect A_i, and zero on the others. We then extend $\phi_{i,\eta}$ to a smooth function on $(\Omega \cup N_\delta(\partial\Omega)) \cup (\overline{B}_\varepsilon(a_1) \cup \cdots \cup \overline{B}_\varepsilon(a_m))$ by setting it equal to zero on the balls. The restriction to Ω of this function is then the trace of a function $\varphi_{i,\eta} \in H^1(\mathbb{R}^2)$ for which

$$\lim_{\eta\to 0} \int_\Omega |\nabla \varphi_{i,\eta}|^2\, dx \le \lim_{\eta\to 0} \int_{\mathbb{R}^2} |\nabla \psi_\eta|^2\, dx = 0.$$

On the other hand, by construction

$$\liminf_{\eta\to 0} \int_{\partial\Omega} \varphi_{i,\eta}^2\, d\mathcal{H}^1 \ge \mathcal{H}^1(\partial A_i),$$

while for $i \neq j$ (since $\partial A_i \cap \partial A_j$ contains at most one point)

$$\int_{\partial\Omega} \varphi_{i,\eta}\varphi_{j,\eta}\, d\mathcal{H}^1 = 0.$$

We thus infer that (6.15) holds true, so that the step complete.

Step 6: Conclusion. Assume now in addition that F is strictly increasing with respect to its arguments, and let Ω_{opt} be an optimal domain. Then thanks to (6.5) we have $\tilde{\sigma}_k(\Omega_{opt}) > 0$, so that by Proposition 6.3 we deduce that Ω_{opt} is bounded. Let us apply Lemma 6.5 to Ω_{opt}, getting the new domain $\hat{\Omega}_{opt}$. Notice that $|\hat{\Omega}_{opt} \setminus \Omega_{opt}| = 0$, for otherwise we could rescale $\hat{\Omega}_{opt}$ to recover a new admissible domain satisfying the measure constraint on which the shape functional is strictly greater than on Ω_{opt}, a contradiction. The new domain satisfies the properties of Step 2, so that it is given by the union of at most k disjoint Jordan domains whose closures intersect pairwise in at most one point in view of Step 3 and Step 5. The proof of the theorem is now concluded. □

7. Numerical experiments

In this section we shall give some numerical approximations of the optimal shapes for several spectral functionals, in particular for the maximization of the k-th eigenvalue of the Steklov problem, $k = 1, 2, \ldots, 10$. This last question has been recently addressed in [6] and in [1]. In this section, we have a double purpose: on the one hand we shall consider more general spectral functionals and on the other hand we shall work with non-starshaped domains. Following our existence result we can consider more general functionals (like convex combinations of eigenvalues) and, in two dimensions, we can work with simply connected sets. Star-shapedness is not guaranteed, and for this reason we develop a parametric approach to handle this situation.

In the star-shaped case, it is classical to parametrize the domain by the radial function, which can be seen as a truncated Fourier series with a finite number of coefficients. Shape derivative formulas allow us to write the derivative with respect to each Fourier coefficient and thus, a gradient descent algorithm can be used. This method has been used successfully in spectral optimization in [1], [5], [19], etc.

As in general we do not have any theoretical guarantee that the optimizer lies in the class of the star shaped domains, we introduce a method which allows us to work directly in the class of simply connected two dimensional domains. We consider a general parametrization $\gamma : t \mapsto (\mathbf{x}(t), \mathbf{y}(t))$ for $t \in [0, 2\pi]$. The coordinate functions $\mathbf{x}, \mathbf{y}$ are supposed to be periodic of period 2π. Thus, these functions have the following Fourier series expansions

$$\mathbf{x}(t) = a_0 + \sum_{j=1}^\infty a_j \cos(j\theta) + \sum_{j=1}^\infty b_j \sin(j\theta)$$

$$\mathbf{y}(t) = c_0 + \sum_{j=1}^\infty c_j \cos(j\theta) + \sum_{j=1}^\infty d_j \sin(j\theta).$$

Supposing that the shape Ω bounded by the curve γ, which is regular enough, the coefficients $(a_j), (b_j), (c_j), (d_j)$ decay very rapidly to 0. Thus, we expect that truncating these Fourier series to their first coefficients up to a certain threshold, we don't lose much information on the shape Ω.

In case of variation of the shape by a vector field V, the general shape derivative formula for a simple Steklov eigenvalue provided in [12, Section E] is given by

$$\frac{d\sigma_k}{dV} = \int_{\partial\Omega} (|\nabla_\tau u_k|^2 - |\partial_n u_k|^2 - \sigma_k \mathcal{H}|u_k|^2) V.n \, d\sigma. \tag{7.1}$$

As a consequence, the derivatives of the Steklov eigenvalues with respect to the coefficients $(a_j), (b_j), (c_j), (d_j)$ are:

$$\begin{aligned}
\frac{d\sigma_k}{da_j} &= \int_0^{2\pi} (|\nabla_\tau u_k|^2 - (\partial_n u_k)^2 - \sigma_k \mathcal{H} u_k^2)\mathbf{y}'(\theta)\cos(j\theta)d\theta \\
\frac{d\sigma_k}{db_j} &= \int_0^{2\pi} (|\nabla_\tau u_k|^2 - (\partial_n u_k)^2 - \sigma_k \mathcal{H} u_k^2)\mathbf{y}'(\theta)\sin(j\theta)d\theta \\
\frac{d\sigma_k}{dc_j} &= -\int_0^{2\pi} (|\nabla_\tau u_k|^2 - (\partial_n u_k)^2 - \sigma_k \mathcal{H} u_k^2)\mathbf{x}'(\theta)\cos(j\theta)d\theta \\
\frac{d\sigma_k}{dd_j} &= -\int_0^{2\pi} (|\nabla_\tau u_k|^2 - (\partial_n u_k)^2 - \sigma_k \mathcal{H} u_k^2)\mathbf{x}'(\theta)\sin(j\theta)d\theta,
\end{aligned}$$

where u_k is $L^2(\partial\Omega)$ unit normalized eigenfunction associated to σ_k. All quantities containing the eigenfunction u_k in the above integrals are always evaluated in $(\mathbf{x}(\theta), \mathbf{y}(\theta))$.

We use a gradient descent algorithm implemented in Matlab and we make sure that the curve γ does not self intersect. This is checked at every iteration by testing if a fine polygonal discretization of $\partial\Omega$ has self intersections. In practice, for the maximization of the Steklov eigenvalues, the gradient descent algorithm with small step size prevents the curve γ from self-intersecting.

For the numerical computation of the Steklov spectrum in [6] the author uses fundamental solutions while in [1] the authors use a single layer potential method. The numerical results presented below were announced in the phd thesis [7] and use fundamental solutions (see [2]). The main idea is to consider functions which are harmonic inside the domain Ω and impose the boundary eigenvalue condition in a finite number of points chosen on $\partial\Omega$. We choose to work with linear combinations of radial harmonic functions with centers outside the domain Ω.

Given a general shape Ω, we consider a uniform discretization of its boundary $(x_i)_{i=1}^N$ and introduce the points (y_i) at fixed distance r from $\partial\Omega$ on the exterior normals in (x_i) to $\partial\Omega$. We consider the radial harmonic functions $\phi_i(x) = \log|x - y_i|$ and we search for functions u of the form

$$u = \alpha_1\phi_1 + ... + \alpha_N\phi_N,$$

which satisfy the boundary condition

$$\alpha_1\partial_n\phi_1(x_i) + ... + \alpha_N\partial_n\phi_N(x_i) = \sigma(\alpha_1\phi_1(x_i) + ... + \alpha_N\phi_N(x_i)),$$

for $i = 1...N$. This is a generalized eigenvalue problem and can be solved using the function `eigs` in Matlab, giving all the ingredients for the computation of the spectrum and of the derivative. The precision of this method is justified for regular domains in [6], where a comparison with mesh-based methods is provided. The following result is proved in [6].

Proposition 7.1. *Consider Ω a bounded, open, regular domain, and suppose that u_ε satisfies the following approximate eigenvalue problem:*

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega \\ \partial_n u_\varepsilon = \sigma_\varepsilon u_\varepsilon + f_\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{7.2}$$

Then if $\|f_\varepsilon\|_{L^2(\partial\Omega)}$ is small, there exists a constant C, depending on only on Ω, and a Steklov eigenvalue σ_k satisfying

$$\frac{|\sigma_\varepsilon - \sigma_k|}{\sigma_k} \le C\|f_\varepsilon\|_{L^2(\partial\Omega)}.$$

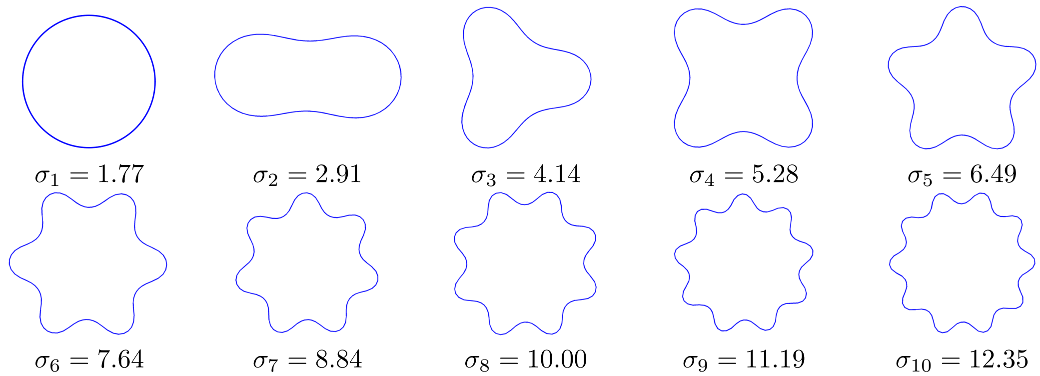

FIGURE 1. Shapes which maximize the k-th Steklov eigenvalue under area constraint, $k = 2, 3, ..., 10$.

We were able to check numerically the well known results concerning the maximization of simple quantities depending on the Steklov eigenvalues due to Weinstock [22], Brock [8], Hersch-Payhe-Schiffer [18]. We tested a wide range of optimization problems which gave rise to some numerical conjectures presented below (the area is assumed to be fixed and is numerically handled by considering the scale invariant quantities $\sigma_k(\Omega)|\Omega|^{\frac{1}{2}}$):

- $\min\left(\frac{1}{\sigma_1} + ... + \frac{1}{\sigma_n}\right)$ is realized by the disk;
- $\max \sigma_1, \max \sigma_1\sigma_2$ are realized by disks.
- **(Conjecture)** the maximizers of σ_k with area constraint are connected and have the symmetry of a regular k-gons. Furthermore, we observe that at the optimum the eigenvalues are multiple, the multiplicity cluster starts at k and has length 3 when k is odd and 2 when k is even. The numerical results for $k \in [2, 10]$ can be seen in Figure 1. Our results agree with those obtained in [1] with different methods.
- **(Conjecture)** the product $\sigma_1\sigma_2...\sigma_n$ is maximized by the disk.
- **(Conjecture)** We say that $A \subset \{0, 1, 2, 3, ...\}$ has the property (P) if $1 \in A$ and $2k \in A \Rightarrow 2k - 1 \in A$. If A has the property (P) then $\sum_{k\in A} \frac{1}{\sigma_k}$ is minimized by the disk. For example $\frac{1}{\sigma_1} + \frac{1}{\sigma_3} + \frac{1}{\sigma_4}$ is minimized by the disk. This fact was verified for various sets A having property (P) with $A \subset \{0, 1, ..., 15\}$.
- **(Conjecture)**, see [18]. $\sum_{k=1}^{n} \frac{1}{\sigma_{2k-1}\sigma_{2k}}$ is minimized by the disk.

In Figure 2 we find the optimal shapes for several functionals depending on the Steklov spectrum under area constraint. This supports the versatility of the numerical method and of the optimization procedure. Although most of the functionals considered are not of particular interest, we note a few facts concerning the minimizers of sums of Steklov eigenvalues. We see that $\sigma_1 + \sigma_2$ is maximized by a shape with two axes of symmetry. On the other hand, for $k \in [3, 10]$ we observe some numerical evidence that $\sigma_1 + ... + \sigma_k$ is maximized by the disk under area constraint.

8. APPENDIX: ISOPERIMETRIC CONTROL OF THE RELAXED SPECTRUM

In this section we reproduce the arguments of [11] to obtain the isoperimetric inequalities (5.1) and (6.2) for the relaxed Steklov eigenvalues considered in the paper. It is convenient to introduce

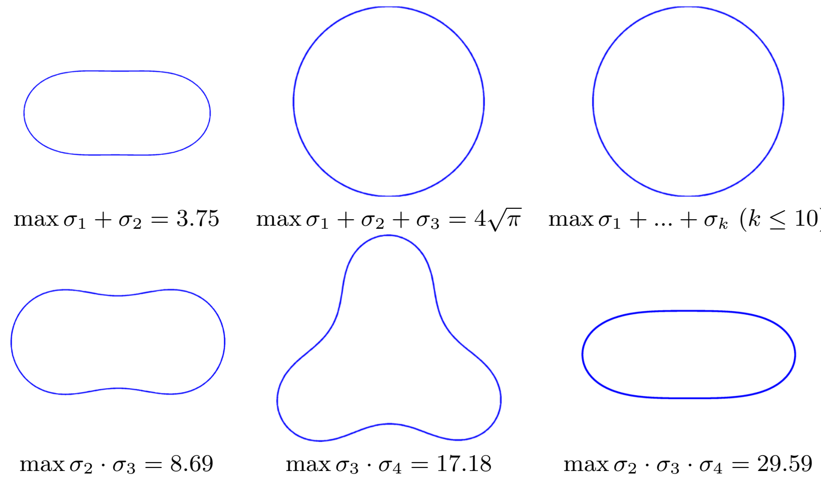

FIGURE 2. Several numerical optimizations of functionals depending on the Steklov spectrum

the following notation: for an annulus with center in $x_0 \in \mathbb{R}^d$

$$A := \{x \in \mathbb{R}^d \,:\, r_1 < |x - x_0| < r_2\},$$

we set

$$2A := \left\{x \in \mathbb{R}^d \,:\, \frac{r_1}{2} < |x - x_0| < 2r_2\right\}.$$

Proof of inequality (5.1). Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter with $|\Omega| = m$. In view of [16, Corollary 3.12] applied to the finite non atomic measure

$$\nu := \mathcal{H}^{d-1} \lfloor \partial^* \Omega,$$

there exists $2k+2$ annuli $A_1, \dots A_{2k+2}$ in $\mathbb{R}^d$ such that for every $i, j = 1, \dots, 2k+2$, $i \neq j$

$$\mathcal{H}^{d-1}(\partial^*\Omega \cap A_i) \geq \gamma_d \frac{\mathcal{H}^{d-1}(\partial^*\Omega)}{2k+2} \tag{8.1}$$

and

$$2A_i \cap 2A_j = \emptyset. \tag{8.2}$$

Here γ_d is a constant depending only on the dimension d of the space. We can reorder the annuli in such a way that

$$|\Omega \cap 2A_i| \leq \frac{|\Omega|}{k+1} \qquad \text{for } i = 1, \dots, k+1. \tag{8.3}$$

If $A_i = \{r_{1,i} < |x - x_i| < r_{2,i}\}$, let us set

$$h_i(x) := \begin{cases} \frac{1}{r_{2,i}} dist(x, \mathbb{R}^d \setminus (2A_i)) & \text{if } x \notin B_{r_{2,i}}(x_i), \\ 1 & \text{if } x \in A_i, \\ \frac{1}{r_{1,i}} dist(x, \mathbb{R}^d \setminus (2A_i)) & \text{if } x \in B_{r_{1,i}}(x_i). \end{cases}$$

Notice that $h_i \in H^1(\mathbb{R}^d)$ with $h_i = 0$ outside $2A_i$. Moreover $\nabla h_i = 0$ in A_i, with

$$|\nabla h_i| = \frac{1}{r_{2,i}} \qquad \text{in } B_{2r_{2,i}}(x_i) \setminus B_{r_{2,i}}(x_i)$$

and

$$|\nabla h_i| = \frac{1}{r_{1,i}} \qquad \text{in } B_{r_{1,i}}(x_i) \setminus B_{\frac{r_{1,i}}{2}}(x_i),$$

so that

$$\alpha_d := \int_{2A_i} |\nabla h_i|^d \, dx$$

is a constant depending only on the dimension d. In view of (8.1) and (8.3), using Hölder inequality and the isoperimetric inequality for sets of finite perimeter we obtain

$$\begin{aligned}
(8.4) \quad \frac{\int_\Omega |\nabla h_i|^2 \, dx}{\int_{\partial^*\Omega} h_i^2 \, dx} &\le \frac{\left(\int_{2A_i} |\nabla h_i|^d \, dx\right)^{\frac{2}{d}} |\Omega \cap (2A_i)|^{\frac{d-2}{d}}}{\mathcal{H}^{d-1}(A_i \cap \partial^*\Omega)} \le \frac{\alpha_d \left(\frac{|\Omega|}{k+1}\right)^{\frac{d-2}{d}}}{\gamma_d \frac{\mathcal{H}^{d-1}(\partial^*\Omega)}{2k+2}} \\
&\le \beta_d (k+1)^{\frac{2}{d}} \frac{|\Omega|^{\frac{d-2}{d}}}{\mathcal{H}^{d-1}(\partial^*\Omega)} \le C_d k^{\frac{2}{d}} \frac{\mathcal{H}^{d-1}(\partial^*\Omega)^{\frac{d-2}{d-1}}}{\mathcal{H}^{d-1}(\partial^*\Omega)} \\
&= C_d k^{\frac{2}{d}} \frac{1}{\mathcal{H}^{d-1}(\partial^*\Omega)^{\frac{1}{d-1}}},
\end{aligned}$$

where C_d depends only on d. Since the functions $h_1, \dots, h_{k+1}$ have disjoint supports in view of (8.2), by considering $S_{k+1} := \operatorname{span}\{h_1, \dots, h_{k+1}\} \subseteq H^1(\mathbb{R}^d)$ we deduce that

$$\tilde{\sigma}_k(\Omega) \le C_d \frac{k^{\frac{2}{d}}}{\mathcal{H}^{d-1}(\partial^*\Omega)^{\frac{1}{d-1}}},$$

so that the proof is concluded. □

Proof of the isoperimetric inequality (6.2). Let $\Omega \subset \mathbb{R}^2$ be an open set with $|\Omega| < +\infty$ and $\mathcal{H}^1(\partial\Omega) < +\infty$. In particular Ω has finite perimeter. We can follow step by step the proof of inequality (5.1) above considering the non atomic finite measure

$$\nu := \mathcal{H}^1 \lfloor \partial\Omega.$$

Notice that the isoperimetric inequality for sets of finite perimeter entails

$$|\Omega| \le \delta_2 \mathcal{H}^1(\partial^*\Omega)^2 \le \delta_2 \mathcal{H}^1(\partial\Omega)^2,$$

where $\delta_2 > 0$, being $\partial^*\Omega \subseteq \partial\Omega$. Then we can repeat the computations in (8.4) obtaining

$$\tilde{\sigma}_k(\Omega) \le C_2 \frac{k}{\mathcal{H}^1(\partial\Omega)},$$

so that the conclusion follows. □

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