Partitions of minimal length on surfaces

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LJK, Grenoble

Objective

Problem

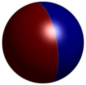
Find numerically partitions which minimize the total length of the boundaries under area constraints

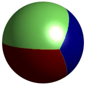
- efficient, flexible numerical method
- compare with existing results

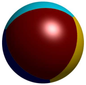
Practical motivation

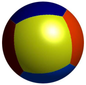
Minimize the cost of hand sewn balls

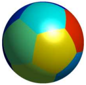
Known results


Bernstein 1905 - two half-spheres


Masters 1996 - the Y partition - angles $2\pi/3$


Engelstein 2009 - 4 triangles - the regular tetrahedron


open problem - 6 squares - the cube


Hales 2002 - the dodecahedron

Qualitative results

F. Morgan

The minimal partitions into cells of fixed areas exists and satisfies the following properties :

- the borders of the cells have constant geodesic curvatures
- the singular points are triple and satisfy the 120° condition.

Previous works

Cox, Flikkema 2010 - Evolver

- 2D partitions : equilateral triangle, square, pentagon, hexagon, circle $N \leq 42$
- spherical partitions $N \leq 32$.

Description of the method (sphere)

- evolution starting from triangles
- topology changes/random search
- for $n \geq 14$: enumerate ALL partitions of the sphere into pentagons and hexagons
- for each partition find the associated local minimum
- keep the candidate with the smallest length

Functional formulation - Euclidean case

$$F_\varepsilon(u) = \varepsilon \int_D |\nabla u|^2 + \frac{1}{\varepsilon}\int_D u^2(u-1)^2, \int_D u = \text{const.}$$

$$F_\varepsilon \xrightarrow{\Gamma} \frac{1}{3}\text{Per}()$$

for the L^1 topology.

The minimisers of F_ε converge towards the minimizers of Per at fixed area when $\varepsilon \to 0$.

Oudet 2011. Same computational region for every phase!

Kelvin's conjecture in 3D.

Pros and cons

Advantages:

- shape $\rightarrow$ function on a fixed domain
- fixed computation grid
- automatic treatment of singular points

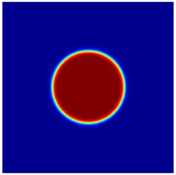
Weak points:

- approximate cost function
- optimal cost depends on ε
- large optimization problems

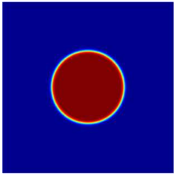
How does the method work?

$$\min_{|\Omega|=1/7} \text{Per}(\Omega).$$

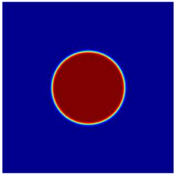
Analytical value: $2\sqrt{\pi/7} = 1.3398$



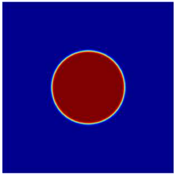
1.3216
$\varepsilon = 1/150$



1.3276
$\varepsilon = 1/200$



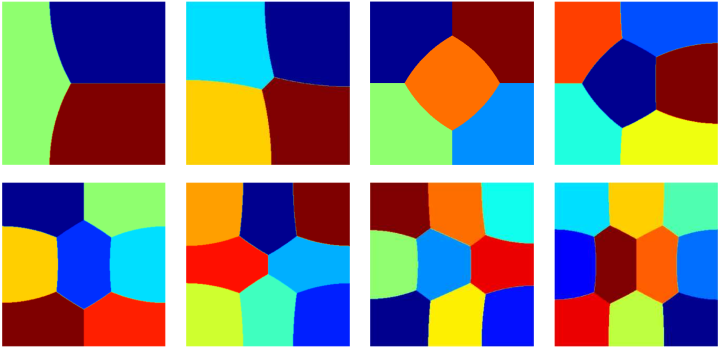
1.3311
$\varepsilon = 1/250$



1.3398
$\varepsilon = 1/300$

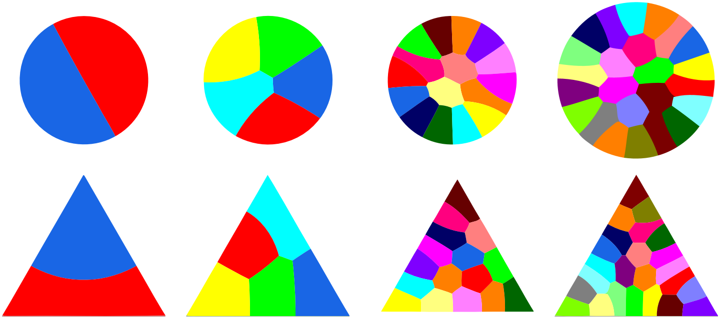
Some examples - 2D partitions

- Numerical method: finite differences
- quasi-Newton (LBFGS) optimization
- n+N constraints
- partition constraint: $\varphi_1 + ... + \varphi_n = 1$



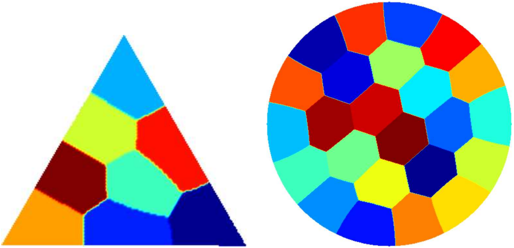
Non-rectangular domains

1. Finite differences - neglect points outside the domain
   - problems near boundary
   - needs high resolution



Non-rectangular domains

2. Finite elements
   - no problems near boundaries



Extend the method to surfaces?

$$\mathrm{Per}(\Omega) \approx \varepsilon \int_S |\nabla_\tau u|^2 + \frac{1}{\varepsilon} \int_S u^2(1-u)^2$$

Γ-convergence theorem ?

- BV spaces on surfaces (tangential divergence)

$$\mathrm{Per}(\omega) = \sup\{ \int_\omega \mathrm{div}_\tau\, g d\sigma : g \in C^1(S; \mathbb{R}^d), |g| \leq 1\} < +\infty$$

Γ-convergence theorem

$$F_\varepsilon(u) = \begin{cases} \int_S \left(\varepsilon|\nabla_\tau u|^2 + \frac{1}{\varepsilon}u^2(1-u)^2\right) & \text{if } u \in H^1(S) \\ +\infty & \text{otherwise} \end{cases}$$

$$F(u) = \begin{cases} \frac{1}{3}\operatorname{Per}(\omega) & \text{if } u = \chi_\omega \in BV(S) \\ +\infty & \text{otherwise} \end{cases}$$

$F_\varepsilon \xrightarrow{\Gamma} F$ for the $L^1(S)$ topology.

True also in the case of partitions

Numerical formulation

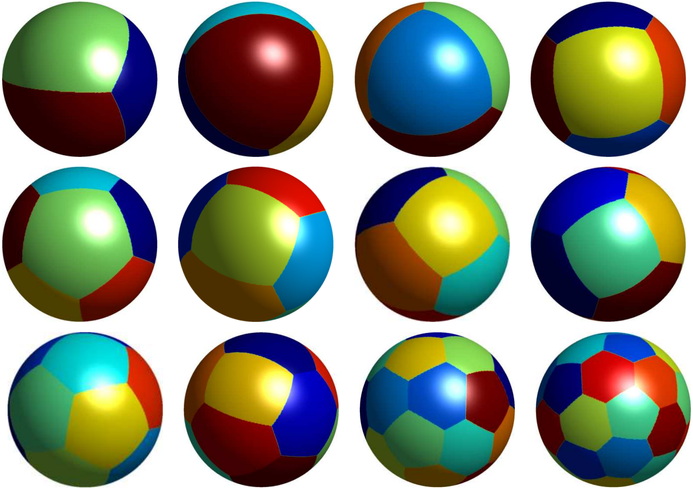
- P_1 finite elements $\to$ stiffness and mass matrices K, M.
- if $v = u(1-u)$ (point-wise multiplication)

$$\varepsilon \int_S |\nabla_\tau u|^2 + \frac{1}{\varepsilon}\int_S u^2(1-u)^2 = \varepsilon u^T K u + \frac{1}{\varepsilon} v^T M v$$

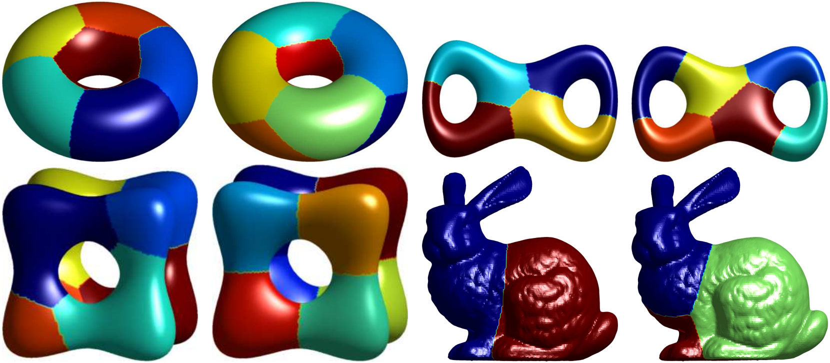
- quasi-Newton (LBFGS) algorithm ($5 \cdot 10^6$ dof)
- partition constraint : $u_1 + ... + u_n = 1$.
- fixed area constraints :

$$\int_S u_i = c \Leftrightarrow (1, 1, ..., 1) M u_i = c.$$

Results - the sphere



Other surfaces



Comparison Cox-Flikkema - spherical case

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$$\int_M K + \int_{\partial M} k_g + \sum \theta_i = 2\pi\chi(M)$$

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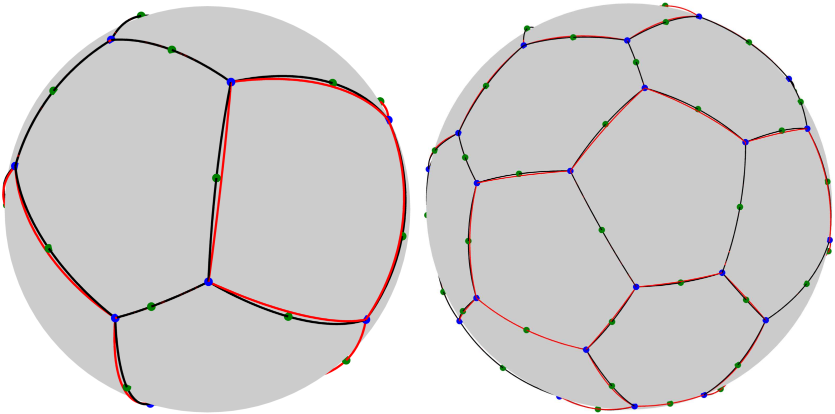
The boundaries are not all geodesics

Comparison Cox-Flikkema - spherical case

- relaxed cost - not precise enough
- extract polyhedral structure : triple points, edges, faces
- constant geodesic curvature $\to$ arcs of circles
- Gauss-Bonnet $\longrightarrow$ area computation
- treatment of the constraints

$$G_\varepsilon((\omega_i)) = \sum_{i=1}^{n} \mathrm{Per}(\omega_i) + \frac{1}{\varepsilon} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\mathrm{Area}(\omega_i) - \mathrm{Area}(\omega_j))^2.$$

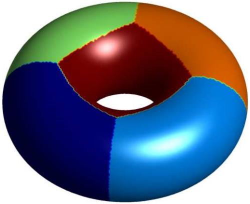
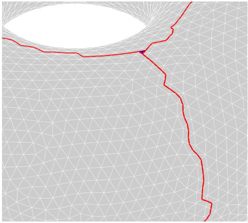
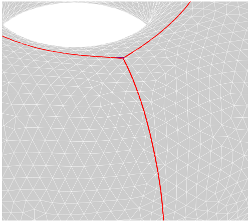
Two examples, $n = 9,\ 20$



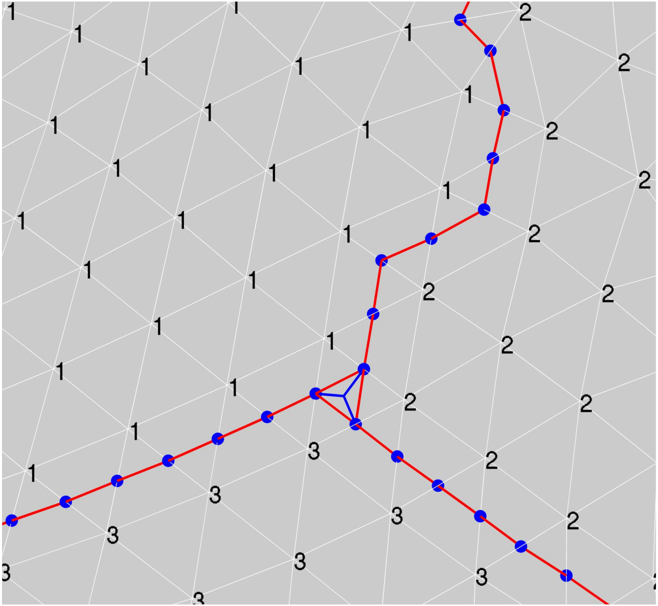
- Same results as Cox-Flikkema
- no need to search the polyhedral configuration
- one single optimization step $n \in [3, 24] \cup \{32\}$.
- a few tests for $n \in [25, 31]$

Cost computation - general surfaces

- extract the contours : $\omega_i \to u_i > \max_{j \neq i} u_j$
- optimization on the triangulated surface

Details



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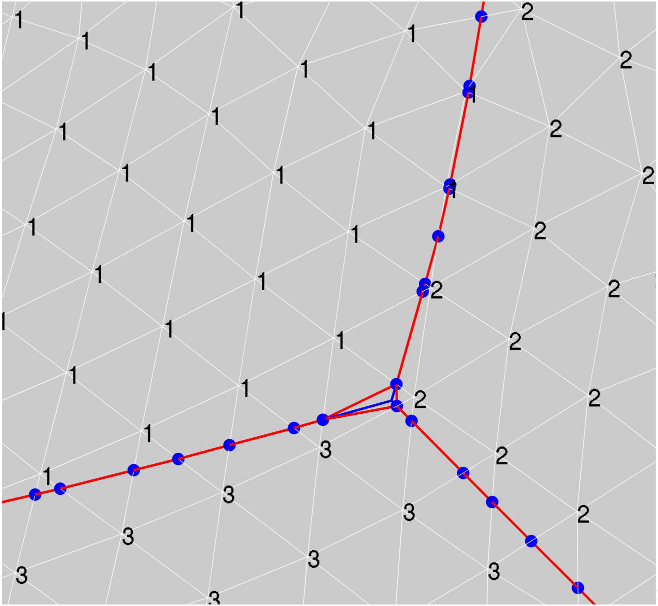

Future work

- asymptotic behavior - large number of cells
- other discretization techniques - spectral methods?
- understand Hales' proof for $n = 12$. see if it works for $n = 6$?

Thank you!