THE NONLOCAL ISOPERIMETRIC PROBLEM FOR POLYGONS: HARDY-LITTLEWOOD AND RIESZ INEQUALITIES

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Abstract. Given a non-increasing and radially symmetric kernel in $L^1_{\rm loc}(\mathbb{R}^2;\mathbb{R}_+)$, we investigate counterparts of the classical Hardy-Littlewood and Riesz inequalities when the class of admissible domains is the family of polygons with given area and N sides. The latter corresponds to study the polygonal isoperimetric problem in nonlocal version. We prove that, for every $N \geq 3$, the regular N-gon is optimal for Hardy-Littlewood inequality. Things go differently for Riesz inequality: while for $N = 3$ and $N = 4$ it is known that the regular triangle and the square are optimal, for $N \geq 5$ we prove that symmetry or symmetry breaking may occur (i.e. the regular N-gon may be optimal or not), depending on the value of N and on the choice of the kernel.

1. Introduction

Given a non-negative and non-increasing radially symmetric function h in $L^1_{\rm loc}(\mathbb{R}^d)$, called in the sequel an *admissible kernel*, for any measurable set $\Omega \subset \mathbb{R}^d$ let

$$J_h(\Omega) := \int_\Omega \int_\Omega h(x-y)\, dx\, dy\,. \tag{1}$$

The classical Riesz rearrangement inequality [42] states in particular that, denoting by Ω^* the ball with the same volume as Ω, it holds $J_h(\Omega) \leq J_h(\Omega^*)$. It is a natural question to ask whether symmetry is preserved when passing to the polygonal setting (in dimension $d = 2$): denoting by $\mathcal{P}_N$ the class of polygons with N sides and area for definiteness equal to π, and by Ω^*_N the regular gon in $\mathcal{P}_N$, this amounts to ask whether

$$\max\big\{J_h(\Omega)\ :\ \Omega \in \mathcal{P}_N\big\} = J_h(\Omega^*_N)\,. \tag{2}$$

As a general fact, transposing isoperimetric-type inequalities with balls as optimal domains into the setting of polygons with a fixed number of sides is indeed a very natural problem. It has been investigated both in the field of geometric measure theory and in the field of mathematical physics, with drastically different levels of difficulties.

For the isoperimetric inequality in geometric measure theory solved by De Giorgi in [20], i.e. the minimization of perimeter under volume constraint, the polygonal version

$$\mathrm{Per}\,(\Omega) \geq \mathrm{Per}(\Omega^*_N) \qquad \forall \Omega \in \mathcal{P}_N \tag{3}$$

is an elementary result, which can be found in several textbooks in convex geometry.

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On the other hand, for isoperimetric-type inequalities in mathematical physics, such as Saint-Venant of Faber-Krahn inequalities (for which we refer to the classical monograph [39] and to the recent book [30]), the polygonal versions

$$\lambda_1(\Omega) \geq \lambda_1(\Omega_N^*) \qquad \text{and} \qquad \tau(\Omega) \leq \tau(\Omega_N^*) \tag{4}$$

are conjectures formulated several decades ago by Pólya-Szegö, who also proved them for $N = 3$ and $N = 4$. Here $\lambda_1(\Omega)$ is the principal eigenvalue of the Dirichlet Laplacian in Ω, while $\tau(\Omega)$ is the torsional rigidity of Ω (namely the $L^1(\Omega)$-norm of the unique solution to the equation $-\Delta u = 1$ in $H_0^1(\Omega)$). The inequalities analogue to (4) have been proved for every $N \geq 3$ for the logarithmic capacity [44] and for the Cheeger constant [10] (for related results, see also [5, 11, 32]). At present, (4) are open for any $N \geq 5$, and they can be included among the major open problems in shape optimization. Their validity is also related to a conjecture by Caffarelli and Lin [14], about the asymptotical optimality of the hexagonal honeycomb for related optimal partition problems, see [13].

To some extent, Riesz inequality can be viewed as a kind of "bridge" between the classical isoperimetric inequality (3) and the physical inequalities (4), as it is intimately connected to each of them. It is useful to briefly explain these connections, before introducing our results.

The relation between Riesz inequality and the classical isoperimetric inequality is easily individuated. Indeed, the functional J_h in (1) differs just by a change of sign and a translation from the *nonlocal h-perimeter*, defined by

$$P_h(\Omega) := \int_\Omega \int_{\Omega^c} h(x-y)\, dx\, dy\,, \tag{5}$$

where the quantity $h(x-y)$ is interpreted as an interaction density between two points $x \in \Omega$ and $y \in \Omega^c := \mathbb{R}^d \setminus \Omega$. A suitable scaling of h in (5) allows to recover the usual notion of perimeter via an asymptotic formula. The concept of nonlocal perimeter has been first introduced in [7], and in recent times it has been widely developed, especially concerning the fractional kernel $h(x) = |x|^{-(d+s)}$, $s \in (0,1)$, and concerning bounded integrable kernels (see respectively the seminal papers [15, 16] and the recent monograph [36]). In particular, the optimality of balls in the nonlocal isoperimetric inequality has been proved in the fractional case [26, 25, 22, 35], for kernels which are not radially symmetric and decreasing [19], and for some Minkowski type nonlocal perimeters [18]. In this perspective, (2) is equivalent to

$$\min \left\{ P_h(\Omega) \ : \ \Omega \in \mathcal{P}_N \right\} = P_h(\Omega_N^*)\,.$$

To sketch the relation between the polygonal Riesz inequality and the physical isoperimetric inequalities (4), let us focus for simplicity on the Saint-Venant inequality. It is known that the torsional rigidity satisfies (see [17, Section 4])

$$\tau(\Omega) = \int_0^{+\infty} \Big(\lim_{m\to+\infty} \int_\Omega \cdots \int_\Omega \prod_{i=1}^m p_{t/m}(x_i - x_{i-1}) dx_0 \ldots dx_m \Big) dt\,, \tag{6}$$

where p_t denotes the heat kernel

$$p_t(x) = \frac{e^{-\frac{|x|^2}{4t}}}{4\pi t}\,. \tag{7}$$

A probabilistic reformulation in the same vein can be given also for the principal frequency, see again [17, Section 4]. Then the classical Saint-Venant and Faber-Krahn inequalities, with the ball as optimal domain, can be obtained as a consequence of the general rearrangement inequality for multiple integrals due to Brascamp-Lieb-Luttinger [8, Theorem 1.2], which in 2-dimensions reads

$$\int_{\Omega}\dots\int_{\Omega}\prod_{i=1}^{k} h_i\Big(\sum_{j=1}^{m} a_{ij}x_j\Big)\,dx_1\dots dx_m \le \int_{\Omega^*}\dots\int_{\Omega^*}\prod_{i=1}^{k} h_i^*\Big(\sum_{j=1}^{m} a_{ij}x_j\Big)\,dx_1\dots dx_m\,; \tag{8}$$

here h_i are measurable non-negative functions on $\mathbb{R}^2$ vanishing at infinity, h_i^* are their symmetric decreasing rearrangements, $\{a_{ij}\}$ are real numbers, Ω is a set of finite Lebesgue measure in $\mathbb{R}^2$, and Ω^* is the ball with the same area as Ω.

Hence, a possible approach to Pólya-Sezgö conjectures (4) would be to prove a polygonal version of the result by Brascamp-Lieb-Luttinger, stating that the inequality (8) remains true when the integrals at the left hand side are extended to a polygon $\Omega \in \mathcal{P}_N$, and the integrals at the right hand side are extended to Ω_N^*. Of course one has to start from small values of m. For $m = 2$ (and a suitable choice of h_i and $\{a_{ij}\}$), this corresponds exactly to study the Riesz inequality (10). But the problem turns out to be nontrival even for $m = 1$. Indeed in this case it amounts investigate the validity of the following polygonal version of the classical Hardy-Littlewood inequality (see [28, Chapter 10])

$$\int_{\Omega} h(x)\,dx \le \int_{\Omega_N^*} h(x)\,dx \qquad \forall \Omega \in \mathcal{P}_N\,. \tag{9}$$

Aim of this paper is to attack the polygonal Hardy-Littlewood inequality (9) and the polygonal Riesz inequality

$$J_h(\Omega) \le J_h(\Omega_N^*)\,, \text{ or equivalently } P_h(\Omega) \ge P_h(\Omega_N^*) \qquad \forall \Omega \in \mathcal{P}_N\,. \tag{10}$$

To the best of our knowledge, very few results are available in this respect in the literature. Concerning the inequality (9), within the restricted setting of convex polygons, and for particular kernels, is mentioned as an open question by Fejes-Tóth in [21]. Still in the restricted setting of convex polygons, and for the kernel $h(x) = |x|$, the inequality is proved in [37]. Concerning the inequality (10), for $N = 3, 4$ (triangle and quadrilaterals), though not explicitly stated it can be deduced via Steiner symmetrization from Lemma 3.2 in [8]. More recently, in [6] Bonacini, Cristoferi and Topaloglu have considered the strictly related problem of characterizing "critical" triangles and quadrilaterals. More precisely, a polygon $\Omega \in \mathcal{P}_N$ is said to be critical for J_h if, for some positive constant c, it holds

$$\frac{d}{d\varepsilon} J_h(\Omega_\varepsilon)\Big|_{\varepsilon=0} = c\frac{d}{d\varepsilon}|\Omega_\varepsilon|\Big|_{\varepsilon=0} \tag{11}$$

whenever $\{\Omega_\varepsilon\}$ are obtained from Ω by one of the following elementary movements: either the rotation of one side with respect to its midpoint, or the parallel translation of one side with respect to itself (see the Appendix in Section 9 for the detailed definitions). Clearly, a polygon maximizing J_h over $\mathcal{P}_N$ must be a critical polygon. In [6] it is proved that, under some weak assumptions on h, the regular triangle and the square are respectively the unique critical triangle and quadrilateral. It is also conjectured that the same rigidity property holds true for any $N \geq 5$ and that, consequently, the inequality (10) remains true for every $N \geq 5$.

In this paper we prove that the polygonal Hardy-Littlewood inequality (9) holds true for every admissible kernel h and without any convexity assumption on the admissible polygons (see Theorem 6). The situation concerning the polygonal Riesz inequality (10) is more delicate, because a key point turns out to be the choice of the kernel. In this respect, the above discussion motivates the assertion that the heat kernel is of special relevance. We point out that, for $h = p_t$, the corresponding functional $J_h(\Omega)$, namely the quantity

$$Q_\Omega(t) := \int_\Omega \int_\Omega p_t(x-y)dxdy \tag{12}$$

is the so-called *heat content* of the set Ω at time t. It represents the quantity of heat kept by the set Ω once it is warmed at constant temperature 1 and its heat is left to diffuse in the plane. We refer to [40] for related isoperimetric properties and to [45, 46] for related asymptotic expansions in the polygonal setting. A natural heuristic way to investigate the validity of the polygonal inequality (10) for $h = p_t$, with t sufficiently small or t sufficiently large, consists in looking at the limiting behaviours of $Q_\Omega(t)$ as $t \to 0^+$ and as $t \to +\infty$. In the limit as $t \to 0^+$, by [46, Theorem 1] for any polygon Ω it holds

$$Q_\Omega(t) = |\Omega| - \frac{2}{\sqrt{\pi}} |\partial\Omega| t^{\frac{1}{2}} + O(t)\,,$$

where the $O(t)$ term can be expressed in terms of the inner angles of Ω. Such asymptotic expansion, combined with the classical isoperimetric inequality (3), suggests that inequality (10) should hold when $h = p_t$ with t sufficiently small. On the other hand, in the limit as $t \to +\infty$, starting from the asymptotics of p_t and looking at the leading term, we obtain

$$4\pi t Q_\Omega(t) = |\Omega|^2 - \frac{1}{4t} \int_\Omega \int_\Omega |x-y|^2 \, dx \, dy + O\Big(\frac{1}{t^2}\Big)\,,$$

so that the inequality (10) should hold for $h = p_t$ with t sufficiently large provided it holds for the quadratic kernel $h(x) = |x|^2$.

This observation drew our attention to study the inequalities (10) for the quadratic kernel $h(x) = |x|^2$, and more generally for power type kernels

$$h(x) = |x|^k\,, \qquad \text{with } k > 0\,. \tag{13}$$

Notice carefully that such kernels are not admissible in the sense specified at the beginning of the paper, since they are increasing. Thus one should write inequalities (10) for $h = M - |x|^k$, with the constant $M = M(\Omega)$ chosen so large that $h \geq 0$ on Ω. Equivalently, this amounts to rewrite the reverse inequalities of (10) for $h = |x|^k$. Another family of "simple" kernels, which are of natural interest since their linear combinations can serve to approximate any smooth admissible kernel, are those of the form

$$h(x) = \chi_{B_r(0)}(x) \tag{14}$$

that we call in the sequel *characteristic kernels*.

Our results about Riesz polygonal inequality are mainly focused on the two families of kernels in (14) and in (13). For characteristic kernels we prove that, when r is small enough, inequality (10) holds (see Theorem 1); indeed, in this case we also have a rigidity result characterizing the regular N-gon as the unique critical polygon (see Theorem 3). On the other hand, when r is large enough, a big surprise is coming: for N even, $N \geq 6$, the inequality turns out to be false! See Theorem 2. In particular, the above mentioned conjecture made in [6] is, in general, false. The heuristic reason is that, for r large enough, our problem is equivalent to the problem of finding so-called "largest small N-gons", namely polygons with fixed diameter and maximal area. This is a challenging problem in discrete geometry, for which it is known that symmetry breaking occurs for any $N \geq 6$ even, see Section 2 for more details.

For power-type kernels, by using our polygonal Hardy-Littlewood inequality, we prove that the inequality (10) holds true for $k = 2$ and $k = 4$ (see Theorem 5). The same strategy fails for non-integers k, for odd integers k, as well as for higher even integers k. In fact, as a consequence of Theorem 2, the inequality (10) turns out to be false also for power-type kernels with sufficiently high exponent (still for $N \geq 6$ even).

The conclusion which stems from our analysis is that optimal polygons for the nonlocal isoperimetric inequality turn out to be sensitive to the choice of the kernel h, as it may produce either symmetry or symmetry breaking. This is a highly unexpected phenomenon, which makes the study of the nonlocal isoperimetric inequality quite intriguing: indeed, in the light of our results, the problem becomes to understand which are specifically the kernels yielding symmetry for every $N \geq 3$. We suspect that this is the case for the heat kernel; we don't have a proof of this fact, but we give some affirmative numerical results for polynomial approximations of p_t (see Section 3).

Outline of the paper. The paper is organized as follows: in Section 2 we state our main results, complemented with some comments on their proofs and a list of related open problems; in Section 3 we explore numerically some of these open problems; in the subsequent sections we give the proofs of the results stated in Section 2; finally in the Appendix we provide some first and second order shape derivatives which are used in the proofs.

2. Main results

Our main results about the nonlocal polygonal isoperimetric inequality for characteristic kernels read as follows. Below, when $h = \chi_{B_r(0)}$, we set for brevity

$$P_r(\Omega) := \int_\Omega \int_{\Omega^c} \chi_{B_r(0)}(x-y)\, dx\, dy \tag{15}$$

Theorem 1 (symmetry). *For every $N \geq 3$, there exists $r' = r'(N) > 0$ such that*

$$\forall r \leq r' , \quad \min \Big\{ P_r(\Omega) \ : \ \Omega \in \mathcal{P}_N \Big\} = P_r(\Omega_N^*) \, . \tag{16}$$

The fact that r' above is in general finite is related to the result of Reinhardt asserting that, when $N \geq 6$ is even, the regular N-gon is *not* a minimizer of the diameter under an area constraint [41, 43]. In fact we have the following:

Theorem 2 (symmetry breaking)**.** *For every N even, $N \geq 6$, setting $r'' = r''(N) := \min\{\operatorname{diam}\Omega \,:\, \Omega \in \mathcal{P}_N\}$, we have*

$$\forall r \in [r'', \operatorname{diam}\Omega_N^*)\,, \quad \min\Big\{P_r(\Omega) \;:\; \Omega \in \mathcal{P}_N\Big\} < P_r(\Omega_N^*)\,. \tag{17}$$

Moreover, still for $r \in [r'', \operatorname{diam}\Omega_N^*)$ the minimum in (17) *equals $\pi^2(r^2-1)$, and it is attained at a polygon $\Omega_N^\sharp$ with diameter r'', which is not the regular polygon.*

Comparing Theorems 1 and 2 shows that, for characteristic kernels, the optimality of the regular N-gon does depend on the value of r (at least for $N \geq 6$ even). This brings the study of the nonlocal isoperimetric inequality for polygons into the more complex perspective of understanding for which N and for which kernels symmetry or symmetry breaking occurs.

Before stating some partial results in that direction, let us give some short comments about the proofs of Theorems 1 and 2.

The proof of Theorem 1 is performed in a first stage for convex polygons and then it is extended to the general case. For convex polygons, we argue by contradiction. The idea is that, in the regime of a small r, if a convex polygon Ω minimizes the r-perimeter over $\mathcal{P}_N$ and it has r-perimeter strictly smaller than Ω_N^*, then Ω must be close to Ω_N^* (this follows from a uniform asymptotic estimate for the r-perimeter as $r \to 0$, where the classical perimeter appears in the leading term). In particular, close to Ω_N^* there would be a r-critical polygon, that is a polygon satisfying the stationarity condition (11), for $h = \chi_{B_r(0)}$. When r is sufficiently small, this is not possible thanks to a symmetry result for critical polygons that we state separately in Theorem 3 below, since it may have an independent interest. The second part of the proof dealing with arbitrary polygons requires some more refined arguments, in particular since minimizing sequences may converge to a "generalized polygon" (precisely in the sense of Definition 12), possibly containing self-intersections in its boundary. Roughly speaking, the idea is to reduce the problem to a situation similar to the convex setting: this is achieved by exploiting triangulations in order to identify local concentrations of mass, and by localizing our estimates near the sides of the limit polygon where there is no accumulation of vertices. We refer to Section 4 for the detailed proof.

Let us now state the afore mentioned symmetry result for r-critical polygons. To that aim, it is convenient to reformulate more explicitly the shape derivative in the left hand side of (11). This has been done in [6], but to make the paper self-contained we enclose a proof in the Appendix, see Lemma 19. The outcome is the following: if $\{\Omega_\varepsilon\}$ are obtained from Ω respectively by rotating the side $[A_iA_{i+1}]$ with respect to its midpoint M_i, or by a parallel movement of such side with respect to itself, the stationarity condition (11) amounts to ask that, setting $v_\Omega(x) := \int_\Omega h(x-y)dy$, it holds

$$\int_{A_i}^{M_i} v_\Omega(x)\,|xM_i|\,dx - \int_{M_i}^{A_{i+1}} v_\Omega(x)\,|xM_i|\,dx = 0 \tag{18}$$

$$\int_{A_i}^{A_{i+1}} v_\Omega(x)\,dx = c\,\mathcal{H}^1([A_iA_{i+1}])\,. \tag{19}$$

The result below states that the validity of eqs (18)-(19) for $i = 1, \dots, N$ enforces symmetry, provided the support of the kernel is small enough. It can be viewed as a polygonal version of the Alexandrov-type symmetry recently proved in [12, Corollary 7]. The proof is obtained by using a reflection argument which is reminiscent of [6, 24].

Theorem 3 (symmetry for r-critical polygons)**.** *Let h be an admissible kernel and let $\Omega \in \mathcal{P}_N$ satisfy equations* (18)-(19) *for every $i = 1, \dots, N$. When $N > 3$, assume further that* $\operatorname{spt}(h) \subseteq B_r(0)$*, with r such that*

(20) $\partial\Omega \cap B_r(x)$ *is contained into a pair of consecutive sides of* $\Omega \qquad \forall x \in \partial\Omega\,.$

Then Ω is a regular N-gon.

The proof of Theorem 2 is obtained in a completely different way; indeed, it follows as a rather straightforward consequence of a result by Reinhardt asserting that, when $N \geq 6$ is even, the regular N-gon is *not* a minimizer of the diameter under an area constraint (while, for N odd, the regular N-gon is a minimizer), see [41, 43] and the expository paper [38]. In the particular case $N = 6$, the optimal hexagon $\Omega_6^\sharp$ was found by Graham [27, 4], see Figure 1. Its construction can be done as follows. First fix two points $A(0,0)$ and $D(0,-1)$ at distance one, and then, denoting $c = d - b$ we determine the other four vertices $B(-0.5, c)$, $C(-x, b)$, $E(x, b)$, and $F(0.5, c)$: taking x as a parameter, b and d are found from the relations $x^2 + b^2 = 1$ and $(x + 0.5)^2 + d^2 = 1$. For $b = 0.939053346$ and $d = 0.536702650$, a numerical value for x is given by $x = 0.343771453$. [1]

Denoting by H_G and H_R suitable scalings of Graham and regular hexagons, we have:

- At *fixed diameter*, the area of the Graham hexagon is greater than the area of the regular one, the ratio of the areas being $\frac{\text{Area}(H_G)}{\text{Area}(H_R)} = 1.039201$.
- At *fixed area*, the diameter of the Graham hexagon is smaller than the diameter of the regular one and the ratio of their diameters is $\frac{\text{diam}(H_G)}{\text{diam}(H_R)} = 0.980957$.

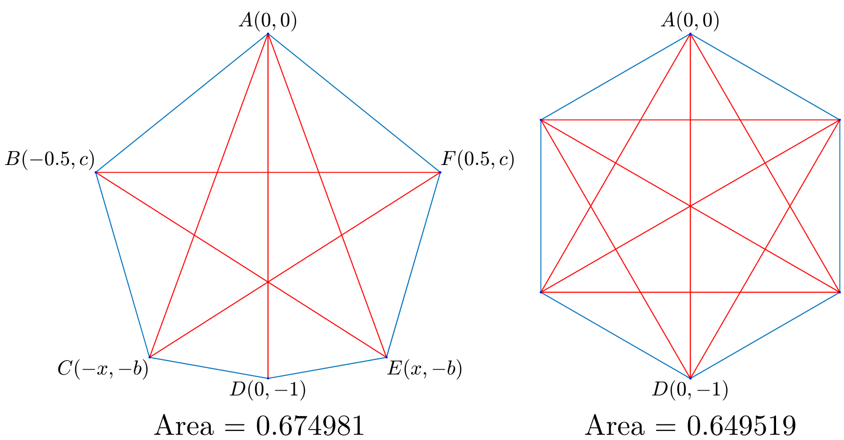

FIGURE 1. The Graham hexagon (left) and the regular hexagon (right) with unit diameter.

[1] The construction is taken from the MathWorld page `https://mathworld.wolfram.com/GrahamsBiggestLittleHexagon.html`

For $N > 6$ even, the determination of a polygon minimizing the diameter under an area constraint is a challenging problem in discrete geometry, which remains, to the best of our knowledge, open. Let us just mention that, among equilateral polygons, it has been recently proved that the regular polygon is optimal [1]. Among arbitrary polygons, it has been proved in [23] that for every N even the optimal polygon enjoys the following property conjectured by Graham: its skeleton (namely the collection of diameters connecting any two vertices) is not an asterisk but consists of a $(N-1)$-cycle and one additional edge. Moreover, for $N = 8$, very accurate numerical solutions have been proposed in [2] and [29]. We also refer to the recent paper [3] for more recent symbolic calculations and for further bibliography.

We now come back to the problem of minimizing the h-perimeter over $\mathcal{P}_N$. We are going to focus on some specific kernels which are not characteristic functions. In this respect, let us point out that the phenomenon of symmetry breaking is not a prerogative of characteristic kernels. Indeed from Theorem 2 and keeping the same notation as in its statement, we easily obtain the following

Corollary 4. *Let N even, $N \geq 6$.*

(i) *If h is a smooth admissible kernel close enough to $\chi_{B_r}(0)$, with $r \in [r'', \operatorname{diam} \Omega_N^*)$, symmetry breaking occurs, i.e. $P_h(\Omega_N^\sharp) < P_h(\Omega_N^*)$.*

(ii) *If $h(x) = |x|^k$, there exists $\overline{k}$ such that symmetry breaking occurs for every $k \geq \overline{k}$, i.e. $P_h(\Omega_N^\sharp) > P_h(\Omega_N^*)$. In particular, for $N = 6$ we have $\overline{k} \leq 2832$.*

Regarding statement (ii) above recall that, since power-type kernels $h(x) = |x|^k$ with $k > 0$ are increasing, the corresponding inequalities (10) must be reversed. Dealing with such kernels, in view of Corollary 4, the question becomes whether the regular polygon is a maximizer of the nonlocal perimeter at least for small k. We show that the answer is affirmative in the particular cases $k = 2$ and $k = 4$:

Theorem 5 (symmetry for power-type kernels)**.** *Let $h(x) = |x|^2$ or $h(x) = |x|^4$. For every $N \geq 3$, the regular poygon Ω_N^* maximizes P_h over $\mathcal{P}_N$. In equivalent terms, we have the following polygonal Riesz inequalities*

$$\int_\Omega \int_\Omega |x-y|^2 \, dx \, dy \geq \int_{\Omega_N^*} \int_{\Omega_N^*} |x-y|^2 \, dx \, dy \,, \qquad \forall \Omega \in \mathcal{P}_N \,,\ N \geq 3 \,; \tag{21}$$

$$\int_\Omega \int_\Omega |x-y|^4 \, dx \, dy \geq \int_{\Omega_N^*} \int_{\Omega_N^*} |x-y|^4 \, dx \, dy \,, \qquad \forall \Omega \in \mathcal{P}_N \,,\ N \geq 3 \,. \tag{22}$$

The proof of Theorem 5 relies on the idea to reduce the study of inequalities 21-(22) to the study of Hardy-Littlewood type polygonal inequalities. More precisely, by writing explicitly the polynomials $|x-y|^2$ and $|x-y|^4$, it turns out that the minimization of their double integral over $\Omega \times \Omega$ is equivalent to the minimization of the single integrals $\int_\Omega |x|^2$ and $\int_\Omega |x|^4$. We are thus led to the following question: is it true that Ω_N^* minimizes over $\mathcal{P}_N$ an integral functional of the type $\int_\Omega |x|^k$? More in general, for any admissible kernel h, one is led to investigate the following Hardy-Littlewood type maximization problem

$$\max \Big\{ \int_\Omega h(x) \, dx \ : \ \Omega \in \mathcal{P}_N \Big\} \,. \tag{23}$$

As mentioned in the Introduction, in the restricted setting of convex polygons, problem (23) is mentioned as an open question by Fejes Tóth in [21] when h is a characteristic kernel. In this case, it amounts to solve the following purely geometric problem: find the polygon in $\mathcal{P}_N$ which maximizes the overlap with the ball $B_r(0)$. Despite its elementary formulation, the solution to such geometric problem is far from being immediate, and it is also the heart of the matter in order to solve problem (23) for arbitrary kernels. As in the proof of Theorem 1, the difficulty comes mainly from the fact that maximizing sequences of polygons may converge to "generalized polygons" with self-intersections in their boundary. We overcome this difficulty via an ad hoc geometric construction, allowing to reduce ourselves to deal with star-shaped polygons; once made this restriction, we can take advantage of first order optimality conditions, which enable us to arrive at the regular N-gon.

Theorem 6 (polygonal Hardy-Littlewood inequality)**.** *Let h be an admissible kernel. For every $N \geq 3$, we have*

$$\int_\Omega h(x)\,dx \leq \int_{\Omega_N^*} h(x)\,dx\,, \qquad \forall \Omega \in \mathcal{P}_N\,. \tag{24}$$

Applying inequality (24) allows us to prove (21)-(22), but the same strategy is not successful to obtain the analogous inequality

$$\int_\Omega\int_\Omega |x-y|^k\,dx\,dy \geq \int_{\Omega_N^*}\int_{\Omega_N^*} |x-y|^k\,dx\,dy\,, \qquad \forall \Omega \in \mathcal{P}_N\,,\ N \geq 3 \tag{25}$$

for non-integers, or odd integers, or higher exponents k. We are just able to prove that (25) continues to hold in some very specific situations, that we gather in the statement below:

Lemma 7. *Inequality* (25) *holds in the following cases:*

(i) $k = 6$, $N = 8$, *under the restriction that Ω is convex and axisymmetric;*

(ii) $k \geq 1$, $N \geq 3$, *under the restriction that Ω is a linear image of Ω_N^*.*

Remark 8. Theorem 6 allows to extend the result in [37] by Morgan and Bolton about the optimality of the hexagonal economic regions for the location problem to other kernels than the average distance, for instance power-type kernels.

Clearly our results raise many new questions, some of which may be very challenging. A short list is given below.

Open problems

(A) Characteristic kernels: Determine or estimate the radius r' in Theorem 1.

(B) Power-type kernels: Determine for which values of k the inequality (25) holds.

(C) Gaussian kernel $h(x) = e^{-|x|^2}$: Determine whether Ω_N^* minimizes the h-perimeter over $\mathcal{P}_N$ for any $N \geq 3$. (Alternatively, in terms of the heat content defined in (12), does the inequality $Q_\Omega(t) \leq Q_{\Omega_N^*}(t)$ $\forall \Omega \in \mathcal{P}_N$ hold for any $N \geq 3$ and every $t > 0$?)

(D) Arbitrary admissible kernels h: Determine whether Ω_N^* minimizes the h-perimeter over $\mathcal{P}_N$ for every N odd ($N \geq 5$) and whether, under some suitable assumptions on h, the same holds for every N even ($N \geq 6$).

(E) More general kernels: Explore what happens also for kernels which are not locally integrable, but induce a finite perimeter on the class of polygons, as it is for instance the case for the fractional kernel (see e.g. [34, Corollary 1.2]).

3. Numerical results about problems (B) and (C)

In this section we bring some numerical evidence related to open problems (B) and (C). The numerical results are summarized below. Some of them (mainly on the local minimality) could be turned into analytical ones provided the approximations would be controlled and the numerical computations certified. Let us point out that, as soon as the kernel is of polynomial type with sufficiently small degree, the computations we perform are accurate up to rounding errors in double precision. The computations use quadrature rules which are exact for low degree polynomials. This is explained in Section 3.3.

3.1. **About problem (B).** We made multiple numerical optimizations with randomized initialization, in order to minimize over $\mathcal{P}_N$ the functional

$$\int_\Omega \int_\Omega |x-y|^k \, dx \, dy. \tag{26}$$

We used the constrained optimization algorithm `interior-point` from the Matlab `fmincon` routine. The computations were performed for $N \in \{5, 6, 7, 8, 9, 10\}$, and for $k \in \{6, 8, 10, 12\}$. All simulations led to the regular polygon.

Next, in order to extract information about the local minimality of the regular N-gon, we looked at the sign of the eigenvalues of the Hessian matrix of the scale-invariant functional defined for all polygons with N sides by

$$|\Omega|^{-(k+4)/2} \int_\Omega \int_\Omega |x-y|^k \, dx \, dy.$$

Needless to say, since the above functional is invariant under rigid motions, several zero eigenvalues must be expected, so that local minimality is gained as soon as the other eigenvalues are strictly positive.

We computed the Hessian matrix of our functional under vertices displacement, by using formula (86) in the Appendix and the classical Hessian formula for the area functional which can be found e.g. in [5, Section 2]. We obtained 4 eigenvalues equal to zero (corresponding to translations, rotations, and homotheties) and $2N-4$ eigenvalues which are strictly positive.

The computations were performed for $N \in \{5, 6, 7, 8, 9, 10\}$, and for k even, $k \leq 24$.

3.2. **About problem (C).** Differently from power-type kernels, the Gaussian kernel is no longer homogeneous under homotheties. Hence, after rescaling, it not restrictive to consider the heat kernel $h(x) = e^{-|x|^2/t}$ at different times, and work with polygons with fixed diameter equal to 1 instead of polygons with fixed area π. Actually, dealing with polygons with fixed diameter turns out to be convenient in order to control the approximation made when the power series expansion of the heat kernel

$$h(x,y) = \sum_{k=0}^{\infty} \frac{1}{k!} \Big(-\frac{|x-y|^2}{t} \Big)^k$$

is replaced by its partial sums

$$h_Q(x,y) = \sum_{k=0}^{Q} \frac{1}{k!} \Big(-\frac{|x-y|^2}{t} \Big)^k . \tag{27}$$

Notice that, in the case $Q = 0$, problem (C) becomes trivial since $h_0 = 1/t$, and also in the case $Q = 1$ the regular N-gon maximizes $J_1(\Omega)$ for every N thanks to Theorem 5.

The idea is then to look at what happens for higher values of Q such that the approximation of h by h_Q is sufficiently good. If x, y belong to a polygon with unit diameter and taking $t \geq 1$, by the inequality $|e^z - \sum_{k=0}^{Q} \frac{z^k}{k!}| \leq \frac{|z|^{Q+1}}{(Q+1)!}$ holding for any $z \in \mathbb{C}$ with $\mathrm{Re} z < 0$, we have

$$|h(x,y) - h_Q(x,y)| \leq \frac{1}{(Q+1)!}.$$

In particular, for $Q = 12$ (which in our computational strategy described in Section 3.3 below corresponds to a quadrature rule of order 24) the inverse of $(Q + 1)!$ is bounded from above by 1.6×10^{-10}. Consequently, for polygons with unit diameter (having area at most π), the global numerical error done in evaluating J_{h_Q} in place of J_h is bounded from above by $\pi^2 \times 1.6 \times 10^{-10}$. Similar estimates yields global errors smaller than 10^{-7} when replacing the gradient and the Hessian of J_h by their analogues for J_{h_Q}, according to the integral formulas in Appendix.

Then, we fix our attention on the functional J_{h_Q} for $Q = 12$. Clearly, working with such functional brings us back to a polynomial setting as in case of problem (B) discussed above, with the difference that now J_{h_Q} is no longer homogeneous with respect to scalings. Hence we consider the Hessian matrix associated with the functional

$$\int_\Omega \int_\Omega h_Q(x,y)\, dx\, dy - \ell_Q |\Omega|, \tag{28}$$

being ℓ_Q a Lagrange multiplier chosen so that the regular N-gon under consideration is a critical point. The Lagrange multiplier ℓ_Q is the ratio of the norms of the gradient of $\int_\Omega \int_\Omega h_Q(x,y) dx dy$ and the gradient of the area; notice indeed that for the regular N-gon these gradients are collinear for symmetry reasons. The Hessian matrix is obtained again by using formula (86). We investigate the sign of its eigenvalues corresponding to eigenvectors orthogonal to the gradient of the area (a space of dimension $2N - 1$). We obtained 3 eigenvalues equal to zero (corresponding to translations and rotations) and $2N - 4$ eigenvalues which are strictly negative.

The computations where performed for $N \in \{5,6,7,8,9,10\}$ and for a few choices of $t \in [1,100]$, including the endpoints. Numerically, we observe that the Hessian eigenvalues vary monotonically with t: they are negative and have a decreasing absolute value as t increases. Therefore, we conjecture that their sign remains negative for all values of t in the considered range.

Furthermore, the smallest absolute value of non-zero eigenvalues, which is obtained for $N = 10$ and $t = 100$, is larger than 10^{-4}. The above discussion about the error estimates done when replacing the heat kernel by its polynomial approximation h_Q indicates that, also for the heat kernel, the regular N-gon is a local maximizer under area constraint.

These simulations motivate us to conjecture that the regular N-gon is a local maximizer of the heat content for every $t \in \mathbb{R}$.

Let us also mention that the same computations above were made also for $Q \in [2, 11]$: for $Q \in [2,5]$, $N \in [5,10]$ and various choices of $t \in [1,100]$ an oscillatory behavior can be observed, namely the Hessian at the regular N-gon may have positive or negative eigenvalues; however, for $Q \geq 6$ the behavior stabilizes and the non-zero eigenvalues become strictly negative.

3.3. **Computational strategy.** Let us now briefly explain the strategy adopted for the computations in Sections 3.1 and 3.2. When Ω is a polygon, and k is a positive even integer, functionals of the type $\int_\Omega \int_\Omega |x-y|^k \, dx \, dy$ can be computed explicitly in terms of the coordinates of the vertices, and the same assertion holds for the integrals involved in the shape derivatives of such functionals. However, the resulting expressions are difficult to interpret and implement. Thus we choose to adopt a different approach, based on quadrature rules. Given a N-gon Ω, we split it into triangles $T_1, ..., T_N$ (using an inner node) and we decompose the energy as

$$\int_\Omega \int_\Omega h(x,y) dx dy = \sum_{i,j=1}^{N} \int_{T_i} \int_{T_j} h(x,y) dx dy \,,$$

so that we can focus on the computation of integrals made over a product of triangles. A quadrature rule for an integral over a triangle T is an approximation of the form

$$\int_T f(x) dx \approx \sum_{i=1}^{M} w_i f(P_i) \,, \tag{29}$$

where $P_1, ..., P_M$ are points in T (expressed, for instance, using barycentric coordinates in the triangle T), and $w_1, ..., w_M$ are the associated weights. A quadrature rule is said to be of order k if the approximation (29) is exact when f is a polynomial of total degree at most equal to k. For any degree k, there exist quadrature rules of such degree, the number of quadrature points being increasing with respect to k.

An example of a triangulation and choice of quadrature points of degrees 6 and 12 for a regular hexagon is shown in Figure 2.

Handling quadratures rules involving double integrals is more complex, but relies on the same principles. In this case, given two triangles T_1, T_2 with corresponding quadrature points $P_1, ..., P_M$, $Q_1, ..., Q_M$ (having the same barycentric coordinates) and

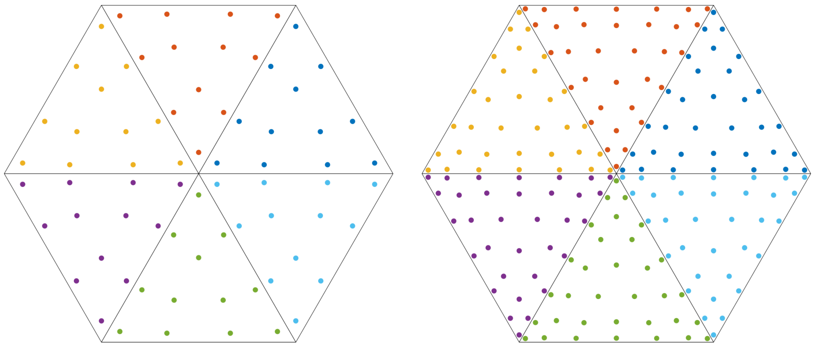

FIGURE 2. Triangulation and corresponding quadrature points for a regular hexagon. The quadrature rules have degrees 6 and 12, respectively.

weights $w_1, ..., w_M$, we have

$$\int_{T_1 \times T_2} h(x,y)dxdy \approx \sum_{i,j=1}^{M} w_i w_j h(P_i, Q_j); \tag{30}$$

as above, the quadrature rule is of order k if the approximation (30) is exact when $h(x,y)$ is a polynomial of total degree at most k.

In order to generate quadrature rules required in our computations, we used the Matlab toolbox Quadtriangle (accessed in November 2022). We used non product rules, included in the referenced toolbox up to degree 25.

4. PROOF OF THEOREM 1

We proceed to prove the result first in the simplified setting of convex polygons and then in the general case.

4.1. **Proof of Theorem 1 in the convex setting.** To prepare the proof, it is useful to introduce the set $\Delta_{r,s}$ defined by

$$\Delta_{r,s} := \Gamma^+_{0,s} \cap B_r(0)\,,$$

where

$$\Gamma^+_{0,s} := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \;:\; x_2 \geq s \right\}$$

Clearly $\Delta_{r,s}$ is empty set for $s > r$, while for $s \in\; [0, r]$ it is given a circular segment of radius r and apothem s, see Figure 3.

For $s \in [0, r]$, the Lebesgue measure of $\Delta_{r,s}$ is immediately determined as

$$|\Delta_{r,s}| = r^2 \arccos\left(\frac{s}{r}\right) - s\sqrt{r^2 - s^2}\,. \tag{31}$$

Below we state a simple geometric lemma which plays a key role in the proof; it provides a lower bound for the r-perimeter of a convex N-gon and an upper bound for the r-perimeter of a regular N-gon. They will be exploited in the limit of a vanishing radius. We focus on the case $N \geq 5$, since for $N = 3, 4$ the equality in (16) is valid for every $r > 0$.

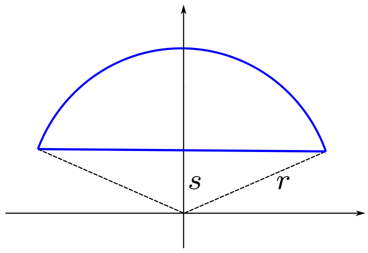

FIGURE 3. The set $\Delta_{r,s}$.

Lemma 9. *(i) Let Ω be a convex polygon in $\mathcal{P}_N$. Assume that, for every side S_i of Ω, denoting by ℓ_i its length and by θ_i, θ_{i+1} its adjacent inner angles, it holds*

$$\ell_i - r\mathrm{cot}\left(\frac{\theta_i}{2}\right) - r\mathrm{cot}\left(\frac{\theta_{i+1}}{2}\right) > 0 \qquad \forall i = 1, \dots, N\,. \tag{32}$$

Then

$$P_r(\Omega) \geq \frac{2}{3}|\partial\Omega| r^3 - \frac{4}{3}\sum_{i=1}^{N} \cot\left(\frac{\theta_i}{2}\right) r^4 =: \Phi_r(\Omega)\,. \tag{33}$$

Moreover, if (32) fails for a family of sides (of cardinality at most $N-3$), the inequality (33) holds with $\Phi_r(\Omega)$ replaced by $\Phi_r(\widehat{\Omega})$, where $\widehat{\Omega}$ is the convex polygon obtained by eliminating any such side and prolonguing its two consecutive sides.

(ii) Let Ω_N^* denote a regular N-gon, with $N \geq 5$. Assume that, denoting by ℓ the length of its sides, it holds

$$\ell - 2r > 0\,. \tag{34}$$

Then

$$P_r(\Omega_N^*) \leq \frac{2}{3}|\partial\Omega_N^*| r^3 + 4N\left(\pi - \frac{1}{3}\right) r^4\,. \tag{35}$$

Proof. (i) Setting $\Omega_s := \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) > s\}$, we have

$$P_r(\Omega) = \int_\Omega |\Omega^c \cap B_r(x)|\, dx = \int_0^r ds \int_{\partial\Omega_s} |\Omega^c \cap B_r(y)|\, d\mathcal{H}^1(y)\,. \tag{36}$$

Thanks to assumption (32), for every $i = 1, \dots, N$ and every $s \in (0, r)$, $\partial\Omega_s$ contains a segment Γ_i of positive length, made by points y such that $\Omega^c \cap B_r(y)$ intersects $\partial\Omega$ only along the side S_i and is congruent to $\Delta_{r,s}$. For the length of this segment we have the following lower bound:

$$|\Gamma_i| \geq \ell_i - r\mathrm{cot}\left(\frac{\theta_i}{2}\right) - r\mathrm{cot}\left(\frac{\theta_{i+1}}{2}\right),$$

and, for points $y \in \Gamma_i$, it holds

$$\int_0^r |\Omega^c \cap B_r(y)|\, ds = \int_0^r |\Delta_{r,s}|\, ds = \frac{2}{3} r^3\,, \tag{37}$$

where the last equality follows from (31) and an elementary integration. Hence,

$$P_r(\Omega) \geq \sum_{i=1}^{N} \Big(\ell_i - r\cot\Big(\frac{\theta_i}{2}\Big) - r\cot\Big(\frac{\theta_{i+1}}{2}\Big)\Big)\frac{2}{3}\pi r^3 = \frac{2}{3}|\partial\Omega| r^3 - \frac{4}{3}\sum_{i=1}^{N}\cot\Big(\frac{\theta_i}{2}\Big) r^4 \,.$$

In case assumption (32) fails for some index, we repeat the proof above with the following only modification: in correspondence of any index for which (32) is false, we remove that side from Ω and we consider the polygon $\widehat{\Omega}$ defined as in the statement. By construction, for every side $\widehat{S}_i$ of $\widehat{\Omega}$ and every $s \in (0, r)$, $\partial\Omega_s$ contains a segment $\widehat{\Gamma}_i$ of positive length (parallel to $\widehat{S}_i$), made by points y such that $\Omega^c \cap B_r(y)$ is congruent to $\Delta_{r,s}$. For the length of this segment we have now the following lower bound:

$$|\widehat{\Gamma}_i| \geq \widehat{\ell}_i - r\cot\Big(\frac{\widehat{\theta}_i}{2}\Big) - r\cot\Big(\frac{\widehat{\theta}_{i+1}}{2}\Big)\,,$$

where $\widehat{\ell}_i$, $\widehat{\theta}_i$, and $\widehat{\theta}_{i+1}$ denote the length of $\widehat{S}_i$, and it adjacent angles. Summing over all the sides of $\widehat{\Omega}$, we find the lower bound $P_r(\Omega) \geq \Phi_r(\widehat{\Omega})$.

(ii) We write the equality (36) for Ω_N^*. For every $s \in (0, r)$, the set of points y at distance s from $\partial\Omega_N^*$ contains N segments of positive length, bounded from above by the positive quantity $\ell - 2r$, made by points y such that $\Omega^c \cap B_r(y)$ is congruent to $\Delta_{s,r}$. For points y in such segments, the equality (37) holds. For points $x \in \Omega$ such that $B_r(x)$ meets more than one side of $\partial\Omega_N^*$, we simply estimate from above $|(\Omega_N^*)^c \cap B_r(x)|$ by πr^2. The measure of these points is bounded from above by $4Nr^2$. We end up with

$$P_r(\Omega_N^*) \leq N(\ell - 2r)\frac{2}{3}r^3 + 4Nr^2(\pi r^2)\,.$$

□

We are now ready to prove Theorem 1 for convex polygons. We argue by contradiction. Assume the statement is false. Then, there exists an infinitesimal sequence of radii $\{r_k\}$ and a sequence of convex polygons $\{\Omega_k\} \subset \mathcal{P}_N$ such that

$$P_{r_k}(\Omega_k) < P_{r_k}(\Omega_N^*)\,. \tag{38}$$

Here and in the remaining of the proof, Ω_N^* denotes a regular N-gon of area π. By possibly passing to a subsequence and up to translations, the sequence of convex polygons $\{\Omega_k\}$ admits a limit Ω_0 in the Hausdorff complementary topology. There are two possibilities: either $\Omega_0 = \emptyset$, or $\Omega_0 \neq \emptyset$. Let us show that both cases lead to a contradiction.

Case 1): $\Omega_0 = \emptyset$. Let us consider the sequence of (possibly empty) convex polygons contained into Ω_k defined by

$$\omega_k := \Big\{x \in \Omega_k \ :\ \mathrm{dist}(x, \Omega_k^c) \geq \frac{r_k}{2}\Big\}\,.$$

Up to a subsequence, we may distinguish to subcases: either $\sup_k |\partial\omega_k| < +\infty$, or $|\partial\omega_k| \to +\infty$.

Case 1a): $\sup_k |\partial\omega_k| < +\infty$. Up to a further subsequence, either the convex polygons ω_k are empty, or they converge to a segment. Anyhow, we have that $|\Omega_k \setminus \omega_k| \to \pi$. For every $x \in (\Omega_k \setminus \omega_k)$, the intersection $\Omega^c \cap B_{r_k}(x)$ contains a set congruent to $\Delta_{r_k, \frac{r_k}{2}}$. Hence

$$P_{r_k}(\Omega_k) \geq |\Omega_k \setminus \omega_k| |\Delta_{r_k, \frac{r_k}{2}}| = |\Omega_k \setminus \omega_k| \Big(\frac{\pi}{3} - \frac{\sqrt{3}}{4} \Big) r_k^2 \,, \tag{39}$$

where in the last equality we have used (31). On the other hand, for k sufficiently large assumption (34) is fulfilled and hence the inequality (35) in Lemma 9 holds with $r = r_k$. In view of this fact, and since $|\Omega_k \setminus \omega_k| \to \pi$, the inequality (39) contradicts (38) in the limit as $r_k \to 0$.

Case 1b): $|\partial \omega_k| \to +\infty$. We have

$$P_{r_k}(\Omega_k) \geq |\partial \omega_k| \int_0^{\frac{r_k}{2}} |\Delta_{r_k, s}| \, ds = \frac{|\partial \omega_k|}{24} (16 + 4\pi - 9\sqrt{3}) r_k^2 \,. \tag{40}$$

Since (35) holds with $r = r_k$, and since $|\partial \omega_k| \to +\infty$, the inequality (40) contradicts (38) in the limit as $r_k \to 0$.

Case 2): $\Omega_0 \neq \emptyset$. We apply Lemma 9 (i) to the sequence of convex polygons Ω_k. We observe that, since $r_k \to 0$, for k large enough assumption (32) is certainly satisfied, except possibly for certain indices corresponding to sides of infinitesimal length. Thus we have

$$P_{r_k}(\Omega_k) \geq \Phi_{r_k}(\Omega_k) \qquad \text{(or alternatively } P_{r_k}(\Omega_k) \geq \Phi_{r_k}(\widehat{\Omega}_k)) \,.$$

We observe that the coefficients of the polynomial function $\Omega \mapsto \Phi_r(\Omega)$ only depend on the perimeter and on the inner angles of the polygon Ω (see (33)); moreover, the same holds for the polynomial function $\Omega \mapsto \Phi_r(\widehat{\Omega})$, because the perimeter and the inner angles of $\widehat{\Omega}$ can be easily expressed in terms of the perimeter and the inner angles of Ω. Now, since Ω_k converge to Ω_0, the perimeter and the inner angles of Ω_k converge respectively to the perimeter and to the inner angles of Ω_0. We conclude that, for k sufficiently large, the following lower bound holds:

$$P_{r_k}(\Omega_k) \geq \frac{2}{3} |\partial \Omega| r_k^3 - C_0 r_k^4 \,,$$

where C_0 is a fixed constant independent of k. By combining the above lower bound with Lemma 9 (ii) (which applies since its assumption (34) is satisfied for k sufficiently large), we obtain that

$$\frac{2}{3} |\partial \Omega| r_k^3 - C_0 r_k^4 \leq \frac{2}{3} |\partial \Omega_N^*| r_k^3 + 4N \Big(\pi - \frac{1}{3} \Big) r_k^4$$

and hence

$$\limsup_{k \to +\infty} |\partial \Omega_k| \leq |\partial \Omega_N^*| \,.$$

By the classical isoperimetric inequality for convex polygons, this implies that $\Omega_0 = \Omega_N^*$. To conclude, we observe that it is not restrictive to assume that Ω_k is a minimizer of the r_k-perimeter over the class of convex polygons in $\mathcal{P}_N$ with area π. (Notice that such a minimizer exists for any r_k sufficiently small, because otherwise a maximizing sequence of polygons would degenerate, yielding a contradiction by the same arguments used to deal with Case 1) above.) Then we have found a sequence of critical polygons for the

r_k-perimeter, converging to Ω_N^*, and satisfying (38). Since $r_k \to 0$, this contradicts Theorem 3. □

4.2. **Proof of Theorem 1 in the general case.** Also in this case we prepare the proof with a geometric lemma. For every $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and every $s > 0$, we set

$$\Gamma_{\theta,s} := \Big\{ x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_2 = (\tan\theta) x_1 + s \Big\}$$
$$\Gamma^+_{\theta,s} := \Big\{ x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_2 \geq (\tan\theta) x_1 + s \Big\}$$
$$\Gamma^-_{\theta,s} := \Big\{ x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_2 \leq (\tan\theta) x_1 + s \Big\}$$

Given a family of straight lines $\{\Gamma_{\theta_i,s_i}\}$, for $i = 1, \dots, 2q-1$ which do not intersect each other in $B_r(0)$, with $\theta_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $0 \leq s_1 < \cdots < s_{2q-1}$, consider the union of strips

$$\Sigma := \Gamma^-_{\theta_1,s_1} \cup \bigcup_{i=1}^{q-1} \left(\Gamma^+_{\theta_{2i},s_{2i}} \cap \Gamma^-_{\theta_{2i+1} s_{2i+1}}\right), \tag{41}$$

see Figure 4. Recalling that that $\Delta_{r,s}$ is the set defined at the beginning of Section 4.1, we prove the following estimate for the measure of $\Sigma^c \cap B_r(0)$:

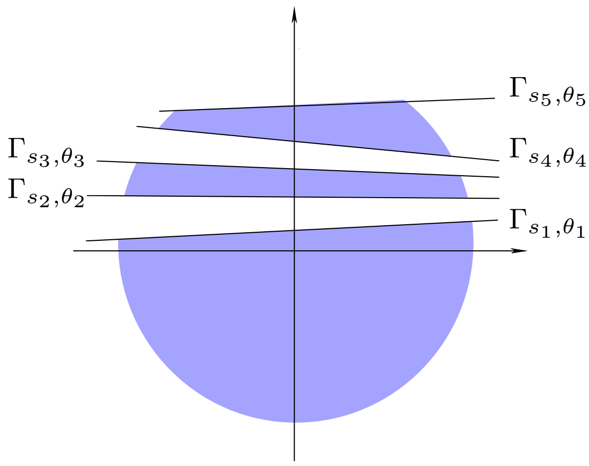

FIGURE 4. The intersection of the set Σ in (41) with $B_r(0)$

Lemma 10. *Let Σ be defined by (41). Setting $\overline{s} := \mathcal{H}^1\big(\Sigma \cap (\{0\} \times [0, +\infty))\big)$, it holds*

$$|\Sigma^c \cap B_r(0)| \geq |\Delta_{r,\overline{s}}| - 4r^2 \sum_{i=1}^{2q-1} |\theta_i| \,. \tag{42}$$

Proof. In order to prove (42), it is not restrictive to assume that $\overline{s} \in [0, r]$, since otherwise it is trivially satisfied. We notice first that, for every $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and for every $s \in [0, r]$, it holds

$$|\Delta_{r,s}| - 4|\theta|r^2 \leq |\Gamma^+_{\theta,s} \cap B_r(0)| \leq |\Delta_{r,s}| + 4|\theta|r^2 . \tag{43}$$

Namely, for $\theta = 0$, (43) holds with equality signs, by the definition of $\Delta_{r,s}$. For $\theta \neq 0$, it holds since the symmetric difference between $\Gamma^+_{\theta,s} \cap B_r(0)$ and $\Delta_{r,s}$ is contained into the union of the two sets

$$B_r(0) \cap (\Gamma^+_{\theta,s} \cap \Gamma^-_{0,s}) \quad \text{and} \quad B_r(0) \cap (\Gamma^+_{0,s} \cap \Gamma^-_{\theta,s}) ,$$

and each of these sets has measure bounded from above by $\frac{|\theta|}{2}(2r)^2$. Now, the inequality (42) is a consequence of (43) and of the monotonicity of the map

$$[0, r] \ni s \mapsto \mathcal{H}^1(\Gamma_{0,s} \cap B_r(0)),$$

which implies that, for $0 \leq s \leq a < b$, it holds $|\Delta_{r,a} \setminus \Delta_{r,b}| \leq |\Delta_{r,s} \setminus \Delta_{r,s+b-a}|$. □

Remark 11. In Lemma 10, inequality (42) remains trivially valid replacing Σ by any subset $\widetilde{\Sigma} \subseteq \Sigma$. We shall use this argument for subsets $\widetilde{\Sigma}$ of the type

$$\widetilde{\Sigma} := (\Gamma^+_{\theta_0,s_0} \cap \Gamma^-_{\theta_1,s_1}) \cup \bigcup_{i=1}^{q-1} \left(\Gamma^+_{\theta_{2i},s_{2i}} \cap \Gamma^-_{\theta_{2i+1},s_{2i+1}}\right) , \tag{44}$$

where $s_0 \leq 0$ and Γ_{θ_0,s_0} does not intersect any Γ_{θ_i,s_i} with $i = 1, \ldots, 2q-1$, in $B_r(0)$.

As a further preliminary, let us give the following definition, that was already used by the second and third authors in [10].

Definition 12. A *generalized polygon with at most N-sides* is the limit in the H^c_{loc} topology of a sequence $\{\Omega_n\}$ of classical polygons with at most N sides (meant as *open* polygons) such that $\limsup_n |\Omega_n| < +\infty$.

Recall that the convergence of $\{\Omega_n\}$ to Ω in the H^c_{loc} topology means that, for every ball B, we have $\lim_n d_{H^c}(\Omega_n \cap B, \Omega \cap B) = 0$, d_{H^c} being the *Hausdorff complementary distance*, namely

$$d_{H^c}(\Omega_n \cap B, \Omega \cap B) := \sup_{x \in \mathbb{R}^2} \left|\text{dist}(x, (\Omega_n \cap B)^c) - \text{dist}(x, (\Omega \cap B)^c)\right| ,$$

(dist stands for the Euclidean distance from a closed set).

As a consequence of well-known properties of such topology (see for instance [9, 31]), any generalized polygon is an open set of finite Lebesgue measure, which is simply connected (as its complement is connected), but possibly disconnected. Any connected component is delimited by a finite number of line segments, which are pairwise joined at their endpoints to form a closed path, possibly containing self-intersections, given by points or line segments.

We now ready to prove Theorem 1 for arbitrary polygons. As in the convex case, we argue by contradiction, and we denote by Ω^*_N a regular N-gon or area π. If the statement is false there exists an infinitesimal sequence of positive radii $\{r_k\}$ and a sequence of polygons $\{\Omega_k\} \subset \mathcal{P}_N$ such that

$$P_{r_k}(\Omega_k) < P_{r_k}(\Omega^*_N)\,. \tag{45}$$

To achieve the proof it is enough to show that

$$\Omega_k \stackrel{H^c}{\to} \Omega^*_N\,. \tag{46}$$

Indeed, the convergence (46) implies in particular that Ω_k is convex for k large enough, which is a contradiction since we have already proved the statement for convex polygons.

In order to prove (46), we consider for every k a triangulation of Ω_k made by $N-2$ disjoint (open) triangles $\{T^1_k, \dots, T^{N-2}_k\}$, with vertices and sides belonging to the family of vertices and diagonals of Ω_k, such that

$$\overline{\Omega}_k = \overline{T}^1_k \cup \cdots \cup \overline{T}^{N-2}_k\,.$$

Up to subsequences (which here and in the sequel are not relabeled), there exist sequences of vectors $\{y^j_k\}$ and sets T^j (which may be either a triangle or the empty set), such that

$$T^j_k - y^j_k \stackrel{H^c}{\to} T^j \qquad \forall j = 1, \dots, N-2\,. \tag{47}$$

For every $j = 1, \dots, N-2$, one of the following three situations occurs:

(a) $T^j = \emptyset$ and $|T^j_k| \to m^j > 0$

(b) $T^j = \emptyset$ and $|T^j_k| \to 0$

(c) $T^j \neq \emptyset$

For convenience, we divide the remaining of the proof in three steps.

Step1: Situation (a) cannot occur.

Assume by contradiction that we are in situation (a) for some sequence T^j_k. Hereafter, we omit for simplicity the index j. Then, up to a subsequence and to a rigid motion, the vertices of T_k are given by

$$(-\ell_k, 0)\,, \quad (\ell_k, 0)\,, \quad (a_k, b_k), \qquad \text{with } b_k > 0\,,$$

where the horizontal side of length $2\ell_k$ is the longest one and, as $k \to \infty$,

$$\ell_k \to +\infty \quad \text{ and } \quad \ell_k b_k \to m > 0\,.$$

We divide the segment $[-\ell_k, \ell_k]$ into $2N$ equal segments, of length $\frac{\ell_k}{N}$. At least one of them, say up to a translation the segment $[-\frac{\ell_k}{2N}, \frac{\ell_k}{2N}]$, is such that the half strip

$$S_k := \big[-\frac{\ell_k}{2N}, \frac{\ell_k}{2N}\big] \times [0, +\infty)$$

does not contain any other vertex of Ω_k. On the other hand, S_k is crossed by a certain number of sides of Ω_k, including two sides of T_k. Thus, with the notation introduced at the beginning of Section 4.2, there exist angles $\theta_i^k \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and positive numbers s_i^k, for $i = 0, \dots, 2M_k - 1$ such that $0 = s_0^k < s_1^k < s_2^k < \dots s_{2M_k-1}^k$, and

$$\Omega_k \cap S_k = \bigcup_{i=0}^{M_k-1} \left(\Gamma^+_{\theta_k^{2i}, s_k^{2i}} \cap \Gamma^-_{\theta_k^{2i+1}, s_k^{2i+1}}\right)$$

$$T_k \cap S_k = \Gamma^+_{\theta_k^0, s_k^0} \cap \Gamma^-_{\theta_k^1, s_k^1} .$$

Up to subsequences, for every $i = 0, \dots, M_k - 1$, we have

$$\theta_k^{2i}, \theta_k^{2i+1} \to \theta^i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \qquad s_k^{2i}, s_k^{2i+1}, \to s^i \in [0, +\infty] .$$

Moreover, by construction it holds

$$0 = s^0 \le s^1 \le \dots \le s^{M_k - 1} .$$

Hence we may define q as the largest index in $\{0, \dots, M_k - 1\}$ such that $s^q = 0$. Notice that, as a consequence, we also have

$$\theta^0 = \theta^1 = \dots = \theta^q = 0. \tag{48}$$

We set

$$\Sigma_k := \Gamma^-_{\theta_k^1, s_k^1} \cup \bigcup_{i=1}^{q} \left(\Gamma^+_{\theta_k^{2i}, s_k^{2i}} \cap \Gamma^-_{\theta_k^{2i+1} s_k^{2i+1}}\right) .$$

Let us observe that, if $q < M_k - 1$, then

$$\liminf_{k \to +\infty} \operatorname{dist}\Big(\Sigma_k \cap \big(\big[-\frac{\ell_k}{4N}, \frac{\ell_k}{4N}\big] \times [0, +\infty)\big), (S_k \cap \Omega_k) \setminus \Sigma_k\Big) > 0. \tag{49}$$

Then, the idea is to adopt similar arguments as in the convex case, just by using Lemma 10 in place of Lemma 9. More precisely, we consider the set

$$\omega_k := \Big\{x = (x_1, x_2) \in \Sigma_k \cap \big(\big[-\frac{\ell_k}{4N}, \frac{\ell_k}{4N}\big] \times [0, +\infty)\big) \ : \ s_x \ge \frac{r_k}{2}\Big\},$$

where

$$s_x := \mathcal{H}^1\Big(\Sigma_k \cap \big(\{x_1\} \times [0, +\infty)\big)\Big) .$$

In view of Lemma 10 and of the strict inequality (49), for k large enough and for every $x \in \Sigma_k \cap \big(\big[-\frac{\ell_k}{4N}, \frac{\ell_k}{4N}\big] \times [0, +\infty)\big)$, it holds

$$|\Omega_k^c \cap B_{r_k}(x)| \ge |\Delta_{r_k, s_x}| - 4r_k^2 \sum_{i=0}^{2q+1} |\theta_k^i|.$$

Then we follow the same proof as in Case 1) of Section 4.1 to get a contradiction. More precisely, denoting by γ_k the projection of ω_k on the horizontal axis, we distinguish the two cases $\sup_k \mathcal{H}^1(\gamma_k) < +\infty$, and $\mathcal{H}^1(\gamma_k) \to +\infty$.

Assume that $\sup_k \mathcal{H}^1(\gamma_k) < +\infty$. This implies that $|\omega_k| \to 0$. Thus, setting

$$E_k := \Big(T_k \cap \big(\big[-\frac{\ell_k}{4N}, \frac{\ell_k}{4N}\big] \times [0, +\infty)\big)\Big) \setminus \omega_k ,$$

we have

$$\liminf_k |E_k| \geq \liminf_k \frac{|T_k|}{16N^2} = \frac{m}{16N^2} > 0 \tag{50}$$

(where the first inequality holds by a proportion argument, which works since the side of length $2\ell_k$ was assumed to be the longest one of T_k). Then we estimate $P_{r_k}(\Omega_k)$ as follows:

$$P_{r_k}(\Omega_k) \geq \int_{E_k} \Big(|\Delta_{r_k, \frac{r_k}{2}}| - 4r_k^2 \sum_{i=0}^{2q+1} |\theta_k^i| \Big) dx$$

In view of (50), recalling that $|\Delta_{r_k, \frac{r_k}{2}}| = \Big(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\Big) r_k^2$, and taking into account that, by (48), $\theta_i^k \to 0$ for every $i = 0, \dots, 2q+1$, we conclude that

$$\liminf_{k \to +\infty} \frac{P_{r_k}(\Omega_k)}{r_k^3} = +\infty\,, \tag{51}$$

in contradiction with (35) and (45).

If $\mathcal{H}^1(\gamma_k) \to +\infty$, then we estimate $P_{r_k}(\Omega_k)$ as follows:

$$P_{r_k}(\Omega_k) \geq \mathcal{H}^1(\gamma_k) \frac{r_k}{2} \Big(|\Delta_{r_k, \frac{r_k}{2}}| - 4r_k^2 \sum_{i=0}^{2q+1} |\theta_k^i| \Big),$$

so that (51) is again valid, in contradiction with (35) and (45).

Step 2: Identification of local concentrations. Finally, only situations (b) and (c) can occur. Since any sequence of triangles in situation (b) does not affect the limit of the sequence $\{\chi_{\Omega_k}\}$ in $L^1_{loc}(\mathbb{R}^2)$, in order to describe the geometry of local concentrations, we focus only on the sequences of triangles in situation (c). For any pair of such sequences $\{T_k^i\}, \{T_k^j\}$, we consider the corresponding sequences of vectors $\{y_k^i\}, \{y_k^j\}$ such that (47) holds, and we look at whether the distances $\|y_k^i - y_k^j\|$ remain bounded or diverge as $k \to +\infty$. This way we define an equivalence relation on the family of sequences of triangles in situation (c), which splits them into a finite number p of equivalence classes. By construction, for $i = 1, \dots, p$, there exist sequences of vectors $\{\widetilde{y}_k^i\}$, with $\|\widetilde{y}_k^i - \widetilde{y}_k^j\| \to +\infty$ for $i \neq j$, such that

$$\Omega_k - \widetilde{y}_k^i \overset{H^c_{loc}}{\longrightarrow} \Omega^i_\sharp\,, \tag{52}$$

where $\Omega^1_\sharp, \dots, \Omega^p_\sharp$ are generalized polygons, with a total number of sides not larger than N and total area equal π, i.e.,

$$\sum_{i=1}^p |\Omega^i_\sharp| = \pi\,.$$

Then we consider the open sets with polygonal boundary obtained as $\Omega^i_0 := \mathrm{Int}(\overline{\Omega^i_\sharp})$; we observe that

$$\sum_{i=1}^p |\partial \Omega^i_0| \geq |\partial \Omega^*_N|, \tag{53}$$

with equality if and only if $p = 1$ and $\Omega_0^1 = \Omega_N^*$. Indeed, any open connected component U of set Ω_0^i has a boundary which is union of closed polygonal lines, each one with at most N edges. Then, by removing every bounded connected component of $\mathbb{R}^2 \setminus U$ and rescaling the set thus obtained by a factor less than 1, it is possible to decrease the perimeter by preserving the area. Then the classical polygonal isoperimetric inequality ensures that $\sum_{i=1}^p |\partial\Omega_0^i|$ is not smaller than the sum of the perimeters of p regular N-gons with total area π, and (53) follows from the sub-additivity of the map $\mathbb{R}^+ \ni t \to \sqrt{t}$.

Step 3: We prove that

$$\liminf_{k\to+\infty} \frac{P_{r_k}(\Omega_k)}{r_k^3} \geq \frac{2}{3}\sum_{i=1}^p |\partial\Omega_0^i|. \tag{54}$$

The above lower bound, combined with the upper bound inequality (35) and with the assumption (45), will imply that (53) holds with equality sign. This implies in particular that, for k large enough, the sets Ω_k must be convex. As we have seen in Section 4.1, this contradicts (45).

Let us prove (54). We fix an index $i \in \{1, \dots, p\}$ and we localize our estimates around the set Ω_0^i; we may also assume without loss of generality that the corresponding vectors $\widetilde{y}_k^i$ in (52) are equal to zero. Choosing $R_i > 0$ such that $\overline{\Omega_\sharp^i} \subset B_{R_i}(0)$, from (52) we have

$$\Omega_k \cap B_{R_i}(0) \xrightarrow{H^c} \Omega_\sharp^i .$$

Dropping the index i for simplicity of notation, we have to show that

$$\liminf_{k\to+\infty} \frac{1}{r_k^3}\int_{B_R(0)\cap\Omega_k}\int_{\Omega_k^c} \chi_{B_{r_k}(0)}(x-y)dxdy \geq \frac{2}{3}|\partial\Omega_0|. \tag{55}$$

We focus our analysis around a fixed side of Ω_0. Its endpoints are limit of vertices of Ω_k, but its interior as well may contain some accumulation points of vertices of Ω_k. These accumulation points divide our side into several segments (at most N). We pick one of them, say $[-\ell, \ell] \times \{0\}$, with Ω_0 lying below the segment. From the H^c-convergence, and since the open segment $(-\ell, \ell) \times \{0\}$ does not contain any accumulation point of vertices of Ω_k, if $\varepsilon \in (0, \frac{\ell}{2})$ and $\delta > 0$ are sufficiently small so that $[-\ell, \ell] \times [-\delta, \delta] \subset B_R(0)$, then inside the rectangle $[-\ell + \varepsilon, \ell - \varepsilon] \times [-\delta, \delta]$ the structure of Ω_k is similar to the one of the set Σ in Lemma 10 (cf. (41)).

Precisely, we consider the sides of Ω_k which intersect the rectangle $[-\ell + \varepsilon, \ell - \varepsilon] \times [-\delta, \delta]$, and whose supporting lines $\Gamma_{\theta_k^i, s_k^i}$ satisfy $s_k^i \to 0$ as $k \to +\infty$. Assume that, as the index i goes from 1 to $2q + 1$, those lines are labelled from the bottom to the top. Choosing $\delta' > 0$ such that $[-\ell + 2\varepsilon, \ell - 2\varepsilon] \times [-\delta, \delta']$ does not meet any other side of Ω_k, we can locally represent Ω_k in $[-\ell + 2\varepsilon, \ell - 2\varepsilon] \times [-\delta, \delta']$ as the following union of strips:

$$\Sigma_k = \Big([-\ell + 2\varepsilon, \ell - 2\varepsilon] \times [-\delta, \delta']\Big) \cap \Big(\Gamma^-_{\theta_k^1, s_k^1} \cup \bigcup_{i=1}^q \Gamma^+_{\theta_k^{2i}, s_k^{2i}} \cap \Gamma^-_{\theta_k^{2i+1}, s_k^{2i+1}}\Big).$$

Then, using Lemma 10 (and Remark 11), we get

$$\liminf_{k\to+\infty}\frac{1}{r_k^3}\int_{\Sigma_k}\int_{\Omega_k^c}\chi_{B_{r_k}(0)}(x-y)dxdy \geq$$

$$\liminf_{k\to+\infty}\frac{1}{r_k^3}\int_{-\ell+2\varepsilon}^{\ell-2\varepsilon}dx_1\int_{\{x_2\in\mathbb{R}\,:\,s_x\in[0,r_k]\}}|\Sigma_k^c\cap B_{r_k}(x)|dx_2 \geq$$

$$\liminf_{k\to+\infty}\frac{1}{r_k^3}\int_{-\ell+2\varepsilon}^{\ell-2\varepsilon}dx_1\int_0^{r_k}\big(|\Delta_{r_k,x_2}|-4r_k^2\sum_{i=0}^{2q+1}|\theta_k^i|\big)dx_2 \geq$$

$$\frac{2}{3}(2\ell-4\varepsilon).$$

Inequality (55) follows by repeating the above argument around each side of Ω_0 and letting $\varepsilon\to 0$.

□

5. Proofs of Theorem 2 and of Corollary 4

Proof of Theorem 2. For brevity, let us denote by J_r the functional

$$J_r(\Omega)=\int_\Omega\int_\Omega\chi_{B_r(0)}(x-y)\,dx\,dy\,,$$

so that

$$P_r(\Omega)=|\Omega||B_r(0)|-J_r(\Omega)=\pi^2r^2-J_r(\Omega)\qquad\forall\Omega\in\mathcal{P}_N\,.$$

Clearly, for $\Omega\in\mathcal{P}_N$, we have

$$\begin{cases}J_r(\Omega)=\pi^2 & \text{if } \operatorname{diam}\Omega\leq r\\ J_r(\Omega)<\pi^2 & \text{if } \operatorname{diam}\Omega> r\end{cases}$$

Setting $r'':=\min\{\operatorname{diam}\Omega\,:\,\Omega\in\mathcal{P}_{\rm N}\}$, by Reinhardt's Theorem [41], for $N\geq 6$ even there exists a polygon $\Omega_N^\sharp$, which is *not* a regular N-gon, such that

$$r''=\operatorname{diam}\Omega_N^\sharp<\operatorname{diam}\Omega_{\rm N}^*\,.$$

Therefore, for every $r\geq r''$, we have

$$\pi^2=J_r(\Omega_N^\sharp)>J_r(\Omega)\qquad\forall\Omega\in\mathcal{P}_N\text{ with }\operatorname{diam}\Omega>r$$

(in particular, the above strict inequality holds for $\Omega=\Omega_N^*$ if $\operatorname{diam}\Omega_N^*>r$). It follows that, for every $r\geq r''$, the maximum of J_r over $\mathcal{P}_N$ equals π^2, or equivalently the minimum of P_r over $\mathcal{P}_N$ equals $\pi^2(r^2-1)$, and they are attained at $\Omega_N^\sharp$. □

Proof of Corollary 4 (i) Let r'' and $\Omega_N^\sharp$ as in the statement of Theorem 2. For every $r\in[r'',\operatorname{diam}\Omega_N^*)$, there exists a polygon $\Omega_N^\sharp\in\mathcal{P}_N$ such that

$$J_r(\Omega_N^\sharp)>J_r(\Omega_N^*)\,.$$

Then, it is enough to consider a sequence of non-negative and non-increasing radially symmetric kernels $\{h_n\}$ in $L^1_{\rm loc}(\mathbb{R}^2)$ which converge increasingly to $\chi_{B_r}(0)$. By the monotone convergence theorem, for n large enough we have

$$J_{h_n}(\Omega^\sharp_N) > J_{h_n}(\Omega^*_N)\,.$$

(ii) Again, let r'' and $\Omega^\sharp_N$ be as in the statement of Theorem 2. Set $\varepsilon := [\mathrm{diam}(\Omega^*_N) - \mathrm{diam}(\Omega^\sharp_N)]/3$. If $[x_0, y_0]$ is a diameter of Ω^*_N, for $x \in B_\varepsilon(x_0)$ and $y \in B_\varepsilon(y_0)$ we have

$$|x-y| \geq |x_0-y_0| - |x-x_0| - |y-y_0| \geq \mathrm{diam}(\Omega^*_N) - 2\varepsilon = \mathrm{diam}(\Omega^\sharp_N) + \varepsilon =: r'' + \varepsilon\,.$$

Hence,

$$\begin{aligned} \int_{\Omega^*_N}\int_{\Omega^*_N} |x-y|^k\,dx\,dy &\geq \int_{B_\varepsilon(x_0)\cap\Omega^*_N}\int_{B_\varepsilon(y_0)\cap\Omega^*_N} |x-y|^k\,dx\,dy \\ &\geq |B_\varepsilon(x_0)\cap\Omega^*_N|\,|B_\varepsilon(y_0)\cap\Omega^*_N|(r''+\varepsilon)^k\,. \end{aligned} \tag{56}$$

On the other hand, we have

$$\int_{\Omega^\sharp_N}\int_{\Omega^\sharp_N} |x-y|^k dx\,dy \leq \pi^2\,(r'')^k \tag{57}$$

By comparing (56) and (57) we infer that, for k large enough,

$$\int_{\Omega^*_N}\int_{\Omega^*_N} |x-y|^k\,dx\,dy > \int_{\Omega^\sharp_N}\int_{\Omega^\sharp_N} |x-y|^k\,.$$

We now examine in particular the case $N = 6$. Denoting by H_R the regular hexagon with unit diameter and by H_G the Graham hexagon with the same area, it follows that $\mathrm{diam}(H_G) = d \approx 0.980957$. Then, a direct estimate using $|x-y| \leq d$ gives

$$\int_{H_G}\int_{H_G} |x-y|^k dx\,dy \leq |H_G|^2 d^k.$$

On the other hand, by arguing as done above to obtain (56), we get

$$\int_{H_R}\int_{H_R} |x-y|^k dx\,dy \geq 3|B_\varepsilon(x_0)\cap H_R||B_\varepsilon(y_0)\cap H_R|(d+\varepsilon)^k.$$

Observing that $|B_\varepsilon(x_0)\cap H_R| = \frac{\pi\varepsilon^2}{3}$, $\varepsilon = \frac{1-d}{3}$, and using the numerical value of d shows that, for any $k \geq 2832$, H_G has a lower energy than H_R. □

6. Proof of Theorem 3

We argue in two steps.

Step 1. We claim that, if θ_i denotes inner angle of Ω at the vertex A_i, it holds $\theta_i = \theta_{i+1}$. We prove this claim by contradiction, via a reflection argument. Assume $\theta_{i+1} > \theta_i$. Let H be the symmetry axis of the side $[A_i, A_{i+1}]$, let M_i be their intersection point and, for $x \in [A_i, M_i]$, let x' be its symmetric about H. Given ω in the plane we denote $\omega_\star$ its reflection about H. We have

$$(\Omega\cap B_r(x))_\star \subseteq \Omega\cap B_r(x')\,,\ \text{with strict inclusion for } x \text{ close to } A_i. \tag{58}$$

Indeed, the strict inclusion for x close to A_i readily follows from

$$(\Omega \cap B_r(x))_\star = \Omega_\star \cap B_r(x') \qquad \text{and} \qquad \theta_{i+1} > \theta_i\,.$$

We assert that the inclusion (58) remains valid for all $x \in [A_i, M_i]$, namely that

$$\Omega_\star \cap B_r(x') \subseteq \Omega \cap B_r(x') \qquad \forall x \in [A_i, M_i]\,. \tag{59}$$

Once proved (59), the contradiction required to achieve the proof of Step 1 readily follows. Indeed, recalling that the inclusion becomes strict for x close to A_i, we have

$$\begin{aligned} \int_{A_i}^{M_i} v_\Omega(x)\,|xM_i|\,dx &< \int_{A_i}^{M_i} v_\Omega(x')\,|x'M_i|\,dx \\ &= \int_{M_i}^{A_{i+1}} v_\Omega(x')\,|x'M_i|\,dx'\,, \end{aligned}$$

in contradiction with (18).

It remains to show (59). To that purpose, we distinguish the cases $N = 3$ and $N > 3$.

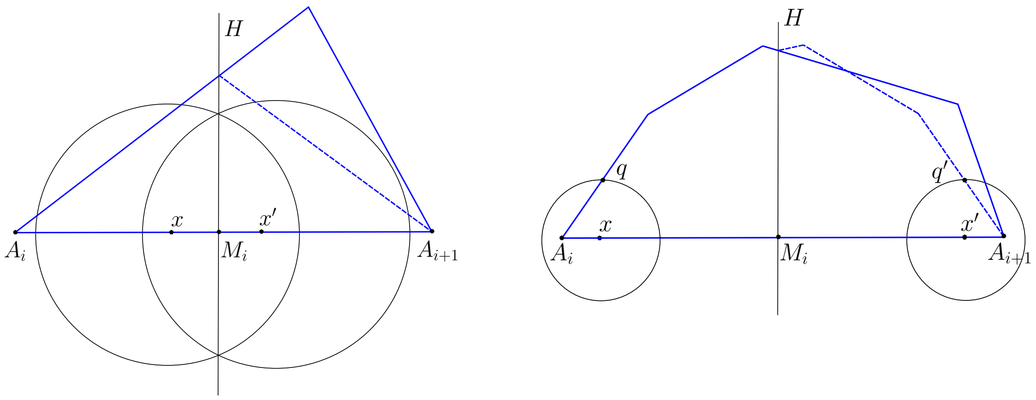

FIGURE 5. The reflection argument for N-gons, with $N = 3$ (on the left) and $N > 3$ (on the right)

For triangles, (59) is straightforward, since the inequality $\theta_{i+1} > \theta_i$ implies that $\Omega_\star \subseteq \Omega$, see Figure 5, left.

For $N > 3$, the inequality $\theta_{i+1} > \theta_i$ does not imply, in general, that $\Omega_\star \subseteq \Omega$, see Figure 5, right. Nevertheless, thanks to assumption (20), in order to check that (59) still holds, it is enough to show that, for any $q \in [A_{i-1}, A_i] \cap B_r(x)$, if q' denotes its symmetric about H, the trapezoid $\mathcal{T}$ with vertices q, q', A_i, A_{i+1} is contained into Ω. Again thanks to assumption (20), the inclusion $\mathcal{T} \subset \Omega$ holds true as soon as $q' \in \overline{\Omega}$. This latter property is true because $q \in [A_{i-1}, A_i]$, and $\theta_{i+1} > \theta_i$.

Step 2. We claim that, if ℓ_i denotes the length of the side $[A_i, A_{i+1}]$, it holds $\ell_i = \ell_{i+1}$. To prove this claim we are going to assume without loss of generality that $N \geq 4$. Indeed, if Ω is a triangle, Step 1 proved above already gives that Ω is equilateral.

We point out that, by assumption (20), it holds

$$r \leq \min_{i=1,\dots,N} \frac{\ell_i}{2}. \tag{60}$$

Indeed, in case $r > \frac{\ell_i}{2}$ for some index i, $\partial\Omega \cap B_r(M_i)$, with M_i mid-point of $[A_i, A_{i+1}]$ would not be contained into two consecutive sides of Ω, against (20).

Next we recall that, by Step 1, all the inner angles θ_i of Ω are equal to a fixed angle θ. Since $N \geq 4$, we have $\theta \geq \pi/2$. Therefore, points $x \in [A_i, A_{i+1}]$ such that $B_r(x)$ intersects another side of Ω (which by (20) is necessarily a consecutive side) are points x whose distance from A_i or from A_{i+1} does not exceed r. For such points, $v_\Omega(x)$ is a function depending on $|xA_i|$ (or $|xA_{i+1}|$), r, and θ, but not on ℓ_i.

Taking (60) into account, we conclude that for a suitable function f, it holds

$$\int_{A_i}^{A_{i+1}} v_\Omega(x)\,dx = \frac{1}{2}\|h\|_{L^1}\big(\ell_i - 2r\big) + f(r,\theta) \qquad \forall i = 1,\dots,N\,,$$

where $\|h\|_{L^1}$ is the L^1 norm of h in $\mathbb{R}^2$. We now enforce condition (19) to deduce

$$\frac{1}{2}\|h\|_{L^1}\big(\ell_i - 2r\big) + f(r,\theta) = c\ell_i \qquad \forall i = 1,\dots,N\,.$$

Clearly this system can be satisfied only if either all the ℓ_i's are equal, or $c = \frac{1}{2}\|h\|_{L^1}$. But the latter equality cannot hold: indeed, by (19), c is the integral mean over a side of Ω of the function $v_\Omega(x)$, and such function is always less than or equal to $\frac{1}{2}\|h\|_{L^1}$, with strict inequality near the vertices. □

7. Proof of Theorem 5 and Lemma 7

Proof of Theorem 5. Taking $h(x) = |x|^2$, for every $\Omega \in \mathcal{P}_N$ we have

$$\begin{aligned}\int_\Omega\int_\Omega |x-y|^2\,dxdy &= \int_\Omega\int_\Omega \Big(x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2x_1y_1 - 2x_2y_2\Big)\,dxdy \\ &= 2\pi \int_\Omega (x_1^2 + x_2^2)\,dx - 2\Big(\int_\Omega x_1\,dx\Big)^2 - 2\Big(\int_\Omega x_2\,dx\Big)^2.\end{aligned}$$

Since the energy is invariant by translations, it is not restrictive to assume that Ω has its baricenter at the origin, and hence the result follows from Theorem 6.

Taking $h(x) = |x|^4$, for every $\Omega \in \mathcal{P}_N$ we have

$$\begin{aligned}\int_\Omega\int_\Omega |x-y|^4\,dxdy &= \int_\Omega\int_\Omega \Big(x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2x_1y_1 - 2x_2y_2\Big)\,dxdy \\ &= \int_\Omega\int_\Omega \big[(x_1^2+x_2^2)^2 + (y_1^2+y_2^2)^2 + 4(x_1y_1+x_2y_2)^2 + 2(x_1^2+x_2^2)(y_1^2+y_2^2) \\ &\qquad -4(x_1y_1+x_2y_2)(x_1^2+x_2^2) - 4(x_1y_1+x_2y_2)(y_1^2+y_2^2)\big]\,dxdy \\ &= 2\pi\int_\Omega |x|^4 + 6\Big(\int_\Omega x_1^2\,dx\Big)^2 + 6\Big(\int_\Omega x_2^2\,dx\Big)^2 \\ &\quad +8\Big(\int_\Omega x_1x_2\,dx\Big)^2 + 4\Big(\int_\Omega x_1^2\,dx\Big)\Big(\int_\Omega x_2^2\,dx\Big),\end{aligned}$$

where the last equality holds up to assuming as done above that Ω has its baricenter at the origin.

Now we use the inequality $6a^2 + 6b^2 + 4ab \geq 4(a+b)^2$, with equality if and only if $a = b$. Applying it for $a = \int_\Omega x_1^2, b = \int_\Omega x_2^2$, we get

$$\begin{aligned}\int_\Omega\int_\Omega |x-y|^4\,dxdy &\geq 2\pi\int_\Omega |x|^4 + 4\Big(\int_\Omega |x|^2\Big)^2 + 8\Big(\int_\Omega x_1x_2\Big)^2\\ &\geq 2\pi\int_{\Omega_N^*} |x|^4 + 4\Big(\int_{\Omega_N^*} |x|^2\Big)^2 + 8\Big(\int_{\Omega_N^*} x_1x_2\Big)^2\\ &= \int_{\Omega_N^*}\int_{\Omega_N^*} |x-y|^4\,dxdy\,,\end{aligned}$$

where the second inequality follows from Theorem 6, and last equality holds since

$$\int_{\Omega_N^*} x_1x_2 = 0 \qquad \text{and} \qquad \int_{\Omega_N^*} x_1^2 = \int_{\Omega_N^*} x_2^2\,. \tag{61}$$

To check these equalities, we decompose a regular N-gon Ω_N^* centred at the origin into N triangles $\{OA_1A_2, ..., OA_NA_1\}$, with $A_i = (p_i, q_i) = (\cos(\alpha + i\frac{2\pi}{N}), \sin(\alpha + i\frac{2\pi}{N}))$.

Concerning the first equality in (61), on each triangle we have

$$\int_{OA_iA_{i+1}} x_1x_2 = \frac{1}{24}(p_iq_{i+1} - p_{i+1}q_i)(2p_iq_i + 2p_{i+1}q_{i+1} + p_iq_{i+1} + p_{i+1}q_i)\,;$$

inserting the expressions of (p_i, q_i) and summing over i we get

$$\int_{\Omega_N^*} x_1x_2 = \frac{1}{24}\sin\frac{2\pi}{N}\sum_{i=1}^N\Big[\sin(2\alpha + (i-1)\frac{4\pi}{N}) + \sin(2\alpha + i\frac{4\pi}{N}) + \sin(2\alpha + i\frac{2\pi}{N})\Big] = 0$$

Concerning the second equality in (61), on each triangle we have

$$\int_{OA_iA_{i+1}} x_1^2 = \frac{1}{12}(p_i^2 + p_ip_{i+1} + p_{i+1}^2)(p_iq_{i+1} - p_{i+1}q_i),$$

$$\int_{OA_iA_{i+1}} x_2^2 = \frac{1}{12}(q_i^2 + q_iq_{i+1} + q_{i+1}^2)(p_iq_{i+1} - p_{i+1}q_i).$$

Inserting the expressions of (p_i, q_i), we see that $(p_iq_{i+1} - p_{i+1}q_i) = \sin\frac{2\pi}{N}$ for every i, so that

$$\int_{\Omega_N^*} x_1^2 = \frac{1}{12}\sin\frac{2\pi}{N}\sum_{i=1}^N(p_i^2 + p_iq_{i+1} + p_{i+1}^2)$$

$$\int_{\Omega_N^*} x_2^2 = \frac{1}{12}\sin\frac{2\pi}{N}\sum_{i=1}^N(q_i^2 + q_iq_{i+1} + q_{i+1}^2)$$

Now,

$$\sum_{i=1}^N(p_i^2 + p_ip_{i+1} + p_{i+1}^2) = 2\sum_{i=1}^N\cos^2\big(\alpha + i\frac{2\pi}{N}\big) + \sum_{i=1}^N\cos\big(\alpha + i\frac{2\pi}{N}\big)\cos(\alpha + (i+1)\frac{2\pi}{N})$$

$$\sum_{i=1}^{N}(q_i^2+q_iq_{i+1}+q_{i+1}^2)=2\sum_{i=1}^{N}\sin^2\big(\alpha+i\frac{2\pi}{N}\big)+\sum_{i=1}^{N}\sin\big(\alpha+i\frac{2\pi}{N}\big)\sin\big(\alpha+(i+1)\frac{2\pi}{N}\big)$$

Using the formulas

$$\cos^2 a=\frac{1+\cos 2a}{2},\qquad 2\cos a\cos b=\cos(a+b)+\cos(a-b)$$

$$\sin^2 a=\frac{1-\cos 2a}{2},\qquad 2\sin a\sin b=\cos(a-b)-\cos(a+b)$$

we conclude that

$$\sum_{i=1}^{N}\cos^2\big(\alpha+i\frac{2\pi}{N}\big)=\sum_{i=1}^{N}\sin^2\big(\alpha+i\frac{2\pi}{N}\big)=\frac{N}{2}$$

$$\sum_{i=1}^{N}\cos\big(\alpha+i\frac{2\pi}{N}\big)\cos\big(\alpha+(i+1)\frac{2\pi}{N}\big)=\sum_{i=1}^{N}\sin\big(\alpha+i\frac{2\pi}{N}\big)\sin\big(\alpha+(i+1)\frac{2\pi}{N}\big)=N\cos\frac{2\pi}{N}.$$

□

Proof of Lemma 7.
(i) If $\Omega\in\mathcal{P}_8$ is axially symmetric, some elementary computations give

$$\begin{aligned}\int_\Omega\int_\Omega|x-y|^6\,dxdy &=2\pi\int_\Omega|x|^6\,dx+18\int_\Omega|x|^2\,dx\int_\Omega|x|^4\,dx\\ &+12\int_\Omega(x_1^2-x_2^2)\,dx\int_\Omega(x_1^4-x_2^4)\,dx\\ &\geq 2\pi\int_{\Omega_8^*}|x|^6\,dx+18\int_{\Omega_8^*}|x|^2\,dx\int_{\Omega_8^*}|x|^4\,dx\\ &+12\int_\Omega(x_1^2-x_2^2)\,dx\int_\Omega(x_1^4-x_2^4)\,dx\,,\end{aligned}$$

where the inequality is obtained by invoking as usual Theorem 6. Since

$$\int_{\Omega_8^*}(x_1^2-x_2^2)\,dx\int_{\Omega_8^*}(x_1^4-x_2^4)=0\,,$$

to conclude the proof it is enough to show that

$$\int_\Omega(x_1^2-x_2^2)\,dx\int_\Omega(x_1^4-x_2^4)\,dx\geq 0\,.$$

To that aim, we exploit the assumption that Ω is an axisymmetric convex octagon. Assuming without loss of generality that Ω has four vertices at the points $(\pm1,0)$, $(0,\pm1)$, and one in the region $\{x_1>0,\ x_2>0,\ x_2\geq x_1,\ x_2\geq 1-x_1\}$, we have that Ω contains an axisymmetric convex octagon Q, which has still four vertices at the points $(\pm1,0)$, $(0,\pm1)$ and one on the straight line $x_1=x_2$. Then

$$\int_\Omega(x_1^2-x_2^2)\,dx\int_\Omega(x_1^4-x_2^4)\,dx\geq\int_Q(x_1^2-x_2^2)\,dx\int_Q(x_1^4-x_2^4)\,dx=0\,.$$

(ii) We can assume without loss of generality that $k > 1$. Indeed, once the inequality is proved for such k, one can pass to the limit as $k \to 1$. The proof is inspired from [32]. In the remaining of this proof, the functional J_h with $h(x) = |x|^k$ will be denoted for brevity by J_k, i.e. we set

$$J_k(\Omega) := \int_\Omega \int_\Omega |x - y|^k \, dx \, dx \, .$$

Our target is to show that, for every real 2×2 volume preserving real matrix M, it holds

$$J_k(M(\Omega_N^*)) \geq J_k(\Omega_N^*) \, .$$

We have $M = ASB$, with S diagonal and $A, B \in O(2)$. Since $J_k(\Omega) = J_k(A(\Omega))$ for any $\Omega \in \mathcal{P}_N$ and any $A \in O(2)$, it is not restrictive to assume that $A = \mathrm{Id}$. Moreover, by considering the regular polygon $B(\Omega_N^*)$ in place of Ω_N^*, we may assume that also $B = \mathrm{Id}$. We are thus reduced to show that

$$J_k(S(\Omega_N^*)) \geq J_k(\Omega_N^*) \, , \qquad \text{where } S = \mathrm{diag}\{\sigma \, , \, \sigma^{-1}\}, \ \sigma \in \mathbb{R}^+ \, .$$

We consider the family of polygons $\Omega_t := S_t(\Omega_N^*)$, with $S_t = \mathrm{diag}\{\sigma^t \, , \, \sigma^{-t}\}$, so that $\Omega_0 = \Omega_N^*$, and $\Omega_1 = S(\Omega_N^*)$. We claim that $J_k(\Omega_N^*) > J_k(\Omega_t)$ for every $t \in (0, 1]$, or equivalently that the map $g_k(t) := J_k(\Omega_t)$ satisfies $g_k(t) > g_k(0)$ for every $t \in (0, 1]$. Indeed, let us show that

$$g_k'(0) = 0 \, , \qquad g_k''(t) \geq 0 \text{ on } (0, 1) \, . \tag{62}$$

By arguing as in [32, Lemma 4.3 and Lemma 4.4], we are allowed to differentiate under the sign of integral. Setting $\Phi_k(r) = r^k$, and writing for brevity r in place of $|S_t(x) - S_t(y)|$, by direct computations we have

$$g_k'(t) = \int_\Omega \int_\Omega \frac{\partial}{\partial t} \Phi_k(|S_t x - S_t y|) \, dx \, dy \, , \qquad g_k''(t) = \int_\Omega \int_\Omega \frac{\partial^2}{\partial t^2} \Phi_k(|S_t x - S_t y|) \, dx \, dy$$

where

$$\begin{aligned} \frac{\partial}{\partial t} \Phi_k(|S_t x - S_t y|) &= \Phi'(r) \, \frac{S_t(x) - S_t(y)}{|S_t(x) - S_t(y)|} \cdot (\dot{S}_t(x) - \dot{S}_t(y)) \\ &= k r^{k-1} \, \frac{S_t(x) - S_t(y)}{|S_t(x) - S_t(y)|} \cdot (\dot{S}_t(x) - \dot{S}_t(y)) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \Phi_k(|S_t x - S_t y|) &= \Big(\Phi_k''(r) - \frac{\Phi_k'(r)}{r} \Big) \Big(\frac{S_t(x) - S_t(y)}{|S_t(x) - S_t(y)|} \cdot (\dot{S}_t(x) - \dot{S}_t(y)) \Big)^2 \\ &+ \frac{\Phi_k'(r)}{r} \Big(|\dot{S}_t(x) - \dot{S}_t(y)|^2 + (S_t(x) - S_t(y)) \cdot (\ddot{S}_t(x) - \ddot{S}_t(y)) \Big) \\ &= k^2 r^{k-2} \Big(\frac{S_t(x) - S_t(y)}{|S_t(x) - S_t(y)|} \cdot (\dot{S}_t(x) - \dot{S}_t(y)) \Big)^2 \\ &+ k r^{k-2} \Big(|\dot{S}_t(x) - \dot{S}_t(y)|^2 + (\log \sigma)^2 |S_t(x) - S_t(y)|^2 \Big) \, . \end{aligned}$$

Hence conditions (62) are satisfied (we exploit here the assumption $k > 1$). □

8. Proof of Theorem 6

We consider the maximization problem

$$\max\left\{ \int_\Omega h(x)\,dx \ :\ \Omega \in \mathcal{P}_N \right\}, \tag{63}$$

and we proceed as follows:

– In Section 8.1 we prove that it is not restrictive to take $h = \chi_{B_r(0)}$ (cf. Proposition 13).

– In Section 8.2 we prove that problem (63) admits a solution, which is a classical star-shaped polygon (cf. Proposition 14).

– In Section 8.3 we prove that, among classical star-shaped polygons, the regular N-gon is optimal (cf. Proposition 17).

The validity of Theorem 6 follows at once by combining Propositions 13, 14, and 17.

In the sequel, we write for brevity B_r in place of $B_r(0)$.

8.1. Reduction to the characteristic kernel.

Proposition 13. *If a polygon solves problem* (63) *when $h = \chi_{B_r}$ for all $r > 0$, then it solves problem* (63) *for every admissible kernel h.*

Proof. Assume Ω^* solves (63) for every $r > 0$. Consider a sequence of radii $0 < r_1 < ... < r_k$ and some positive reals $d_1, ..., d_k$. Then by optimality of Ω^* for every radius r_i, $i = 1, ..., k$ we have

$$\int_\Omega \sum_{i=1}^k d_i \chi_{B_{r_i}} \le \int_{\Omega^*} \sum_{i=1}^k d_i \chi_{B_{r_i}}$$

Since every radially decreasing step function can be written in the form $\sum_{i=1}^k d_i \chi_{B_{r_i}}$ it follows that Ω^* solves (63) for radially decreasing step functions.

Every radially decreasing function h can be written as the limit of an increasing sequence of radial step functions $\{h_n\}_{n\ge1}$. Passing to the limit in the inequalities

$$\int_\Omega h_n(x)dx \le \int_{\Omega^*} h_n(x)dx,$$

shows that Ω^* solves (63) for arbitrary admissible kernels. □

8.2. Existence and reduction to star-shaped polygons.

In view of Proposition 13, we are going to focus our attention on the maximization problem

$$\max\Big\{ \mathcal{E}(\Omega) := |\Omega \cap B_r| \ :\ \Omega \text{ is a polygon with } N \text{ sides with } |\Omega| \le \pi \Big\}. \tag{64}$$

Proposition 14. *Problem* (64) *admits a solution. Moreover, every solution is star-shaped.*

As a preliminary remark, let us observe that there are some ranges for the value of r for which problem (64) can be elementarily solved. More specifically, let $\Omega_N^{r,circ}$ and $\Omega_N^{r,in}$ denote respectively the regular N-gons circumscribed and inscribed to B_r. Then:

– If $\pi \geq |\Omega_N^{r,circ}|$, the maximum in (64) is equal to πr^2, and it is attained either at infinitely many admissible polygons among which the regular N-gon of area π (if the inequality is strict), or uniquely at $\Omega_N^{r,circ}$ (if the inequality holds with equality sign).

– If $\pi \leq |\Omega_N^{r,in}|$, the maximum in (64) is equal to π, and it is attained either at infinitely many admissible polygons among which the regular N-gon of area π (if the inequality is strict), or uniquely at $\Omega_N^{r,in}$ (if the inequality holds with equality sign).

Thus, in the remaining of this section we always tacitly assume that r is chosen so that

$$|\Omega_N^{r,in}| < \pi < |\Omega_N^{r,circ}| \tag{65}$$

(this is just for definiteness, as our proof below works also in the 'trivial' cases left).

The proof of Proposition 14 requires as a key ingredient a geometric construction that we state separately in the next lemma, along with its application in our problem in the subsequent remark.

Lemma 15. *Let Ω be a polygon with N sides such that $|\Omega \cap B_r| > 0$ and $|\Omega \setminus B_r| > 0$. Then there exists another polygon Ω' with N sides, which is star-shaped and satisfies the inclusions*

$$(\Omega' \cap B_r) \supseteq (\Omega \cap B_r) \qquad \textit{and} \qquad (\Omega' \setminus B_r) \subseteq (\Omega \setminus B_r)\,. \tag{66}$$

Remark 16. Let Ω and Ω' be polygons as in Lemma 15. We claim that, starting from Ω', it is easy to construct another star-shaped polygon $\widetilde{\Omega}$, still having N sides, such that

$$|\widetilde{\Omega}| \leq |\Omega| \qquad \text{and} \qquad |\widetilde{\Omega} \cap B_r| \geq |\Omega \cap B_r|\,. \tag{67}$$

Indeed, from the first inclusion in (66) we have $\mathcal{E}(\Omega') = \mathcal{E}(\Omega) + \delta$, with $\delta > 0$. Compare then the areas of Ω' and Ω. In case $|\Omega'| \leq |\Omega|$, we simply define $\widetilde{\Omega} := \Omega'$. In the case left, namely when $|\Omega'| > |\Omega|$, we define $\widetilde{\Omega}$ as the polygon homothetic to Ω' which as the same area as Ω, namely we take $\widetilde{\Omega} := (|\Omega|/|\Omega'|)^{1/2}\Omega'$. Clearly, such $\widetilde{\Omega}$ is a still a star-shaped polygon with N sides, and it is easy to check that that its energy is not less than the energy of Ω. Actually we have:

$$\mathcal{E}(\widetilde{\Omega}) \geq \frac{|\Omega|}{|\Omega'|}\mathcal{E}(\Omega') = \frac{|\Omega|}{|\Omega'|}\big(\mathcal{E}(\Omega) + \delta\big) = \frac{\mathcal{E}(\Omega) + |\Omega \setminus B_r|}{\mathcal{E}(\Omega) + \delta + |\Omega' \setminus B_r|}\big(\mathcal{E}(\Omega) + \delta\big) \geq \mathcal{E}(\Omega)\,,$$

where the last inequality follows after an immediate computation as a consequence of the two inequalities $\delta > 0$ and $|\Omega' \setminus B_r| \leq |\Omega \setminus B_r|$, the latter holding by the second inclusion in (66).

Let us assume for a moment that Lemma 15 holds true, and let us show how Proposition 14 follows. Let $\{\Omega_n\}$ be a maximizing sequence for problem (64). Clearly, up to a subsequence each polygon Ω_n has a non-negligible intersection both with B_r and with its complement (recall we are assuming (65)). Then, for every n, we denote by Ω_n' the star-shaped polygon given by Lemma 15. Proceeding as in Remark 16, we obtain a new polygon $\widetilde{\Omega}_n$ with its star-shapedness centre inside B_r, which satisfies (67). Thus, $\{\widetilde{\Omega}_n\}$ is still a maximizing sequence.

Now, by the compactness and lower semicontinuity properties of the Hausdorff complementary topology [31, Section 2], up to a subsequence, we may assume that

$$\widetilde{\Omega}_n \xrightarrow{H^c_{\rm loc}} \Omega \tag{68}$$

for some Ω which is a generalized polygon with N sides, according to Definition 12, having area at most π. We observe that the perimeter of the sets $(\widetilde{\Omega}_n \cap B_r)$ is uniformly bounded from above (since all the polygons $\widetilde{\Omega}_n$ have a fixed number N of sides). Hence, by the compact embedding of $BV(B_r)$ into $L^1(B_r)$, the energy $\mathcal{E}(\widetilde{\Omega}_n)$ converges to $\mathcal{E}(\Omega)$.

Notice carefully that the limit generalized polygon Ω is still star-shaped (this is precisely the scope reached through the modification of the sequence Ω_n into the sequence $\widetilde{\Omega}_n$).

It may still occur that $\partial\Omega$ contains some self-intersections, but they can only be contact segments between two consecutive sides. Hence, it is enough to remove any such contact segment, in order to transform Ω into a classical polygon, which will be a solution to problem (64). □

We now turn to the most delicate part of the proof, namely the geometric construction in Lemma 15.

Proof of Lemma 15. Let $\Omega \in \mathcal{P}_N$ be a polygon as in the assumptions of the Lemma. To prove the statement, we can further assume with no loss of generality that Ω has no side tangent to B_r and no vertex in ∂B_r. Indeed, if this is not the case, once the Lemma is proved for polygons with no side tangent to B_r and no vertex in ∂B_r, we can approximate Ω (in the Hausdorff complementary topology) by a sequence of polygons $\{\Omega_n\}$ satisfying such additional conditions, and apply the Lemma to each Ω_n: we find a sequence of polygons $\{\Omega'_n\}$, whose limit polygon Ω' (which exists up to passing to a subsequence and is still star-shaped) does the job for Ω.

Thus, let $\Omega \in \mathcal{P}_N$ be a polygon as in the assumptions of the Lemma, which in addition has no side tangent to B_r, and no vertex in ∂B_r. For the sake of clearness, we give first the construction of the polygon Ω' in a simplified situation, namely when the intersection between Ω and ∂B_r consists precisely of N arcs of circle, and then we proceed in the general case.

• *Case when each side of Ω has both its endpoints outside $\overline{B}_r$, and intersects B_r.* (Equivalently, the intersection between Ω and ∂B_r consists precisely of N arcs of circle.) Starting from a fixed endpoint of such an arc, say P_1, and following a counter-clockwise oriented parametrization of ∂B_r, name these arcs $\widehat{P_iQ_i}$, for $i = 1, \dots, N$; none of these arcs is degenerated into a point, since by assumption no side of Ω is tangent to B_r, see Figure 6, left.

For $i = 1, \dots N$, let γ_i be the straight line through Q_i and P_{i+1}, with the convention $P_{N+1} = P_1$, and let π_i be the (open) half-plane determined by γ_i which contains all the points Q_k, P_j for $k \neq i$ and $j \neq i + 1$.

$$\Omega' := \bigcap_{i=1,\dots,N} \pi_i \, . \tag{69}$$

By construction, Ω' is a classical convex polygon in $\mathcal{P}_N$, see Figure 6, right.

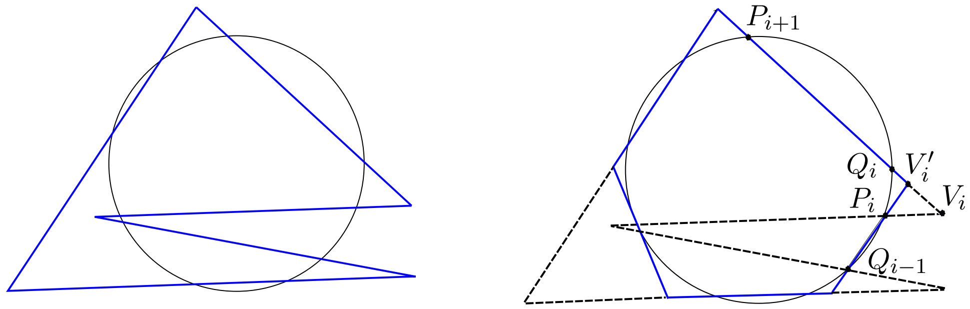

FIGURE 6. Costruction of the polygon Ω' in Lemma 15, case when each side has endpoints outside $\overline{B}_r$ and meets B_r

The first inclusion in (66) is satisfied because any circular segment delimited by the arc $\widehat{P_{i+1}Q_i}$ and the segment $P_{i+1}Q_i$ cannot intersect Ω (otherwise the arc $\widehat{P_{i+1}Q_i}$ would be crossed by a side of Ω).

The second inclusion in (66) is satisfied because, denoting by V_i the common vertex of the two consecutive sides of Ω containing P_i and Q_i, and by V_i' the intersection of the straight lines γ_i (through Q_i and P_{i+1}) and γ_{i-1} (through Q_{i-1} and P_i), it holds

$$\Delta\, Q_i V_i' P_i \ \subseteq \ \Delta\, Q_i V_i P_i \, .$$

This is due to the fact that the point V_i' belongs to both the half-plane determined by the straight line trhough V_i and P_i and containing Q_i, and the half-plane determined by V_i and Q_i and containing P_i.

• *General case.* Consider the intersection between Ω and ∂B_r. Such intersection consists now of M arcs of circle, with $M \leq N$. Starting from a fixed endpoint of such an arc, say P_1, and following a counter-clockwise parametrization of ∂B_r, name these arcs $\widehat{P_iQ_i}$, for $i = 1, \dots, M$. Notice that, if we equip $\partial\Omega$ with an oriented parametrization such that Ω lies on the left of each side, then, at every point Q_i, the side of Ω passing through Q_i is entering into B_r.

For $i = 1, \dots M$, let π_i be the half-planes defined as above. Let also $\{A_1, \dots, A_k\}$ denote the (possibly empty) family of vertices of Ω lying inside B_r (recall that by assumption no vertex of Ω lies on ∂B_r).

We point out that, if one would define Ω' as in (69), none of the two inclusions in (69) would be in general satisfied: the former due to the possible presence of vertices inside B_r, the latter due to the possible presence of sides exterior to B_r (in both cases, with possible self-intersections occuring in $\partial\Omega$).

For this reason, the definition of Ω' is more involved: we are going to construct it as the union of two sets, denoted by Ω'_{in} and Ω'_{out}, which lie respectively inside and outside B_r. Such sets are "curvilinear" polygons, whose boundaries do not contain self-intersections, and consist in a finite number of "sides", meant as arcs of circle lying on ∂B_r or line segments (in case of Ω'_{in}, the segments lie inside B_r, while in case of Ω'_{out} they lie outside). The closures of the two sets Ω'_{in} and Ω'_{out} intersect precisely at the M arcs $\widehat{P_iQ_i}$, so that the set

$$\Omega' := \Omega'_{in} \cup \Omega'_{out} \cup \{\widehat{P_iQ_i} \ : \ i = 1, \dots, M\} \tag{70}$$

turns out to be a classical polygon, which by construction will be a star-shaped one.

Let us specify how Ω'_{in} and Ω'_{out} are defined. We set

$$\Omega'_{in} := \mathrm{conv}\Big(\bigcap_{i=1,\dots,M} \pi_i, A_1, \dots, A_k \Big) \cap B_r \,.$$

where conv denotes the convex envelope. By construction, specifically thanks to the presence of the vertices $\{A_1, \dots, A_k\}$ in the above definition, we have

$$\Omega'_{in} \supseteq (\Omega \cap B_r) \,. \tag{71}$$

In order to define Ω'_{out}, let us choose a point in the interior of Ω'_{in}, say x_0, which does not belong to any of the straight lines supporting the edges of Ω.

Let us denote by $\mathcal{F}$ the family of all the straight lines supporting the non-circular edges of Ω'_{in}. For every $i = 1, \dots, M$, we introduce the set

$$\Delta_i := \Big\{ x + t\overrightarrow{x_0 x} \ : \ x \in \widehat{P_iQ_i}, \ t \in (0, \lambda(x)) \Big\}$$

where

$$\lambda(x) := \inf \Big\{ t > 0 \ : \ x + t\overrightarrow{x_0 x} \in (\partial\Omega \cup \mathcal{F}) \Big\} \,.$$

We define

$$\Omega'_{out} := \bigcup_{i=1}^{M} \Delta_i \,.$$

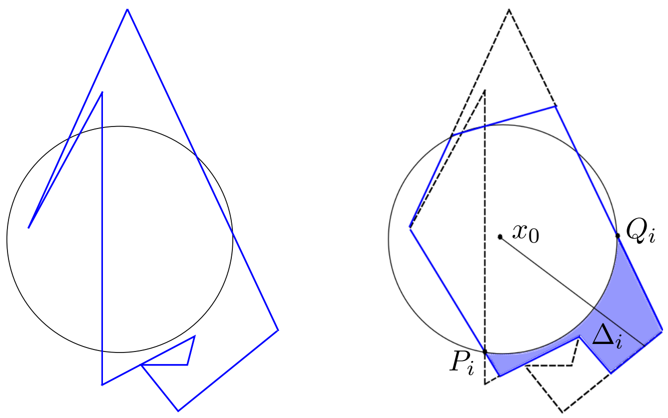

Figure 7. Costruction of the polygon Ω' in Lemma 15, general case

By construction we have

$$\Omega'_{out} \subseteq (\Omega \setminus B_r) \,. \tag{72}$$

In view of (71), and (72), the set Ω' defined in (70) is a classical star-shaped polygon which satisifes both inclusions in (66) as required.

To conclude, it remains to show that $\Omega' \in \mathcal{P}_N$, namely that our procedure respects the constraint on the number of sides. We set

$$\begin{aligned} N_{in} &:= \text{ number of sides of } \Omega \text{ which intersect } B_r \\ N_{out} &:= \text{ number of sides of } \Omega \text{ which do not intersect } B_r \,, \end{aligned}$$

so that $N_{in} + N_{out} = N$. Denoting by $\mathcal{S}'$ the family of sides of Ω', we have

$$\mathcal{S}' = \mathcal{S}'_{in} \cup \mathcal{S}'_{out} \,,$$

where $\mathcal{S}'_{in}$ and $\mathcal{S}'_{out}$ are the families of non-circular sides respectively of Ω'_{in} and of Ω'_{out}. To prove that $\Omega' \in \mathcal{P}_N$ (namely that $\text{card}(\mathcal{S}') \leq N$), we are going to show that

$$\text{card}(\mathcal{S}'_{in}) \leq N_{in} \qquad \text{and} \qquad \text{card}(\mathcal{S}'_{out} \setminus \mathcal{S}'_{in}) \leq N_{out} \,. \tag{73}$$

(i) Counting inside. For every $i = 1, \dots, M$, let us denote by $A_i^1, \dots, A_i^{j_i}$ the vertices of Ω'_{in} lying in the interior of the circular segment delimited by the arc $\widehat{Q_i P_{i+1}}$ and the line segment $Q_i P_{i+1}$. Then the number of (non-circular) sides of Ω'_{in} which join Q_i to P_{i+1} is exactly $1 + j_i$. To any such side, we can associate a side Ω which intersects B_r, in the following way:

- to the side starting at Q_i, we associate the only side of Ω passing through Q_i, which is entering into B_r;
- to the side starting at A_i^k, we associate the side of Ω which starts at A_i^k in the positive orientation of $\partial\Omega$.

Since this association is injective, the first inequality in (73) holds true.

(ii) Counting outside. As above, to any side in $\mathcal{S}'_{out} \setminus \mathcal{S}'_{in}$ we can associate a side of Ω, which in this case does not intersect B_r. Specifically, we distinguish two cases: a side in $\mathcal{S}'_{out} \setminus \mathcal{S}'_{in}$ is either a side of Ω which does not intersect B_r or a newly created one.

- to a side which does not intersect B_r, we associate the corresponding side of Ω; note that the association is injective, because a side of Ω cannot be simultaneously in the boundary of two distinct sets Δ_i's.
- to a newly created side, we associate a side of Ω which is not part of the boundary of $\Delta_{S,i}$ in the following way: a newly created side occurs when some point $A := x + \lambda(x)\overrightarrow{x_0 x}$ is a vertex of Ω and, for $\varepsilon > 0$ sufficiently small, we still have $x + (\lambda(x) + \varepsilon)\overrightarrow{x_0 x} \in \Omega$. This means that A is an endpoint of a side of Ω which is not part of the boundary of Δ_i: this is precisely the side we associate to the newly created one (again, with an injective association).

Thus also the second equality in (73) holds true, and our proof is achived. $\square$

8.3. **Optimality of the regular N-gon.**

Proposition 17. *Problem* (64) *is solved by the regular N-gon of area π.*

Proof. Let Ω be a solution to problem (64), which exists by Proposition 14. As in the proof of Proposition 14, we are going to assume with no loss of generality that the inequalities (65) are satisfied. We are going to prove the result through several claims. We stress that in each of these claims we estimate the variation of the area and of the "energy" $\mathcal{E}(\Omega) := |\Omega \cap B_r|$ when Ω is perturbed by some kind of deformation, preserving the number of sides: this strategy is allowed precisely by the crucial information Ω is a *classical* polygon, whose boundary does not contain self-intersections.

• *No side entirely outside B_r.* Indeed, assume that Ω has a side S which does not intersect B_r. By the right inequality in (65), there exists another side S' which intersects B_r. We move simultaneously S and S', both in a parallell way to themselves, respectively towards the interior and towards the exterior of Ω: the area is preserved while the energy increases, contradicting optimality.

• *No side with one vertex in B_r and one vertex outside $\overline{B}_r$.* Indeed, assume that Ω as such a side S. We perform a rotation of S around its mid-point, in such way that the vertex of S which lies inside B_r moves towards the exterior of Ω: the area is preserved at first order, while the energy increases, again contradicting optimality.

• *No vertex in B_r.* Assume by contradiction that some side S of Ω has an endpoint in B_r. By the previous claim, we know that the other endpoint of S cannot lie outside $\overline{B}_r$, hence it lies either in B_r or on ∂B_r. On the other hand, by the left inequality in (65), we can exclude that all vertices of Ω lie in B_r. We deduce that necessarily Ω contains a chain of consecutive sides, all entirely contained into $\overline{B}_r$, such that the first and the last sides in the chain have one vertex in ∂B_r and the other one in B_r, while all the intermediate sides in the chain are entirely contained into B_r. Now, we can move to ∂B_r all the vertices of the chain lying in B_r so to construct another polygon $\Omega' \in \mathcal{P}_N$ which satisfies the inclusions (66). Starting from this polygon Ω' and arguing as in Remark 16, we find another polygon $\widetilde{\Omega} \in \mathcal{P}_N$, with $|\widetilde{\Omega}| \leq \pi$, which has a strictly larger energy, contradicting optimality.

• *Ω is inscribed into a circle, concentric with B_r, of radius $> r$.* By the previous item, we may associate with each side of Ω a chord of B_r, given by its intersection with B_r (a priori possibly coinciding with the side itself). We perform a rotation of a fixed arbitrary side around its mid point: the first order optimality conditions yield that the mid-point of any side must coincide with the mid-point of the chord associated with it (apply Lemma 19, eq. (79) with $h = \chi_{B_r}$ and eq. (81) from the Appendix in Section 9). We infer that all the vertices of Ω have the same distance from the center of B_r, namely that Ω is inscribed into a circle concentric with B_r. By the left inequality in (65), this circle has radius strictly larger than r.

• *Ω is a regular polygon.* We make a simultaneous parallel movement of two different sides in such a way to preserve the area of Ω. Denoting by ℓ_i the lengths of the sides of Ω and by c_i the lengths of the corresponding chords (obtained by intersecting the sides with B_r), the first order optimality conditions yield

$$\frac{c_i}{\ell_i} = \frac{c_j}{\ell_j} \qquad \text{for every } i \neq j$$

(apply Lemma 19, eq. (82) with $h = \chi_{B_r}$ and eq. (84)). Combined with the previous item, this yields that $\ell_i = \ell_j$ for every i, j, and hence Ω is a regular polygon.

• *Ω is the regular N-gon of area π.* We already know from the previous steps that Ω is a regular polygon, with number of vertices at most N and area at most π.

If Ω has a number of sides strictly less than N, we can add a side just by "cutting" a corner which lies outside B_r. In this case we obtain a new optimal polygon with an edge not intersecting B_r, which contradicts the first step of this proof. Hence, the optimal polygon is a regular N-gon centred with B_r. Then, since regular N-gons centred with B_r are monotone by inclusions, the optimal polygon must have the maximal admissible area, i.e. area equal to π. □

9. Appendix: first and second order shape derivatives

9.1. **General formulas.** When h is an integrand of class $\mathcal{C}^2$, integral energies on $\mathbb{R}^d$ such as

$$\mathcal{E}_h(\Omega) := \int_\Omega h(x)\,dx\,, \qquad \text{or} \qquad J_h(\Omega) := \int_\Omega\int_\Omega h(x-y)\,dx\,dy\,.$$

are twice differentiable with respect to domain perturbations. More precisely, given a Lipschitz velocity field $\theta : \mathbb{R}^d \to \mathbb{R}^d$, let $\Phi_t^\theta(x)$ denote a one parameter family of diffeomorphisms from $\mathbb{R}^d$ into itself with initial velocity θ, i.e. $\Phi_t(x) = x + \theta(x)t + o(t)$.

The first and second order Fréchet shape derivatives of $\mathcal{E}_h$, meant respectively as

$$\begin{aligned}(\mathcal{E}_h)'_\theta(\Omega) &= \lim_{t\to 0}\frac{\mathcal{E}_h(\Phi_t^\theta(\Omega)) - \mathcal{E}_h(\Omega)}{t}\\ (\mathcal{E}_h)''_{\theta,\xi}(\Omega) &= \lim_{t\to 0}\frac{(\mathcal{E}'_h)_\theta(\Phi_t^\xi(\Omega)) - (\mathcal{E}'_h)_\theta(\Omega)}{t}\,,\end{aligned}$$

exist, and their computation as stated in the next lemma is classical, see [31, Theorem 5.2.2, eq. (5.11)], [33, Section 2].

Lemma 18. *Assume h is of class $\mathcal{C}^2$. Then, for any open bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and any Lipschitz deformation $\theta, \xi : \mathbb{R}^d \to \mathbb{R}^d$, it holds*

$$(\mathcal{E}_h)'_\theta(\Omega) = \int_\Omega \operatorname{div}(h\theta) \tag{74}$$

and

$$\begin{aligned}(\mathcal{E}_h)''_{\theta,\xi}(\Omega) = \int_\Omega \Big[&\nabla^2 h\theta\cdot\xi + \nabla h\cdot(\theta\operatorname{div}\xi + \xi\operatorname{div}\theta)\\ &+h\big(\operatorname{div}\theta\operatorname{div}\xi - \tfrac{1}{2}(\nabla^T\theta : \nabla\xi + \nabla^T\xi : \nabla\theta)\big)\Big]\,.\end{aligned} \tag{75}$$

Since the above result is valid in every space dimension, it applies in particular to energies of the type J_h on $\Omega\times\Omega$. In that case, we need to consider "doubled" vector fields of the form $\Theta = (\theta(x), \theta(y)), \Xi = (\xi(x), \xi(y))$, see Section 9.3 for more details.

9.2. **First order shape derivative under rotation/ parallel movement of a side.** We give hereafter the expressions of the first order shape derivatives for the energies

$$\mathcal{E}_h(\Omega) := \int_\Omega h(x)\, dx \qquad \text{and} \qquad J_h(\Omega) := \int_\Omega \int_\Omega h(x-y)\, dx\, dy\,,$$

when a polygon Ω is perturbed by two distinct relevant deformations, preserving the number of sides, which have been previously considered in [10] (see also [6, 24]). We enclose their definitions to make the presentation self-contained. Let Ω be a fixed polygon, and let S be a fixed side of Ω, with consecutive sides S_1 and S_2.

(i) The polygons Ω_ε are obtained from Ω by *rotation of the side S around its mid-point* if they are obtained from Ω by keeping the other sides are fixed, and replacing the three sides (S, S_1, S_2) by the new sides $(S^\varepsilon, S_1^\varepsilon, S_2^\varepsilon)$ described as follows

- S^ε lies on the straight-line obtained by rotating of an oriented angle ε, around the mid-point of S, the straight-line containing S;
- S_1^ε and S_2^ε lie on the same straight-line containing respectively S_1 and S_2;
- the lengths of S^ε, S_1^ε and S_2^ε, are chosen so that the three sides are consecutive.

(ii) The polygons Ω_ε are obtained from Ω by *parallel movemenf of the side S* if they are obtained from Ω by keeping the other sides are fixed, and replacing the three sides (S, S_1, S_2) by the new sides $(S^\varepsilon, S_1^\varepsilon, S_2^\varepsilon)$ described as follows

- S^ε lies on the straight-line parallel to S having signed distance ε from S;
- S_1^ε and S_2^ε lie on the same straight-line containing respectively S_1 and S_2;
- the lengths of S^ε, S_1^ε and S_2^ε, are chosen so that the three sides are consecutive.

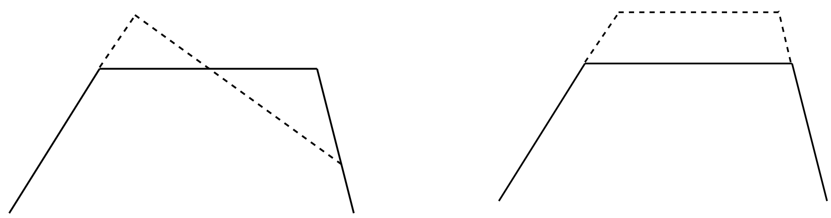

FIGURE 8. Rotation around mid-point (left) and parallel movement (right).

It follows from Lemma 18 that

$$\frac{d}{d\varepsilon}\mathcal{E}_h(\Omega_\varepsilon)\Big|_{\varepsilon=0} = \int_{\partial\Omega} h(x)\, X(x)\cdot \nu_\Omega(x)\, d\mathcal{H}^1(x) \tag{76}$$

$$\frac{d}{d\varepsilon}J_h(\Omega_\varepsilon)\Big|_{\varepsilon=0} = 2\int_{\partial\Omega} v_\Omega(x)\, X(x)\cdot \nu_\Omega(x)\, d\mathcal{H}^1(x)\,, \tag{77}$$

where $v_\Omega(x) = \int_\Omega h(x-y)dy$, $X = \frac{d}{d\varepsilon}\phi_\varepsilon\big|_{\varepsilon=0}$ is the initial velocity of the deformation, and ν_Ω is the unit outward normal defined $\mathcal{H}^1$-a.e. on $\partial\Omega$.

Now, some elementary geometric considerations show that, in cases (i) and (ii), the normal component of the velocity field X is given respectively by

$$X(x)\cdot\nu_\Omega(x) = \begin{cases} |xM| & \text{if } x\in[A_1,M] \\ -|xM| & \text{if } x\in[M,A_2] \end{cases}, \qquad X(x)\cdot\nu_\Omega(x) = 1 \quad \forall x\in[A_1,A_2]\,, \tag{78}$$

where A_1, A_2 are the endpoints of S and M is its midpoint.

We end up with the following

Lemma 19. *(i) If Ω_ε are obtained from Ω by rotation of the side S around its mid-point, it holds*

$$\frac{d}{d\varepsilon}\mathcal{E}_h(\Omega_\varepsilon)\Big|_{\varepsilon=0} = \int_{A_1}^{M} h(x)|xM|\,d\mathcal{H}^1(x) - \int_{M}^{A_2} h(x)|xM|\,d\mathcal{H}^1(x) \tag{79}$$

$$\frac{d}{d\varepsilon}J_h(\Omega_\varepsilon)\Big|_{\varepsilon=0} = 2\Big[\int_{A_1}^{M} v_\Omega(x)|xM|\,d\mathcal{H}^1(x) - \int_{M}^{A_2} v_\Omega(x)|xM|\,d\mathcal{H}^1(x)\Big]\,. \tag{80}$$

In particular, it follows from (79) *taking* $h\equiv 1$ *that*

$$\frac{d}{d\varepsilon}|\Omega_\varepsilon|\Big|_{\varepsilon=0} = 0\,. \tag{81}$$

(ii) If Ω_ε are obtained from Ω by parallel movement of the side S with respect to itself, it holds

$$\frac{d}{d\varepsilon}\mathcal{E}_h(\Omega_\varepsilon)\Big|_{\varepsilon=0} = \int_S h(x)\,d\mathcal{H}^1(x) \tag{82}$$

$$\frac{d}{d\varepsilon}J_h(\Omega_\varepsilon)\Big|_{\varepsilon=0} = 2\int_S v_\Omega(x)\,d\mathcal{H}^1(x)\,. \tag{83}$$

In particular, it follows from (79) *taking* $h\equiv 1$ *that*

$$\frac{d}{d\varepsilon}|\Omega_\varepsilon|\Big|_{\varepsilon=0} = \mathcal{H}^1(S)\,. \tag{84}$$

□

9.3. **Gradient and Hessian under vertices displacement.** Let Ω be a N-gon with vertices $A_1, ..., A_N$, and let $\mathcal{T} = (T_i)_{i=1}^M$ be a triangulation of Ω such that the edges of Ω are edges of some triangles T_i. An example is shown in Figure 9. Following [33, 5], let φ_i denote the piece-wise affine function on the triangulation $\mathcal{T}$ such that $\varphi_i(A_j) = \delta_{ij}$.

Given vectors $\theta_1, \dots, \theta_N$ and $\xi_1, \dots, \xi_N$ which perturb the vertices $A_1, ..., A_N$, respectively, consider the double perturbation fields

$$\Theta(x,y) = \sum_{i=1}^N \begin{pmatrix}\theta_i\varphi_i(x)\\ \theta_i\varphi_i(y)\end{pmatrix} \in \mathbb{R}^2\times\mathbb{R}^2\,, \qquad \Xi(x,y) = \sum_{i=1}^N \begin{pmatrix}\xi_i\varphi_i(x)\\ \xi_i\varphi_i(y)\end{pmatrix} \in \mathbb{R}^2\times\mathbb{R}^2.$$

From the general results recalled in Section 9.1, we have

$$J'_\Theta(\Omega) = \int_\Omega\int_\Omega \nabla h\cdot\Theta + h\,\mathrm{div}\,\Theta$$

$$J''_{\Theta,\Xi}(\Omega) = \int_\Omega\int_\Omega \nabla^2 h\Theta\cdot\Xi + \nabla h\cdot(\Theta\,\mathrm{div}\,\Xi + \Xi\,\mathrm{div}\,\Theta) + h(\mathrm{div}\,\Theta\,\mathrm{div}\,\Xi - \nabla\Theta^T : \nabla\Xi)$$

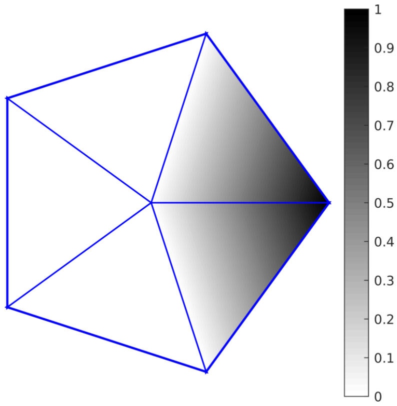

FIGURE 9. Triangulation of the regular polygon used for constructing polygonal deformations together with the piecewise affine function φ_1.

Now, the different terms appearing in the above equalities can be made explicit by using the expressions of Θ, Ξ (the computations are similar to those in [33, Appendix A]). We have the following formulas:

$$\operatorname{div}\Theta = \sum_{i=1}^{N} \theta_i \cdot \varphi_i(x) + \sum_{i=1}^{N} \theta_i \cdot \varphi_i(y)\,;$$

$$\begin{aligned}\operatorname{div}\Xi \operatorname{div}\Theta = &\sum_{i,j=1}^{N} (\theta_i \cdot \nabla\varphi_i(x))(\xi_j \cdot \nabla\varphi_j(x)) + \sum_{i,j=1}^{N} (\theta_i \cdot \nabla\varphi_i(x))(\xi_j \cdot \nabla\varphi_j(y)) \\ &+ \sum_{i,j=1}^{N} (\theta_i \cdot \nabla\varphi_i(y))(\xi_j \cdot \nabla\varphi_j(x)) + \sum_{i,j=1}^{N} (\theta_i \cdot \nabla\varphi_i(y))(\xi_j \cdot \nabla\varphi_j(y)) \\ &= \sum_{i,j=1}^{N} \theta_i \cdot \Big(\nabla\varphi_i(x) \otimes \nabla\varphi_j(x) + \nabla\varphi_i(x) \otimes \nabla\varphi_j(y) \\ &+ \nabla\varphi_i(y) \otimes \nabla\varphi_j(x) + \nabla\varphi_i(y) \otimes \nabla\varphi_j(y)\Big)\xi_j\,;\end{aligned}$$

$$\nabla\Theta^T : \nabla\Xi = \sum_{i,j=0}^{n-1} \theta_i \cdot (\nabla\varphi_j(x) \otimes \nabla\varphi_i(x) + \nabla\varphi_j(y) \otimes \nabla\varphi_i(y))\xi_j\,;$$

$$\begin{aligned}\nabla h \cdot (\Theta \operatorname{div}\Xi + \Xi \operatorname{div}\Theta) = \sum_{i,j=1}^{N} \theta_i \cdot \Big(&\varphi_i(x)(\nabla_x h \otimes \nabla\varphi_j(x)) + \varphi_j(x)(\nabla\varphi_i(x) \otimes \nabla_x h) \\ &+\varphi_i(x)(\nabla_x h \otimes \nabla\varphi_j(y)) + \varphi_j(y)(\nabla\varphi_i(x) \otimes \nabla_y h) \\ &+\varphi_i(y)(\nabla_y h \otimes \nabla\varphi_j(x)) + \varphi_j(x)(\nabla\varphi_i(y) \otimes \nabla_x h) \\ &+\varphi_i(y)(\nabla_y h \otimes \nabla\varphi_j(y)) + \varphi_j(y)(\nabla\varphi_i(y) \otimes \nabla_y h)\Big)\xi_j\,;\end{aligned}$$

$$\nabla^2 h(x,y)\Theta\cdot\Xi=\sum_{i,j=1}^{N}\theta_i\cdot\Big(\varphi_i(x)\varphi_j(x)\nabla^2_{xx}h(x,y)+\varphi_i(x)\varphi_j(y)\nabla^2_{xy}h(x,y)$$
$$+\varphi_i(y)\varphi_j(x)\nabla^2_{yx}h(x,y)+\varphi_i(y)\varphi_j(y)\nabla^2_{yy}h(x,y))dxdy\Big)\xi_i\,.$$

Inserting these formulas into the expressions of $J'_\Theta(\Omega)$ and $J''_{\Theta,\Xi}(\Omega)$, we find that

$$J'_\Theta(\Omega)=B\cdot\theta=\sum_{i=1}^{N}B_i\cdot\theta_i\qquad\text{and}\qquad J''_{\Theta,\Xi}(\Omega)=\theta^T\cdot M\xi=\sum_{i,j=1}^{N}\theta_i\cdot M_{ij}\xi_j\,,$$

where the vector $B=(B_i)_{i=1,\dots,N}$ and the matrix $(M_{ij})_{i,j=1,\dots,N}$, representing respectively the gradient and the Hessian of J with respect to the vertices, are given by

$$B_i=\int_\Omega\int_\Omega(\varphi_i(x)\nabla_x h+\varphi_i(y)\nabla_y h+h(x,y)(\nabla\varphi_i(x)+\nabla\varphi_i(y)))dxdy \tag{85}$$

and

$$\begin{aligned}M_{ij}=&\int_\Omega\int_\Omega h(x,y)(\nabla\varphi_i(x)\otimes\nabla\varphi_j(x)-\nabla\varphi_j(x)\otimes\nabla\varphi_i(x)\\&\qquad+\nabla\varphi_i(x)\otimes\nabla\varphi_j(y)+\nabla\varphi_i(y)\otimes\nabla\varphi_j(x)\\&\qquad+\nabla\varphi_i(y)\otimes\nabla\varphi_j(y)-\nabla\varphi_j(y)\otimes\nabla\varphi_i(y))dxdy\\&+\int_\Omega\int_\Omega(\varphi_i(x)(\nabla_x h\otimes\nabla\varphi_j(x))+\varphi_j(x)(\nabla\varphi_i(x)\otimes\nabla_x h)\\&\qquad+\varphi_i(x)(\nabla_x h\otimes\nabla\varphi_j(y))+\varphi_j(y)(\nabla\varphi_i(x)\otimes\nabla_y h)\\&\qquad+\varphi_i(y)(\nabla_y h\otimes\nabla\varphi_j(x))+\varphi_j(x)(\nabla\varphi_i(y)\otimes\nabla_x h)\\&\qquad+\varphi_i(y)(\nabla_y h\otimes\nabla\varphi_j(y))+\varphi_j(y)(\nabla\varphi_i(y)\otimes\nabla_y h))dxdy\\&+\int_\Omega\int_\Omega(\varphi_i(x)\varphi_j(x)\nabla^2_{xx}h(x,y)+\varphi_i(x)\varphi_j(y)\nabla^2_{xy}h(x,y)\\&\qquad+\varphi_i(y)\varphi_j(x)\nabla^2_{yx}h(x,y)+\varphi_i(y)\varphi_j(y)\nabla^2_{yy}h(x,y))dxdy\,.\end{aligned} \tag{86}$$

Note: On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] C. Audet, Maximal area of equilateral small polygons, Amer. Math. Monthly **124** (2017), no. 2, 175–178.
[2] C. Audet, P. Hansen, F. Messine, and J. Xiong, The largest small octagon, J. Combin. Theory Ser. A **98** (2002), no. 1, 46–59.
[3] C. Audet, P. Hansen, and D. Svrtan, Using symbolic calculations to determine largest small polygons, J. Global Optim. **81** (2021), no. 1, 261–268.
[4] H. Bieri, Ungelöste Probleme: Zweiter Nachtrag zu Nr. 12, Elem. Math. **16** (1961), 105–106.
[5] B. Bogosel and D. Bucur, On the Polygonal Faber-Krahn Inequality, Arxiv arXiv:2203.16409.

[6] M. Bonacini, R. Cristoferi, and I. Topaloglu, Riesz-type inequalities and overdetermined problems for triangles and quadrilaterals, J. Geom. Anal. **32** (2022), no. 2, Paper No. 48, 31.

[7] J. Bourgain, H. Brezis, and P. Mironescu, Another look at Sobolev spaces, Optimal control and partial differential equations, IOS, Amsterdam, 2001, pp. 439–455.

[8] H.J. Brascamp, E.H. Lieb, and J.M. Luttinger, A general rearrangement inequality for multiple integrals, J. Functional Analysis **17** (1974), 227–237.

[9] D. Bucur and G. Buttazzo, Variational methods in shape optimization problems, Progress in Nonlinear Differential Equations and their Applications, 65, Birkhäuser Boston Inc., Boston, MA, 2005.

[10] D. Bucur and I. Fragalà, A Faber-Krahn inequality for the Cheeger constant of N-gons, J. Geom. Anal. **26** (2016), no. 1, 88–117.

[11] D. Bucur and I. Fragalà, Symmetry results for variational energies on convex polygons, ESAIM Control Optim. Calc. Var. **27** (2021), Paper No. 3, 16.

[12] D. Bucur and I. Fragalà, Rigidity for measurable sets, Adv. Math. **414** (2023).

[13] D. Bucur, I. Fragalà, B. Velichkov, and G. Verzini, On the honeycomb conjecture for a class of minimal convex partitions, Trans. Amer. Math. Soc. **370** (2018), no. 10, 7149–7179.

[14] L. A. Caffarelli and F. H. Lin, An optimal partition problem for eigenvalues, J. Sci. Comput. **31** (2007), no. 1-2, 5–18.

[15] L. A. Caffarelli and P.E. Souganidis, A rate of convergence for monotone finite difference approximations to fully nonlinear, uniformly elliptic PDEs, Comm. Pure Appl. Math. **61** (2008), no. 1, 1–17.

[16] L.A Caffarelli, J.M. Roquejoffre, and O. Savin, Nonlocal minimal surfaces, Comm. Pure Appl. Math. **63** (2010), no. 9, 1111–1144.

[17] T. Carroll, Old and new on the bass note, the torsion function and the hyperbolic metric, Irish Math. Soc. Bull. (2001), no. 47, 41–65. MR 1880191

[18] A. Cesaroni, S. Dipierro, M. Novaga, and E. Valdinoci, Minimizers for nonlocal perimeters of Minkowski type, Calc. Var. Partial Differential Equations **57** (2018), no. 2, Paper No. 64, 40.

[19] A. Cesaroni and M. Novaga, The isoperimetric problem for nonlocal perimeters, Discrete Contin. Dyn. Syst. Ser. S **11** (2018), no. 3, 425–440.

[20] E. De Giorgi, Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. Ia (8) **5** (1958), 33–44.

[21] G. Fejes Tóth, On the intersection of a convex disc and a polygon, Acta Math. Acad. Sci. Hungar. **29** (1977), no. 1-2, 149–153.

[22] A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini, Isoperimetry and stability properties of balls with respect to nonlocal energies, Comm. Math. Phys. **336** (2015), no. 1, 441–507.

[23] J. Foster and T. Szabo, Diameter graphs of polygons and the proof of a conjecture of Graham, J. Combin. Theory Ser. A **114** (2007), no. 8, 1515–1525.

[24] I. Fragalà and B. Velichkov, Serrin-type theorems for triangles, Proc. Amer. Math. Soc. **147** (2019), no. 4, 1615–1626.

[25] R. L. Frank, E.H. Lieb, and R. Seiringer, Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators, J. Amer. Math. Soc. **21** (2008).

[26] R. L. Frank and R. Seiringer, Non-linear ground state representations and sharp Hardy inequalities, J. Funct. Anal. **255** (2008), no. 12, 3407–3430.

[27] R. L. Graham, The largest small hexagon, J. Combinatorial Theory Ser. A **18** (1975), 165–170.

[28] G.H. Hardy, J.E. Littlewood, and G. Pólya, Inequalities, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988, Reprint of the 1952 edition.

[29] D. Henrion and F. Messine, Finding largest small polygons with GloptiPoly, J. Global Optim. **56** (2013), no. 3, 1017–1028.

[30] A. Henrot, Shape optimization and spectral theory, De Gruyter, 2017.

[31] A. Henrot and M. Pierre, Shape variation and optimization, EMS Tracts in Mathematics, vol. 28, European Mathematical Society (EMS), Zürich, 2018.

[32] R.S. Laugesen, Minimizing capacity among linear images of rotationally invariant conductors, Anal. Math. Phys. **12** (2022), no. 1, Paper No. 21, 25.

[33] A. Laurain, Distributed and boundary expressions of first and second order shape derivatives in nonsmooth domains, J. Math. Pures Appl. (9) **134** (2020), 328–368. MR 4053037

[34] L. Lombardini, Fractional perimeters from a fractal perspective, Adv. Nonlinear Stud. **19** (2019), no. 1, 165–196.

[35] M. Ludwig, Anisotropic fractional perimeters, J. Differential Geom. **96** (2014), no. 1, 77–93.

[36] J.M. Mazón, J.D. Rossi, and J.J. Toledo, Nonlocal perimeter, curvature and minimal surfaces for measurable sets, Frontiers in Mathematics, Birkhäuser/Springer, Cham, 2019.

[37] F. Morgan and R. Bolton, Hexagonal economic regions solve the location problem, Amer. Math. Monthly **109** (2002), no. 2, 165–172.

[38] M.J. Mossinghoff, A $1 problem, Amer. Math. Monthly **113** (2006), no. 5, 385–402.

[39] G. Pólya and G. Szegö, Isoperimetric Inequalities in Mathematical Physics, Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, N. J., 1951.

[40] M. Preunkert, A semigroup version of the isoperimetric inequality, Semigroup Forum **68** (2004), no. 2, 233–245.

[41] K. Reinhardt, Extremale polygone gegebene Durchmessers, Jahresber. Deutsch. Math.-Verein **31** (1922), 251–270.

[42] F. Riesz, Sur une inégalité intégrale, J. London Math. Soc. **5** (1930), no. 3, 162–168.

[43] J.J Schäffer, Ungelöste Probleme: Nachtrag zu Nr. 12, Elem. Math. **13** (1958), 85–86.

[44] A.Y. Solynin and V. A. Zalgaller, An isoperimetric inequality for logarithmic capacity of polygons, Ann. of Math. (2) **159** (2004), no. 1, 277–303.

[45] M. van den Berg and K. Gittins, On the heat content of a polygon, J. Geom. Anal. **26** (2016), no. 3, 2231–2264.

[46] M. van den Berg and S. Srisatkunarajah, Heat flow and Brownian motion for a region in $\mathbf{R}^2$ with a polygonal boundary, Probab. Theory Related Fields **86** (1990), no. 1, 41–52.

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