Some numerical aspects in spectral partitioning problems

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25/09/2015

Outline

- Objective: numerical analysis of some spectral shape optimization problems for boundary eigenvalue problems (eg. Steklov, Wentzell)
- Challenge: accurate vs fast!

Problems:

- Steklov spectrum;
- Laplace-Beltrami spectrum: optimal partitions on manifolds.

For $\Omega \subset \mathbb{R}^2$, regular enough, simply connected, we consider

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Omega, \end{cases}$$

Steklov eigenvalues:

$$0 = \sigma_0 \leq \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \sigma_3(\Omega) \leq \ldots \to +\infty$$

Variational characterization:

$$\sigma_n(\Omega) = \inf_{S_n} \sup_{u \in S_n \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\sigma}, \quad n = 1, 2, \ldots$$

$S_n \subset H^1(\Omega) \cap \{\int_{\partial\Omega} u = 0\}$;

$$\sigma_k(t\Omega) = \frac{1}{t}\sigma_k(\Omega).$$

Problem

$$\max_{\mathcal{A}} F(\sigma_1, ..., \sigma_k)$$

- F continuous and increasing in every variable;
- $\mathcal{A}$ - constraints: area/perimeter/convexity;

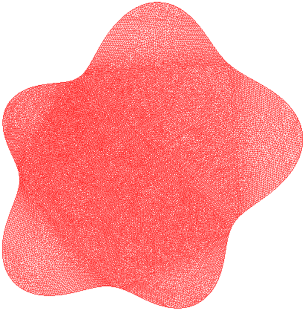
Example : $\max_{|\Omega|=1} \sigma_k(\Omega)$, $\min_{\text{Per}(\Omega)=1} \sum_{l=1}^{k} \frac{1}{\sigma_l(\Omega)}$

Existence results

- Fixed volume + convexity (B.)
- Fixed volume, relaxed formulation (Bucur, Giacomini)

Numerical computation of the Steklov spectrum

FreeFem++ - mesh-based method. For high precision we need many points



Can we be more accurate, use less discretization points and have huge speed improvements?

Fundamental solutions method

"trick"

Work only with functions which solve the PDE exactly in Ω.

- $(y_i)_{i=1}^N$ - the family of points in the exterior of Ω;
- ϕ_i - harmonic radial functions centered at y_i;

$\Rightarrow$ every linear combination

$$u = \alpha_1\phi_1 + ... + \alpha_N\phi_N$$

is harmonic in Ω.

Boundary conditions

$\Rightarrow$ "arrange" (α_i) such that

$$\frac{\partial u}{\partial n} \approx \sigma u \text{ on } \partial\Omega.$$

- $(x_i)_{i=1}^N$ - a discretization of $\partial\Omega$;
- impose boundary conditions at (x_i):

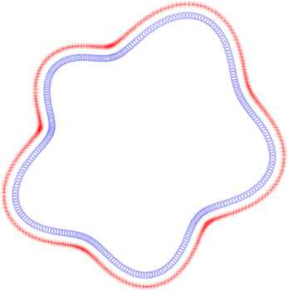
$$\alpha_1 \frac{\partial \phi_1}{\partial n}(x_i) + \ldots + \alpha_N \frac{\partial \phi_N}{\partial n}(x_i) =$$

$$\sigma(\alpha_1\phi_1(x_i) + \ldots + \alpha_N\phi_N(x_i)), \;\; i = 1\ldots N$$

- $\phi_j(x) = \log|x - y_j|$ (analytic formulas) ;
- $A = (\partial_n \phi_j(x_i))_{ij}^N, B = (\phi_j(x_i))_{ij}^N, u = (\alpha_1, ..., \alpha_N)^T$;
- the problem becomes $A\vec{\alpha} = \sigma B\vec{\alpha}$;
- generalized eigenvalue problem : `eigs` (Matlab)

Choice of centers!

- (x_i) are chosen in a uniform way on $\partial\Omega$ (angles or arclength);
- (y_i) are chosen on the normals in (x_i) to $\partial\Omega$ at distance 0.1 of (x_i).



Comparaison FreeFem++

		FreeFem++			
k	MFS	2096▲	33788▲	134898▲	211290▲
1	0.712751	0.714888	0.712886	0.712785	0.712773
2	0.940247	0.942837	0.940411	0.940288	0.940274
3	1.381278	1.38874	1.38175	1.3814	1.38135
4	1.443204	1.45137	1.44372	1.44333	1.44329
5	3.146037	3.15592	3.14665	3.14619	3.14614
6	3.443637	3.45562	3.44438	3.44382	3.44376
7	3.757833	3.78642	3.75962	3.75828	3.75812
8	3.922822	3.95461	3.92478	3.92331	3.92313
9	4.274362	4.32906	4.27774	4.27521	4.2749
10	4.693207	4.75819	4.69723	4.69422	4.69385

Error estimate

- the error seems to be small: quantification ?
- Adaptation of a result of Moler and Payne :

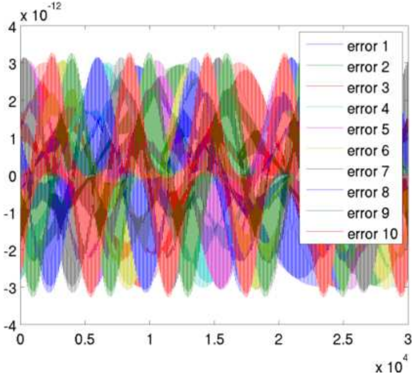
Let Ω be bounded, regular, which satisfies

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial n} = \sigma_\varepsilon u_\varepsilon + f_\varepsilon & \text{on } \partial\Omega. \end{cases}$$

with $\|u_\varepsilon\|_{L^2(\partial\Omega)} = 1$ and $\|f_\varepsilon\|_{L^2(\partial\Omega)} = \delta$ small. Then it exists a rank $k \in \mathbb{N}^*$ such that

$$\frac{|\sigma_\varepsilon - \sigma_k|}{\sigma_k} \leq C\|f_\varepsilon\|_{L^2(\Omega)}.$$

- the unit disk : theoretical prediction : 10^{-12}.

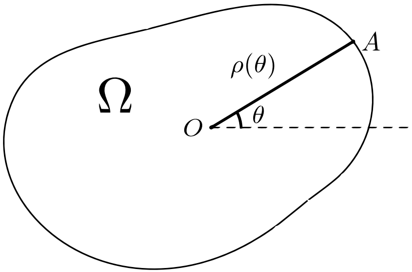


Numerical precision : 10^{-12}.

- in general $\|\partial_n u_{\text{num}} - \sigma_{\text{num}} u_{\text{num}}\|_{L^\infty} = O(10^{-6})$
$\Rightarrow$ precision of order 10^{-6}.

Star shaped sets

$$\Omega \longrightarrow \rho : [0, 2\pi) \rightarrow \mathbb{R}_+.$$



not too restrictive. To be explained later...

Shape derivative - radial case

$$\rho(\theta) \approx a_0 + \sum_{i=1}^{m} a_i \cos(i\theta) + \sum_{i=1}^{m} b_i \sin(i\theta).$$

Dambrine, Kateb, Lamboley, *An extremal eigenvalue problem for the Wentzell-Laplace operator*, Section E

$$\frac{\partial \sigma_k}{\partial a_i} = \int_0^{2\pi} (|\nabla u(\rho(\theta)\vec{r}(\theta))|^2 - (2\sigma_k^2 + \sigma_k \mathcal{H})$$
$$|u(\rho(\theta)\vec{r}(\theta)|^2)\rho(\theta)\cos(i\theta)d\theta$$
$$\frac{\partial \sigma_k}{\partial b_i} = \int_0^{2\pi} (|\nabla u_k(\rho(\theta)\vec{r}(\theta))|^2 - (2\sigma_k^2 + \sigma_k \mathcal{H})$$
$$|u_k(\rho(\theta)\vec{r}(\theta)|^2)\rho(\theta)\sin(i\theta)d\theta.$$

Numerical optimization tests

- Weinstock, Hersch-Payne
- $\min \displaystyle\sum_{i=1}^{n} \frac{1}{\sigma_i(\Omega)|\Omega|^{\frac{1}{2}}}$ realized by the disk
- $\min \displaystyle\sum_{i=1}^{n} \frac{1}{\sigma_{2i-1}(\Omega)\sigma_{2i}(\Omega)|\Omega|}$ realized by the disk
- $\max |\Omega|^{\frac{n}{2}} \prod_{i=1}^{n} \sigma_i(\Omega)$ realized by the disk
- $\min \displaystyle\sum_{i\in A} \frac{1}{\sigma_i(\Omega)|\Omega|^{\frac{1}{2}}}$ realized by the disk, where A has the property : $1 \in A, 2k \in A \Rightarrow 2k-1 \in A$.

Steklov - area constraint

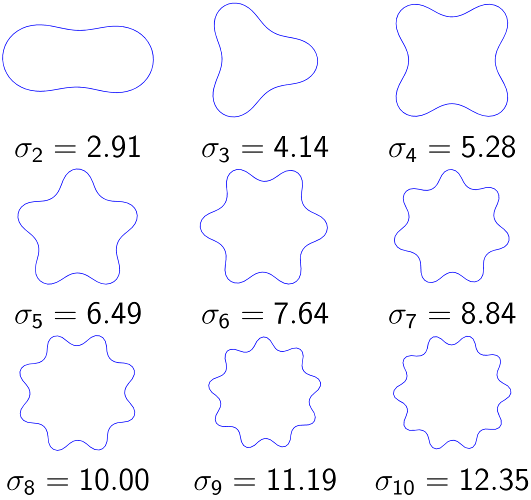


Figure : Shapes which maximize the k-th Steklov eigenvalue under area constraint, $k = 2, 3, ..., 10$.

Steklov - area constraint + convexity

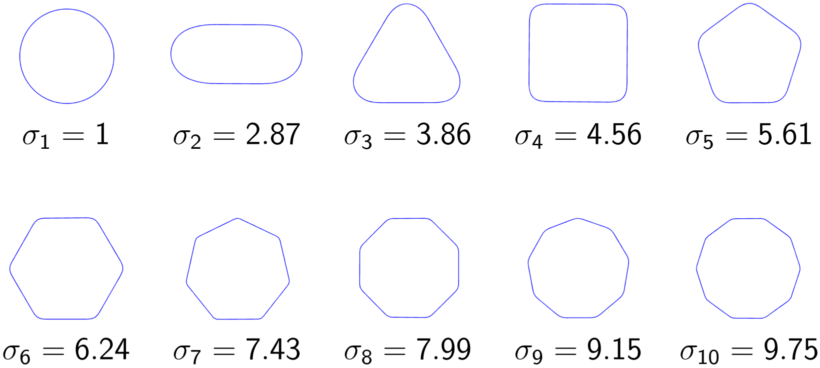


Figure : Convex shapes with unit area which give highest k-th Steklov eigenvalue in our numerical observations

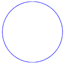
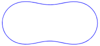
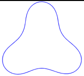
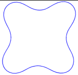
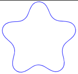
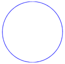
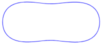
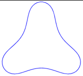
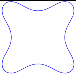
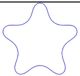
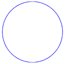
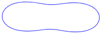
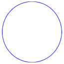
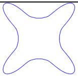
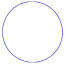
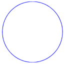
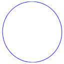
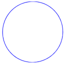
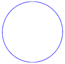
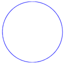
Wentzell problem

$$\begin{cases} -\Delta u = 0 & \text{dans } \Omega, \\ -\beta \Delta_\tau u + \partial u_n = \sigma u & \text{sur } \partial\Omega. \end{cases}$$

- Δ_τ is the Laplace-Beltrami operator;
- adapt method fundamental solutions;
- use the formula: $\Delta u = \Delta_\tau u + \mathcal{H}\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2}$.
- change the boundary condition

$$\sum \alpha_j [(\beta\mathcal{H} + 1)\partial_n \phi_j + \beta \partial_n^2 \phi_j] = \sigma \sum \alpha_j \phi_j.$$

Numerical results - Wentzell

	λ_1	λ_2	λ_3	λ_4	λ_5
$\beta = 0$					
$\beta = 0.1$					
$\beta = 0.5$					
$\beta = 100$					

Laplace-Beltrami spectrum

Objective: study the partitions $(\Omega_i)_{i=1}^n$ of $\mathbb{S}^2$:

$$\min(\lambda_1^{LB}(\Omega_1) + ... + \lambda_1^{LB}(\Omega_n)).$$

Motivation

- verify theoretical conjectures

Fundamental solutions approach

$$\begin{cases} -\Delta_\tau u & = \sigma u \quad \text{on } \mathbb{S}^2 \\ \Delta u & = 0 \quad \text{in } B_1 \end{cases}.$$

We use again the decomposition

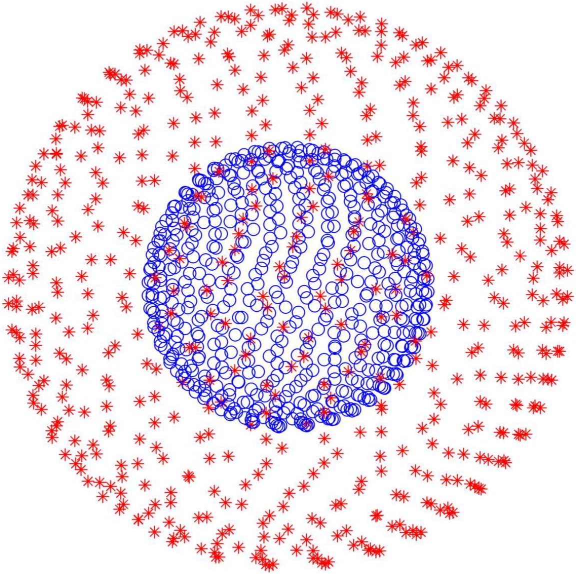
$$\Delta u = \Delta_\tau u + \mathcal{H}\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2},$$

where $\frac{\partial^2 n}{\partial n^2} = (D^2 u.n).n.$

- Fundamental solutions: $\phi_j(x) = 1/|x - y_j|$.
- Boundary condition:

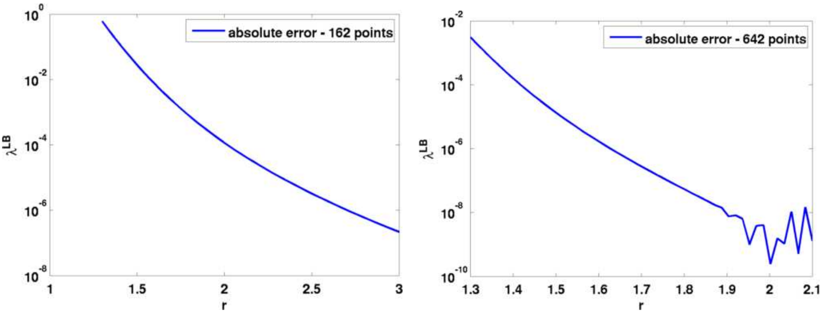
$$\sum \alpha_j (\mathcal{H}\frac{\partial \phi_j}{\partial n} + \frac{\partial^2 \phi_j}{\partial n^2}) = \sigma \sum \alpha_j \phi_j$$

Example source/evaluation points



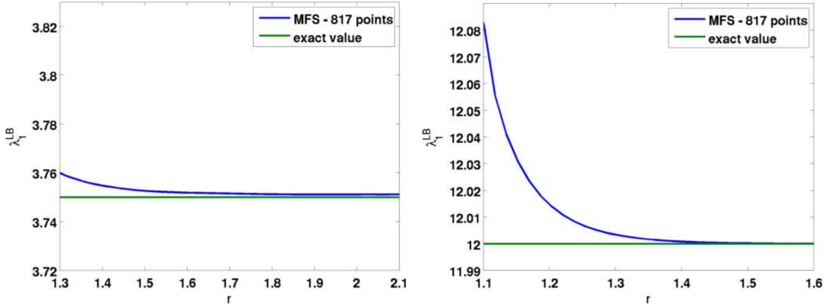
Precision ?

- depends on r (distance between source points and the sphere) and the number of points



Other examples

- vertical slice of angle $2\pi/3$ or triple rectangle triangle

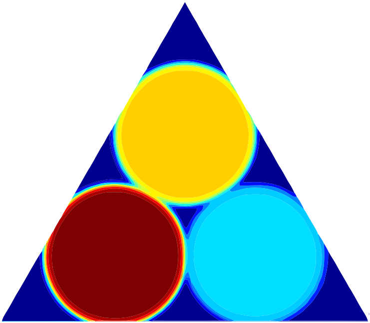


What to do for general subsets of the sphere?

Penalized formulation

$$-\Delta_\tau u + C(1-\varphi)u = \sigma u \text{ on } \mathbb{S}^2$$

- Method used by Bourdin, Bucur, Oudet in order to study numerically spectral optimal partitions in the plane;
- Bogosel, Velichkov, multiphase problem



Discretization

- sphere mesh: evaluation points (x_i) ;
- discrete problem:

$$(K + C\text{diag}(1-\varphi)M)u = \sigma M u,$$

- K: matrix containing the Beltrami terms

$$(\mathcal{H}\frac{\partial \phi_j}{\partial n} + \frac{\partial^2 \phi_j}{\partial n^2})(x_i)$$

- M: matrix containing the values of the fundamental solution functions $\phi_j(x_i)$.
- $\varphi \in [0,1]^N$: the density approximating Ω.

Condition + projection

- Partitioning condition

$$\varphi_1 + ... + \varphi_n = 1.$$

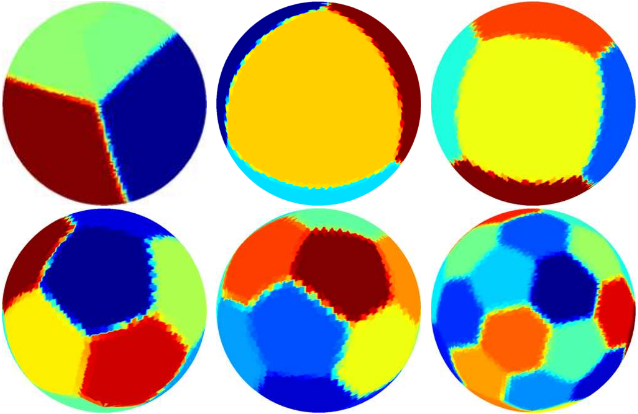
- Projection on the constraint

$$\Pi(\varphi)_{i=1}^n = \left(\frac{|\varphi_i|}{\sum_{j=1}^n |\varphi_j|} \right)_{i=1}^n .$$

Optimization results

- Elliott, Ranner (2014) : $n \in \{3, 4, 5, 6, 7, 8, 16, 32\}$.
- Bogosel (2015) : $n \in \{3, 4, 5, ..., 24, 32\}$.
- for $n \leq 16$ fundamental solutions;
- for $n > 16$ finite element method;

Density results

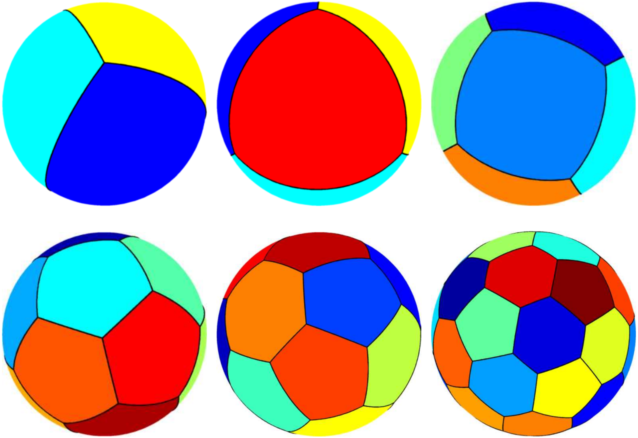


Geodesic arcs

- The partition cells seem to be geodesic polygons;

- Elliott and Ranner observed the same behaviour.

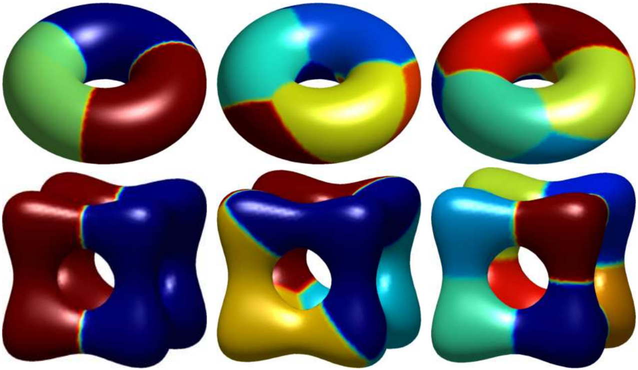
Refined optimization



Some remarks

- in general we have the same results as Elliott and Ranner
- for $n = 16$ we have 4 equal hexagons et 12 equal pentagons (plausible; symmetric structure)
- we observe the same structure as the partitions in equal area cells which minimize the sum of perimeters (Cox-Fikkema)

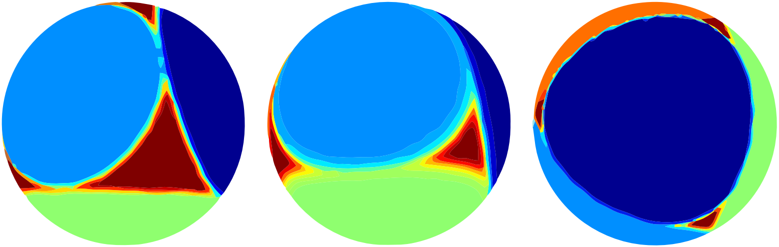
Other shapes



Variations

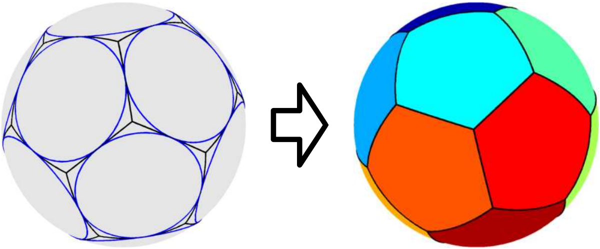
Multiphase problems on surfaces.

$$\min \sum_{i=1}^{n} (\lambda_1^{LB}(\Omega_i) + \alpha \text{Area}(\Omega_i)).$$



Open questions

- which are the optimal partitions for $n = 3, 4, 6, 12$?
  Conjecture : regular partitions;
- is it true that the partition cells are geodesic polygons?
- can we connect the "circle packing" on the sphere to the spectral partitioning problem, via the multiphase problem?



Thank you!