THE STEKLOV SPECTRUM ON MOVING DOMAINS

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ABSTRACT. We study the continuity of the Steklov spectrum on variable domains with respect to the Hausdorff convergence. A key point of the article is understanding the behaviour of the traces of Sobolev functions on moving boundaries of sets satisfying an uniform geometric condition. As a consequence, we are able to prove existence results for shape optimization problems regarding the Steklov spectrum in the family of sets satisfying a ε-cone condition and in the family of convex sets.

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1. INTRODUCTION

For an open, bounded, simply connected set Ω with Lipschitz boundary, we can consider Steklov eigenvalue problem. It is known that the Steklov spectrum of Ω consists of a sequence of the form

$$0 = \sigma_0(\Omega) \leq \sigma_1(\Omega) \leq \sigma_2(\Omega)... \to +\infty.$$

Various optimization problems for functionals depending of the Steklov spectrum, under certain constraints on the geometric properties of Ω, have been studied.

Weinstock [15] observed that $\sigma_1(\Omega)$ is bounded above by $2\pi/\operatorname{Per}(\Omega)$, which means that the disk maximizes the first Steklov eigenvalue in two dimensions, under a perimeter constraint. It is straightforward to see that this implies that the disk also maximizes $\sigma_1(\Omega)$ under area constraint (see Remark 2.4). Girouard and Polterovich proved in [9] that the estimate

$$\sigma_k(\Omega)\operatorname{Per}(\Omega) \leq 2k\pi$$

provided by Hersch, Payne and Schiffer is sharp in the class of simply connected domains, but is not attained in that class. We refer to [9],[10, Section 7.3] for further details.

In general, the known results concerning the optimization of functionals of the Steklov spectrum are proved by identifying an optimizer. Once an optimizer Ω^* is identified, it is proved that the value of the functional on Ω^* is the best possible. In the cases where the optimal shape is not known explicitly, we would like to be able to provide at least an existence result.

First, let's note that in the case of the Steklov eigenvalues, it is only relevant to study optimization problems in which the Steklov eigenvalues are maximized. Indeed, Colbois, El Soufi and Girouard proved in [5] that the Steklov eigenvalues satisfy the bound

$$\sigma_k(\Omega) \leq c_d k^{\frac{2}{d}} \frac{|\Omega|^{\frac{d-2}{d}}}{\operatorname{Per}(\Omega)}. \tag{1.1}$$

Thus, keeping constant volume and increasing the perimeter, we can make the Steklov eigenvalues as low as we want.

A natural way to study optimization problems is to use the classical methods of the calculus of variations. In order to study the problem

$$\max_{\Omega\in\mathcal{A}} \sigma_k(\Omega),$$

where $\mathcal{A}$ is an admissibility class (containing, eventually, some constraints), we need a result concerning the upper semi-continuity of σ_k with respect to some type of convergence.

We mainly deal with the convergence related to the Hausdorff distance, but in a stronger sense which is described in the following. Note that maximizing $\sigma_k(\Omega)$ under perimeter or volume constraint, together with the bound (1.1), means that a maximizing sequence (Ω_n) will have a bound on the perimeters $(\operatorname{Per}(\Omega_n))$. It is well known that a perimeter bound, together with a bounding box constraint implies L^1 compactness of characteristic functions. These considerations allow us to work directly with maximizing sequences converging in the Hausdorff distance and in L^1.

The main results of the article concern inequalities of the type

$$\limsup_{n\to\infty} \sigma_k(\Omega_n) \leq \sigma_k(\Omega), \tag{1.2}$$

under certain regularity assumptions on (Ω_n) and Ω. We work in the framework of sets which satisfy a ε-cone condition, which is equivalent to a uniform Lipschitz property. In particular, this allows us to extend functions in $H^1(\Omega)$ to $H^1(D)$, when $\Omega \subset D$. Another advantage is that we can work with graphs of Lipschitz functions instead of dealing with general sets. We believe that our results could be extended to a more general class of sets described in [14].

We found that in order to prove inequalities of the type (1.2) it is essential to have a result on the lower semi-continuity of traces of Sobolev functions on moving boundaries presented in Proposition 3.2. The main result is stated in Theorem 3.5 and it states that if the sequence of sets (Ω_n) satisfy a ε-cone condition and converge to Ω in the Hausdorff topology then (1.2) holds. Moreover, if the perimeters of Ω_n converge to the perimeter of Ω then we have equality in (1.2). We give a direct proof that the Steklov spectrum of a convex set is close to zero if the diameter is large. This result is a direct consequence of the bound (1.1), but it avoids the use of the technical argument presented in [5]. In the end, we are able to provide existence results in the class of sets satisfying a uniform ε-cone condition, as well as in the class of convex sets. In Figure 2 we present some convex sets obtained numerically for which we have observed the highest, area normalized, k-th Steklov eigenvalue for $k \in [2, 10]$. These shapes were obtained using shape gradients and performing a projection on the convex hull.

As stated above, the semi-continuity result, and the existence results are proved in the class of sets which satisfy a uniform ε-cone condition. It is not clear if these results still hold if this hypothesis is removed and we work in the class of general Lipschitz domains. In the case of the area constraint, Brock proved in [1] that the disk maximizes the first non-trivial Steklov eigenvalue, without any assumptions on the topology of the domain. Ongoing research[1] suggests that in the case of the volume constraint, an existence result can be obtained for a relaxed formulation of the Steklov eigenvalues.

2. Preliminaries

We recall below some theoretical tools needed to prove our results.

2.1. Convergence of sets.

In the study of optimization problems where the variable is the shape of a domain it is often necessary to define a topology on a family of shapes. The choice of such a topology is not obvious, and different situations require different topologies. In our study, we use the Hausdorff complementary convergence on open sets and the L^1 convergence of a of characteristic functions. We recall that the Hausdorff distance between two compact sets K_1, K_2 is given by

$$d_H(K_1, K_2) = \max\{\sup_{x\in K_1} \inf_{y\in K_2} d(x,y),\ \sup_{y\in K_2} \inf_{x\in K_1} d(x,y)\}.$$

If we consider a bounded open set D and the open sets $\Omega_1, \Omega_2 \subset D$ then we define the Hausdorff complementary distance as

$$d_{H^c}(\Omega_1, \Omega_2) = d_H(D \setminus \Omega_1, D \setminus \Omega_2).$$

These two types of convergence are not equivalent in general. Still, it is possible to prove that if we have a bounding box, then any sequence of open sets (Ω_n) has a subsequence converging in the Hausdorff topology to Ω. Furthermore, if the sequence of perimeters of (Ω_n) is bounded, then (Ω_n) has a subsequence which converges in both topologies presented above. We will consider this combined convergence, which provides, in addition to the properties

[1] Private communication with D. Bucur, A. Giacomini (work in progress)

of the Hausdorff convergence, continuity for the volume and lower semi-continuity for the perimeter.

2.2. **Uniform cone condition.** We recall the following definition from [11, Chapter 2].

Definition 2.1. Let y be a point in $\mathbb{R}^d$, ξ a unit vector and $\varepsilon > 0$. We define the cone $C(y, \xi, \varepsilon)$ of vertex y, direction ξ and dimension ε by

$$C(y, \xi, \varepsilon) = \{x \in \mathbb{R}^d : \langle z - y, \xi\rangle \geq \cos\varepsilon|z - y| \text{ and } 0 < |z - y| < \varepsilon\}.$$

We say that an open set Ω has the *ε-cone condition* if for every $x \in \partial\Omega$ there exists a unit vector ξ_x such that for every $y \in \overline{\Omega} \cap B(x, \varepsilon)$ we have $C(y, \xi_x, \varepsilon) \subset \Omega$.

In the proof of our results we use the fact that sets which have the ε-cone condition can be represented locally as the graph of a Lipschitz function. Theorem 2.4.7 from [11] assures us that the ε-cone condition is equivalent to the following uniform Lipschitz condition.

Definition 2.2. We say that a subset Ω of $\mathbb{R}^d$ has a uniform Lipschitz boundary if there are some uniform constants L, a, r such that for any point $x_0 \in \partial\Omega$ there exists an orthonormal system of coordinates S with origin at x_0, a cylinder $K = B_{d-1}(x_0, r) \times (-a, a)$, and a function $\varphi : B_{d-1}(x_0, r) \to [-a, a]$ which is Lipschitz, with constant L and $\varphi(0) = 0$ such that

$$\partial\Omega \cap K = \{(y, \varphi(y)) : y \in K\},$$
$$\Omega \cap K = \{(y, x_N) \in K : x_N > \varphi(y)\}.$$

One advantage of working with sets satisfying an ε-cone condition is the fact that the two types of sets convergence defined before are connected. The Hausdorff complementary convergence of a sequence of sets implies the convergence of characteristic functions in $L^1(D)$ to the same limit. We refer to [11, Theorem 2.4.10] for a proof. Furthermore, if Ω satisfies a ε-cone condition, then the constants L, a, r in the above theorem depend only on ε.

The following proposition mentions an interesting property of the sets which satisfy an ε-cone condition. Using the fact that the boundary of such a set has a local representation as the graph of a Lipschitz function, we can find a bound on the perimeter.

Proposition 2.3. Suppose D is a bounded, open set in $\mathbb{R}^d$ and suppose that $\Omega \subset D$ satisfies a ε-cone condition. Then $\mathrm{Per}(\Omega)$ is uniformly bounded by a constant which depends only on ε and D.

Proof: The above remarks, allow us to say that for every $x_0 \in \partial\Omega$ there exist a cylinder K of the form $B_{d-1}(x_0, r) \times (-a, a)$ centred at x_0 such that $\partial\Omega \cap K$ is the graph of a Lipschitz function with Lipschitz constant L. Furthermore, L, a, r depend only on ε. Note that the perimeter of Ω restricted to K, denoted $\mathrm{Per}_K(\Omega)$, can be expressed as

$$\mathrm{Per}_K(\Omega) = \int_{B_{d-1}(x_0,r)} \sqrt{1 + |\nabla\varphi(x)|^2}dx \leq |B_{d-1}(x_0, r)|\sqrt{1 + L^2}.$$

Therefore, in every such cylinder K, the relative perimeter of Ω is bounded by a constant which depends only on ε.

We claim that the boundary of Ω can be covered with M such cylinders K, where M depends on D. To see this, we propose the following construction. Choose $x_1 \in \partial\Omega$ and let K_1 be the associated cylinder, like in Definition 2.2. At step n, choose $x_n \notin K_1 \cup ... \cup K_{n-1}$ and denote K_n its corresponding cylinder. This operation must end at some point, since pairwise distances between x_i and x_j, with $i \neq j$ are bounded below by a constant $c = \min\{a, r\}$ depending on ε.

To see that there exist a maximal number of points inside D satisfying this property, it is enough to cover D with cubes with a diameter $c' < c$. Obviously, since D is bounded, it is possible to cover D with a finite number M of such cubes. Each cube can contain at most one of the points x_i, since it's diameter is smaller than c. Therefore, the above construction ends in at $n \leq M$ steps.

As a consequence

$$\mathrm{Per}(\Omega) \leq \sum_{i=1}^{n} \mathrm{Per}_{K_i}(\Omega) \leq M|B_{d-1}(x_0,r)|\sqrt{1+L^2}.$$

Thus, the perimeter of Ω is uniformly bounded by a constant depending on ε and D. □

2.3. **Steklov Spectrum.** Let Ω be a simply-connected bounded planar domain with Lipschitz boundary. The *Steklov* eigenvalue problem is

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \sigma u \text{ on } \partial\Omega, \end{cases}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative. The spectrum of the Steklov problem is discrete and its eigenvalues

$$0 = \sigma_0 < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \sigma_3(\Omega) \leq ... \to +\infty$$

satisfy the following variational characterization

$$\sigma_n(\Omega) = \min_{S_n} \max_{u \in S_n \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\sigma},\ n = 1, 2, ...$$

The infimum is taken over all n-dimensional subspaces S_n of $H^1(\Omega)$ that are orthogonal to constants on $\partial\Omega$, i.e. $\int_{\partial\Omega} u d\sigma = 0$.

The Steklov eigenvalues behave well under domain dilatation. Indeed, if we denote $t\Omega$ an image of Ω by a homothety of ratio $t > 0$ then we have

$$\sigma_k(t\Omega) = \frac{1}{t}\sigma_k(\Omega). \tag{2.1}$$

Remark 2.4. In view of property (2.1), the quantities $\sigma_k(\Omega)\,\mathrm{Per}(\Omega)$ and $\sigma_k(\Omega)|\Omega|^{1/2}$ are scale invariant. Thus maximizing $\sigma_k(\Omega)$ under perimeter constraint is equivalent to the problem

$$\max \sigma_k(\Omega)(\mathrm{Per}(\Omega))^{\frac{1}{d-1}},$$

and maximizing $\sigma_k(\Omega)$ under volume constraint is equivalent to the problem

$$\max \sigma_k(\Omega)|\Omega|^{1/d}.$$

Combining the above formulations with the classical isoperimetric inequality, we can conclude that if the ball maximizes σ_k, or another well behaving function of the Steklov spectrum, under a perimeter constraint, then the ball also maximizes the same function under volume constraint.

3. Stability of Steklov Spectrum under Hausdorff Convergence

We recall the following result, which can be found in a similar form in in [6, Theorem 2.3.1]. The weak L^2 convergence coupled with the convergence of a certain integral sequence implies strong L^2 convergence.

Lemma 3.1. Let Ω be a measurable subset of $\mathbb{R}^n$ and suppose $F : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function of class C^1, i.e. it exists $\mu > 0$ such that

$$F(y) \geq F(x) + \nabla F(x) \cdot (y - x) + \mu|y - x|^2,$$

for every $x, y \in \mathbb{R}^n$. Furthermore, we assume that F has the property that if $u \in L^2(\Omega;\mathbb{R}^n)$ then $\nabla F(u)$ is also in $L^2(\Omega;\mathbb{R}^n)$. Let (u_k) be a sequence in $L^2(\Omega,\mathbb{R}^n)$ such that $u_k \rightharpoonup u$ in $L^2(\Omega,\mathbb{R}^n)$. Suppose the following inequality holds:

$$\limsup_{k\to\infty} \int_\Omega F(u_k)dx \leq \int_\Omega F(u)dx$$

Then

$$u_k \to u \text{ in } L^2(\Omega;\mathbb{R}^n).$$

Proof: For every x we have

$$F(u_k(x)) \geq F(u(x)) + \nabla F(u(x)) \cdot (u_k(x) - u(x)) + \mu |u_k(x) - u(x)|^2.$$

Integrating on Ω we have

$$\int_\Omega F(u_k(x))dx \geq \int_\Omega F(u(x))dx + \int_\Omega \nabla F(u(x)) \cdot (u_k(x) - u(x))dx + \mu |u_k - u|^2_{L^2(\Omega;\mathbb{R}^n)}. \quad (3.1)$$

Note that since $\nabla F(u)$ is in $L^2(\Omega; \mathbb{R}^n)$ and $u_k \rightharpoonup u$ weakly in $L^2(\Omega; \mathbb{R}^n)$ we have

$$\lim_{n\to\infty} \int_\Omega \nabla F(u(x)) \cdot (u_k(x) - u(x))dx = 0,$$

Taking $n \to \infty$ in (3.1) and using the hypothesis we obtain

$$0 \geq \mu \limsup_{n\to\infty} \|u_k - u\|_{L^2(\Omega;\mathbb{R}^n)},$$

which implies that $u_k \to u$ strongly in $L^2(\Omega; \mathbb{R}^n)$. □

We apply this Lemma in the case where $F = \sqrt{1 + \|x\|^2}$. This function is not strongly convex on all $\mathbb{R}^n$, but it is strongly convex on every bounded open set. Furthermore, $\nabla F = \frac{x}{\sqrt{1+|x|^2}}$ so F satisfies all the hypotheses of Lemma 3.1.

The following general proposition is a central result of the article, that will allow us to prove a result of shape continuity for the Steklov spectrum. It allows us pass to the limit when considering traces of a weakly H^1 convergent sequence on moving boundaries that converge in the Hausdorff distance. A similar result has been proved in [4] for the more restrictive class of convex domains.

Proposition 3.2. (Convergence of traces) Let D be an open, bounded subset of $\mathbb{R}^d$. Suppose $(\Omega_n), \Omega \subset D$ are open, connected sets which satisfy a uniform ε-cone property and $\Omega_n \xrightarrow{H^c} \Omega$.

(A) For every $(u_n) \subset H^1(D)$ which converges weakly to u in $H^1(D)$ we have

$$\liminf_{n\to\infty} \int_{\partial\Omega_n} |u_n|^p \geq \int_{\partial\Omega} |u|^p$$

(B) Consider $p \in [1, 2]$. Then $\mathrm{Per}(\Omega_n) \to \mathrm{Per}(\Omega)$ if and only if for every $(u_n) \subset H^1(D)$ which converges weakly to u in $H^1(D)$ we have

$$\int_{\partial\Omega_n} |u_n|^p \to \int_{\partial\Omega} |u|^p.$$

Proof: We start with part (B). Note that if the integral convergence holds for any $(u_n), u$ such that $u_n \rightharpoonup u$, then taking $u_n, u \equiv 1$ we obtain exactly $\mathrm{Per}(\Omega_n) \to \mathrm{Per}(\Omega)$.

To prove the converse implication, suppose $\mathrm{Per}(\Omega_n) \to \mathrm{Per}(\Omega)$. First, let's note that is enough to prove convergence result for a subsequence of (u_n). Indeed, from the trace theorem, we know there exists a constant C which depends uniformly on L (see, for example, [7]), such that

$$\|u_n\|_{L^2(\partial\Omega_n)} \leq C\|u_n\|_{H^1(\Omega)}.$$

The fact that u_n converges weakly in $H^1(D)$ implies that (u_n) is bounded in $H^1(D)$ and, by the above inequality, (u_n) is bounded in $L^2(\partial\Omega)$. Furthermore, if $p < 2$, the fact that Ω_n have finite perimeter, $(\mathrm{Per}(\Omega_n))$ is bounded and $2/p > 1$ allows us to conclude, via the Hölder inequality, that $(\int_{\partial\Omega_n} |u_n|^p)$ is also bounded. If we prove the convergence for a subsequence, then any other convergent subsequence will have the same limit, so the whole sequence will converge. This means that in the course of the proof we may pass to a subsequence of (u_n, Ω_n) whenever it is necessary.

Consider the open sets $U_{x_0} = B(x_0, r) \times (-a, a)$ given for each x_0 by Definition 2.2. These open sets cover $\partial\Omega$ which is compact. Thus we can extract a finite cover $\{U_1, ..., U_N\}$. We can assume, that for n great enough, each $\partial\Omega_n$ is representable as the graph of a Lipschitz function in the same coordinate system as $\partial\Omega$. We refer to [11, Chapter 2] for more details.

Consider a partition of unity $\phi_1, ..., \phi_N$ subordinated to the cover $\{U_1, ..., U_N\}$. It remains to prove that

$$\int_{\partial\Omega_n \cap U_i} |u_n|^p \phi_i d\sigma \to \int_{\partial\Omega \cap U_i} |u|^p \phi_i d\sigma.$$

Since $u_n \rightharpoonup u$ in $H^1(D)$ implies $u_n\phi \rightharpoonup u\phi$ in $H^1(D)$, we can drop the ϕ_i in the above limit and look only at integrals of u_n and u.

Denote by $g_n, g : B = B(x_0, r) \to \mathbb{R}$ the functions whose graphs represent the boundaries of $\partial\Omega_n, \partial\Omega$, respectively, in an orthogonal coordinate system in a neighbourhood if x_0. Note that B has dimension $d-1$ so when we speak of almost every $x \in B$ we will mean up to a set of $\mathcal{H}^{d-1}$ measure zero. The fact that $\Omega_n \xrightarrow{H^c} \Omega$ implies $\|g_n - g\|_\infty \to 0$. Since g, g_n are Lipschitz continuous functions, they are differentiable almost everywhere and $|\nabla g|, |\nabla g_n| \leq L$, where L is their common Lipschitz constant. Denote by v the function u after the change of variables in the new orthogonal coordinate system. It remains to prove that

$$\int_B |v_n(x, g_n(x))|^p \sqrt{1 + |\nabla g_n(x)|^2} dx \to \int_B |v(x, g(x))|^p \sqrt{1 + |\nabla g(x)|^2} dx.$$

The condition $\mathrm{Per}(\Omega_n) \to \mathrm{Per}(\Omega)$, the fact that $\mathcal{H}^{d-1}(\Omega_n \cap U_i) = 0$ and the lower semi-continuity of the perimeter under L^1 convergence imply that

$$\lim_{n\to\infty} \mathrm{Per}(\Omega_n \cap U_i) \geq \mathrm{Per}(\Omega \cap U_i),$$

and

$$\lim_{n\to\infty} \mathrm{Per}(\Omega_n \setminus U_i) \geq \mathrm{Per}(\Omega \setminus U_i).$$

This, in turn implies that we have equality, namely

$$\lim_{n\to\infty} \mathrm{Per}(\Omega_n \cap U_i) = \mathrm{Per}(\Omega \cap U_i).$$

Translated into the considered coordinate system this becomes

$$\lim_{n\to\infty} \int_B \sqrt{1 + |\nabla g_n(x)|^2} dx = \int_B \sqrt{1 + |\nabla g(x)|^2} dx.$$

Furthermore, considering measurable sets of the form $V = B' \times [-a, a]$ and the fact that $\mathrm{Per}(\Omega_n \cap V) \to \mathrm{Per}(\Omega \cap V)$, we deduce that

$$\lim_{n\to\infty} \int_{B'} \sqrt{1 + |\nabla g_n(x)|^2} dx = \int_{B'} \sqrt{1 + |\nabla g(x)|^2} dx, \tag{3.2}$$

for every measurable set $B' \subset B$.

Since v_n is a $H^1(D)$ function, for almost every $x \in B$ we have

$$v_n(x, g_n(x)) = v_n(x, g(x)) + \int_{g(x)}^{g_n(x)} \frac{\partial v_n}{\partial y}(x, y) dy.$$

To simplify the computations, we denote $J_n(x) = \sqrt{1 + |\nabla g_n(x)|^2}$, $J(x) = \sqrt{1 + |\nabla g|^2}$. We obviously have $J_n(x), J(x) \in [1, \sqrt{1 + L^2}]$. We use the inequality

$$||a + h|^p - |a|^p| \leq p(|h||a|^{p-1} + |h|^p|),$$

which is trivial for $p = 1$ and is a direct consequence of the mean value theorem applied to the function $t \mapsto |t|^p$ when $p > 1$.

Thus we have

$$\begin{aligned}
&\left|\int_B |v_n(x,g_n(x))|^p J_n(x)dx - \int_B |v_n(x,g(x)|^p J_n(x)dx\right| \\
\leq &\int_B ||v_n(x,g_n(x))|^p - |v_n(x,g(x))|^p| J_n(x)dx \\
\leq &p\int_B \left|\int_{g(x)}^{g_n(x)} \frac{\partial v_n}{\partial y}(x,y)dy\right|^p J_n(x)dx && (A_n)\\
+&p\int_B |v_n(x,g(x))|^{p-1}\left|\int_{g(x)}^{g_n(x)} \frac{\partial v_n}{\partial y}(x,y)dy\right| J_n(x)dx && (B_n)
\end{aligned}$$

Study of (A_n)**.** Since we only know bounds on the L^2 norm of the gradient of v_n, we apply Cauchy-Schwarz inequality and then Hölder's inequality to get

$$\begin{aligned}
A_n \leq &p\int_B |g_n(x)-g(x)|^{\frac{p}{2}}\left|\left[\int_{g(x)}^{g_n(x)} \frac{\partial v_n^2}{\partial y}(x,y)dy\right]^{\frac{1}{2}}\right|^p J_n(x)dx \\
\leq &p\|g_n - g\|_\infty^{\frac{p}{2}}\sqrt{1+L^2}\int_B \left[\int_{g(x)}^{g_n(x)} \frac{\partial v_n^2}{\partial y}(x,y)dy\right]^{\frac{p}{2}} dx \\
\leq &p\|g_n - g\|_\infty^{\frac{p}{2}}\sqrt{1+L^2}\left(\int_B\int_{g(x)}^{g_n(x)} \frac{\partial v_n^2}{\partial y}(x,y)dy\right)^{\frac{p}{2}} |B|^{1/q} \\
\leq &C'\|g_n - g\|_\infty^{\frac{p}{2}}\|\nabla u_n\|_{H^1(D)}^p,
\end{aligned}$$

where C' is a constant which depends on B, p, L and q is chosen such that $\frac{p}{2}+\frac{1}{q}=1$. As a consequence of the fact that $\|g_n - g\|_\infty \to 0$ we have $(A_n)\to 0$.

Study of (B_n)**.** We apply Hölder's inequality for p and its conjugate $\frac{p}{p-1}$

$$\begin{aligned}
B_n \leq &p\int_B |v_n(x,g(x))|^{p-1}\left|\int_{g(x)}^{g_n(x)} \frac{\partial v_n}{\partial y}(x,y)dy\right| J_n(x)dx \\
\leq &p\sqrt{1+L^2}\left(\int_B |v_n(x,g(x))|^p dx\right)^{\frac{p-1}{p}}\left(\int_B \left|\int_{g(x)}^{g_n(x)} \frac{\partial v_n}{dy}(x,y)dy\right|^p\right)^{\frac{1}{p}} dx
\end{aligned}$$

Using arguments similar as in the study of (A_n) we can see that the last integral is bounded by a term of the form $C\|g_n - g\|_\infty^{\frac{1}{2}}$. To conclude that $(B_n)\to 0$ it remains to justify that the first integral is bounded. For this, we apply again Hölder's inequality for $\frac{2}{p}\geq 1$ and its conjugate q to get

$$\int_B |v_n(x,g(x))|^p dx \leq \left(\int_B v_n^2(x,g(x))dx\right)^{\frac{p}{2}} |B|^{\frac{1}{q}}.$$

Using the trace theorem on $\partial\Omega$ we have

$$\int_B v_n^2(x,g(x))dx \leq \int_B (v_n^2(x,g(x))J(x)dx \leq \int_{\partial\Omega} u_n^2 \leq C\|u_n\|_{H^1(D)}^2.$$

This finishes the proof of the fact that $(B_n)\to 0$.

To conclude the proof of (B), it is enough to prove that

$$\lim_{n\to\infty}\int_B |v_n(x,g(x))|^p J_n(x)dx = \int_B |v(x,g(x))|^p J(x)dx$$

First, let's note that the fact that $u_n \to u$ in $L^2(\partial\Omega)$ implies $v_n(x,g(x)) \to v(x,g(x))$ for almost every $x\in B$.

Since g_n, g have Lipschitz constants bounded by L, and B is a bounded set, we deduce that $|\nabla g_n(x)|$ is bounded in $L^2(B)$. Moreover since $\|g_n - g\|_\infty \to 0$ we have that $g_n \to g$ in $L^2(B)$. This means that (g_n) is bounded in $H^1(B)$, so it has a subsequence ∇g_{n_k} that converges weakly in $H^1(B)$ to a function h. This means that $g_n \to h$ in $L^2(B)$ and $\nabla g_n \rightharpoonup \nabla h$ in $L^2(B)$. Since $g_n \to h$ and $g_n \to g$ in $L^2(B)$, we must have $h = g$.

Thus, up to a subsequence, we have $\nabla g_n \rightharpoonup \nabla g$ in $L^2(B;\mathbb{R}^n)$ and (3.2) gives

$$\lim_{n\to\infty} \int_\Omega F(\nabla g_n) = \int_\Omega F(\nabla g),$$

where $F(x) = \sqrt{1+|x|^2}$ is a strictly convex function, if we consider it defined on $\{x \in \mathbb{R}^n : \|x\| \leq L\}$. Thus we can apply Lemma 3.1 and find that $\nabla g_n \to \nabla g$ strongly in $L^2(B;\mathbb{R}^n)$. Passing to a subsequence and relabelling, we can assume that $\nabla g_n \to \nabla g$ almost everywhere in B.

We define the measures $\mu_n = J_n(x)dx,\ \mu = J(x)dx$. We note that property (3.2) implies that μ_n converges set-wise to μ. We use the terminology defined in [13, Chapter 11, Section 4]. This allows us to use versions of the integral convergence theorems provided in the above reference. We recall these results in Remark 3.3.

Using the bounds on J_n, J we have

$$|v_n(x, g(x))|^p \leq \sqrt{1+L^2}|v_n(x, g(x))|^p \frac{J(x)}{J_n(x)}.$$

Since $u_n \to u$ in $L^2(\partial\Omega)$ and $\mathrm{Per}(\Omega)$ is finite, we have

$$|v_n(x, g(x))|^p J(x) \to |v(x, g(x))|^p J(x)$$

in $L^1(B)$, for every $p \in [1, 2]$. This means that

$$\lim_{n\to\infty} \int_B |v_n(x, g(x))|^p \frac{J(x)}{J_n(x)} d\mu_n \to \int_B |v(x, g(x))|^p d\mu.$$

Furthermore, since $J_n \to J$ almost everywhere, it follows that, up to a subsequence, $|v_n(x, g(x))|^p \frac{J(x)}{J_n(x)} \to |v(x, g(x))|^p$ almost everywhere.

Applying a generalized integral convergence theorem, stated in Remark 3.3 (ii), we deduce that

$$\lim_{n\to\infty} \int_B |v_n(x, g(x))|^p d\mu_n = \int_B |v(x, g(x))|^p d\mu.$$

This finishes the proof of part (B).

For part (A) the proof is the same, except the last part where instead of applying the integral convergence theorem we apply the variant of Fatou's Lemma presented in Remark 3.3 (i). Note that general, the measures μ_n do not necessarily converge set-wise to μ. We have the weaker hypothesis $\liminf_{n\to\infty} \mu_n(B') \geq \mu(B')$, which combined with the estimate $\mu_n(B') \leq \sqrt{1+L^2}\mu(B')$ is enough to reach the same conclusions. □

Remark 3.3. Let Ω be a measurable set. Suppose $f_n(x) \to f(x)$ for almost every $x \in \Omega$. Consider the measures μ_n, μ defined on Ω which satisfy for every measurable set $A \subset \Omega$ the equality

$$\lim_{n\to\infty} \mu_n(A) = \mu(A).$$

Following the terminology found in [13, Chapter 11, Section 4] we say that μ_n converges *set-wise* to μ.

(i) If $(f_n), f$ are non negative functions we have

$$\int_\Omega f d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu_n$$

(ii) If there exist functions g_n such that g_n are integrable with respect to μ_n, $|f_n| \leq g_n$, $g_n \to g$ almost everywhere, and

$$\lim_{n\to\infty} \int_\Omega g_n d\mu_n = \int_\Omega g d\mu < \infty$$

then

$$\lim_{n\to\infty}\int_\Omega f_n d\mu_n = \int_\Omega f d\mu.$$

For the part (i), the hypothesis $\mu_n(A)\to\mu(A)$ for every measurable set A can be relaxed to

$$\liminf_{n\to\infty}\mu_n(A)\geq\mu(A),\ \mu_n(A)\leq C\mu(A),$$

where $C>0$ is a constant.

Remark 3.4. It will be necessary to apply Proposition 3.2 part (B) in the case $p=1$ without the absolute values. Under the same hypothesis we want to prove that

$$\lim_{n\to\infty}\int_{\partial\Omega_n} u_n = \int_{\partial\Omega} u.$$

To achieve this it is enough to note that if $u_n \rightharpoonup u$ in $H^1(D)$ then $u_n^+ \rightharpoonup u^+$ and $u_n^- \rightharpoonup u^-$ in $H^1(D)$. We have denoted by u^+, u^- the positive, respective the negative part of u. We refer to [11, Corollary 3,1,12] for a proof of this result. We apply Proposition 3.2 for $u_n^+ \rightharpoonup u^+$ and $u_n^- \rightharpoonup u^-$ to find that

$$\lim_{n\to\infty}\int_{\partial\Omega_n} u_n^+ = \int_{\partial\Omega} u^+$$

and

$$\lim_{n\to\infty}\int_{\partial\Omega_n} u_n^- = \int_{\partial\Omega} u^-.$$

Subtracting these two equalities we get the desired result.

The above proposition helps us to prove the following shape continuity result for the Steklov spectrum. A general approach has been described in [2] in the case where the operators are defined on a common space. Another similar result is presented in [4] for the first biharmonic Steklov eigenvalue in the particular case of convex open sets.

Theorem 3.5. (Shape Stability for the Steklov spectrum) Let D be a bounded open subset of $\mathbb{R}^d$. Suppose $(\Omega_n), \Omega\subset D$ are open sets which satisfy a uniform ε-cone condition and $\Omega_n \xrightarrow{H^c} \Omega$.

(A) The following inequality holds:

$$\limsup_{n\to\infty}\sigma_k(\Omega_n)\leq\sigma_k(\Omega).$$

(B) If $\mathrm{Per}(\Omega_n)\to\mathrm{Per}(\Omega)$ then for every $k\geq 1$ we have

$$\lim_{n\to\infty}\sigma_k(\Omega_n)=\sigma_k(\Omega).$$

Proof: We start with part (B). We divide the proof in two parts:

$$\limsup_{n\to\infty}\sigma_k(\Omega_n)\leq\sigma_k(\Omega) \tag{3.3}$$

and

$$\liminf_{n\to\infty}\sigma_k(\Omega_n)\geq\sigma_k(\Omega) \tag{3.4}$$

For an open set Ω we denote by $V(\Omega)$ the space of functions on $H^1(\Omega)$ which are orthogonal to constants in $L^2(\partial\Omega)$. Note that if Ω has finite perimeter then $V(\Omega)$ is closed under weak convergence in $H^1(\Omega)$ (Straightforward application of Proposition 3.2 together with Remark 3.4).

1. Proof of (3.3)**.** Let $\varepsilon>0$ and consider a k-dimensional subspace S_k of V such that

$$\sigma_k(\Omega)+\varepsilon\geq\max_{u\in S_k\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2}{\int_{\partial\Omega}u^2}.$$

Let $\{u_1,..,u_k\}$ an orthonormal basis for S_k. Since $S_k\subset H^1(\Omega)$ and Ω has Lipschitz boundary, each u_i can be extended to $\tilde{u}_i\in H^1(D)$.

For $n \geq 1$ we modify each $\tilde{u}_i$ in order to make them admissible as test functions on Ω_n. To do this, we modify them with a constant term in order to have zero averages on $\partial\Omega_n$. This is possible since Ω_n has finite perimeter and we can simply define $u_i^n = \tilde{u}_i - c_i^n$, where c_i^n is a constant defined by $0 = \int_{\partial\Omega_n}(\tilde{u}_i - c_i^n)d\sigma = \int_{\partial\Omega_n} \tilde{u}_i d\sigma - c_i^n \operatorname{Per}(\Omega_n)$. Therefore $c_i^n = \frac{1}{\operatorname{Per}(\Omega_n)}\int_{\partial\Omega_n}\tilde{u}_i d\sigma$. Since $\operatorname{Per}(\Omega_n) \to \operatorname{Per}(\Omega) > 0$ and $\int_{\partial\Omega_n}\tilde{u}_i d\sigma \to \int_{\partial\Omega} u_i d\sigma = 0$, we find that $\lim_{n\to\infty} c_i^n = 0$ for $i = 1, ..., k$. This implies that $u_i^n \to \tilde{u}_i$ in $H^1(D)$.

For n great enough, the functions u_i^n span a k-dimensional subspace $S_k^n \subset H^1(D)$ which is admissible as a test subspace for $\sigma_k(\Omega_n)$. This implies that

$$\sigma_k(\Omega_n) \leq \max_{u\in S_k^n\setminus\{0\}} \frac{\int_{\Omega_n}|\nabla u|^2}{\int_{\partial\Omega_n} u^2} = \frac{\int_{\Omega_n}|\nabla v_n|^2}{\int_{\partial\Omega_n} v_n^2},$$

where we have denoted v_n a choice of the maximizers of the Rayleigh quotient on S_k^n. The maximizer v_n exists since S_k^n is finite dimensional.

Consider now $u_0 \in S_k$ arbitrary. Then there exist coefficients $a_1, ..., a_k$ such that

$$u_0 = a_1u_1 + ... + a_ku_k.$$

Consider also the functions $u_0^n \in S_k^n$ defined by

$$u_0^n = a_1u_1^n + ... + a_ku_k^n.$$

It easily follows that $u_0^n \to \tilde{u}_0$ in $H^1(D)$, since they differ only by a constant term which converges to 0 as $n \to \infty$. The maximality property of (v_n) implies

$$\frac{\int_{\Omega_n}|\nabla u_0^n|^2}{\int_{\partial\Omega_n}(u_0^n)^2} \leq \frac{\int_{\Omega_n}|\nabla v_n|^2}{\int_{\partial\Omega_n} v_n^2}. \tag{3.5}$$

We want to prove that $\limsup_{n\to\infty}\sigma_k(\Omega_n) \leq \sigma_k(\Omega)$. Without loss of generality, we can assume that $\lim_{n\to\infty}\sigma_k(\Omega_n)$ exists. If not, we take a subsequence which realizes the $\limsup$. We can find a decomposition $v_n = b_1^nu_1^n + ... + b_k^nu_k^n$. Since the Rayleigh quotient is scale invariant, we can choose the coefficients such that $|b_i^n| \leq 1$. Using a diagonal argument we can choose a subsequence of v_n such that $b_i^n \to b_i$ for $i = 1, ..., m$. Up to relabelling the sequence, we can assume that $v_n \to v$ in $H^1(D)$ where v is given by

$$v = b_1\tilde{u}_1 + ... + b_k\tilde{u}_k.$$

Taking $n \to +\infty$ in inequality (3.5) and using Proposition 3.2 we obtain that

$$\frac{\int_\Omega|\nabla u_0|^2}{\int_{\partial\Omega} u_0^2} \leq \frac{\int_\Omega|\nabla v|^2}{\int_{\partial\Omega} v^2}.$$

Since u_0 was chosen arbitrary, we have that

$$\max_{u_0\in S_k\setminus\{0\}}\frac{\int_\Omega|\nabla u_0|^2}{\int_{\partial\Omega} u_0^2} \leq \frac{\int_\Omega|\nabla v|^2}{\int_{\partial\Omega} v^2}.$$

The restriction of v to Ω is also in S_k, so the above inequality is, in fact, an equality.

We have just proved that

$$\limsup_{n\to\infty}\sigma_k(\Omega_n) \leq \lim_{n\to\infty}\frac{\int_{\Omega_n}|\nabla u_n|^2}{\int_{\partial\Omega_n} u_n^2} = \frac{\int_\Omega|\nabla v|^2}{\int_{\partial\Omega} v^2} = \max_{u\in S_k\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2}{\int_{\partial\Omega} u^2} \leq \sigma_k(\Omega) + \varepsilon.$$

Taking $\varepsilon \to 0$ we obtain the $\limsup$ inequality.

2. Proof of (3.4)**.** Consider $\varepsilon > 0$ and subspaces S_k^n of $H^1(D)$ such that

$$\sigma_k(\Omega_n) + \varepsilon \geq \max_{u\in S_k^n\setminus\{0\}}\frac{\int_{\Omega_n}|\nabla u|^2}{\int_{\partial\Omega_n} u^2}. \tag{3.6}$$

We want to prove that $\liminf_{n\to\infty}\sigma_k(\Omega_n) \geq \sigma_k(\Omega)$. We can assume that the limit exists by taking a subsequence which realizes it. Consider for each S_k^n an orthonormal basis $\{u_1^n, ..., u_k^n\}$. Up to choosing a diagonal subsequence, we can assume that each (u_i^n) converges weakly in

$H^1(D)$ to some u_i, $i = 1, ..., k$. Using Proposition 3.2 and Remark 3.4 it follows that $\int_{\partial\Omega} u_i = 0$, so $S_k = \text{Span}\{u_1, ..., u_k\}$ is admissible as a test space for $\sigma_k(\Omega)$.

Take $u = a_1u_1 + ... + a_ku_k \in S_k \setminus \{0\}$. Then $v_n = a_1u_1^n + ... + a_ku_k^n \in S_m^n \setminus \{0\}$ satisfies $v_n \rightharpoonup u$ in $H^1(D)$. The inequality (3.6) implies that

$$\sigma_k(\Omega_n) + \varepsilon \geq \frac{\int_{\Omega_n} |\nabla v_n|^2}{\int_{\partial\Omega_n} v_n^2}.$$

The weak convergence of (v_n) to u and Proposition 3.2 imply that

$$\liminf_{n\to\infty} \int_{\Omega_n} |\nabla v_n|^2 \geq \int_\Omega |\nabla u|^2 \text{ and } \lim_{n\to\infty} \int_{\partial\Omega_n} v_n^2 = \int_{\partial\Omega} u^2.$$

As a consequence, we have

$$\liminf_{n\to\infty} \sigma_k(\Omega_n) + \varepsilon \geq \frac{\int_{\Omega_n} |\nabla u|^2}{\int_{\partial\Omega_n} u^2}.$$

Since u was chosen arbitrary, we can take the maximum for $u \in S_k \setminus \{0\}$ in the right hand side of the above inequality and we get

$$\liminf_{n\to\infty} \sigma_k(\Omega_n) + \varepsilon \geq \max_{u\in S_k\setminus\{0\}} \frac{\int_{\Omega_n} |\nabla u|^2}{\int_{\partial\Omega_n} u^2} \geq \sigma_k(\Omega).$$

Taking $\varepsilon \to 0$ we obtain

$$\liminf_{n\to\infty} \sigma_k(\Omega_n) \geq \sigma_k(\Omega).$$

Combining the two parts of the proof we conclude that under the hypotheses we considered we have

$$\lim_{n\to\infty} \sigma_k(\Omega_n) = \sigma_k(\Omega).$$

in order to prove part (A) we argue by contradiction. Suppose that $\limsup_{n\to\infty} \sigma_k(\Omega_n) > \sigma_k(\Omega)$. The variational formulation implies the existence of some $\varepsilon > 0$ and a k dimensional subspace S_k of $V(\Omega)$ such that up to a subsequence we have

$$\lim_{n\to\infty} \sigma_k(\Omega_n) > \sigma_k(\Omega) + \varepsilon > \max_{u\in S_k} \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2}.$$

Therefore, for n great enough we have

$$\sigma_k(\Omega_n) > \sigma_k(\Omega) + \varepsilon > \max_{u\in S_k} \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2}.$$

Consider a basis $\{u_1, ..., u_k\}$ of S_k. Like in the proof of part (B), we construct the functions u_i^n which are perturbations by constants of H^1 extensions of u_i to the whole D such that $\int_{\partial\Omega_n} u_i^n = 0$. In this way we construct the k-dimensional subspaces $S_k^n = \{u_1^n, ...u_k^n\}$ which are admissible as test spaces for $\sigma_k(\Omega_n)$. Thus we have

$$\max_{u\in S_k^n} \frac{\int_{\Omega_n} |\nabla u|^2}{\int_{\partial\Omega_n} u^2} \geq \sigma_k(\Omega_n) > \sigma_k(\Omega) + \varepsilon > \max_{u\in S_k} \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2}.$$

Denote v_n a choice of maximizers of the Rayleigh quotient on S_k^n. We have the representation $v_n = b_1^n u_1^n + ... + b_k^n u_k^n = b_1^n \tilde{u}_1 + ... + b_k^n \tilde{u}_k - (b_1^n c_1^n + ... + b_k^n c_k^n)$. Like in the first part we have $c_i^n = \frac{1}{\text{Per}(\Omega_n)} \int_{\partial\Omega_n} \tilde{u}_i d\sigma$, and we can choose the coefficients (b_i^n) such that $|b_i^n| \leq 1$. Note that in this setting we do not necessarily have $c_i^n \to 0$ as $n \to \infty$, but there is a uniform bound for (c_i^n). We can choose a subsequence and relabel it such that $v_n \to b_1\tilde{u}_1 + ... + b_k\tilde{u}_k - C = u_0 - C$ in $H^1(D)$.

Using Proposition 3.2 part (B), we have

$$\liminf_{n\to\infty} \int_{\partial\Omega_n} v_n^2 \geq \int_{\partial\Omega} (u_0 - C)^2 = \int_{\partial\Omega} u_0^2 - 2C \int_{\partial\Omega} u_0 + C^2 \,\text{Per}(\Omega) \geq \int_{\partial\Omega} u_0^2,$$

since $\int_{\partial\Omega} u_0 = 0$. Furthermore, the fact that $v_n \to u_0 - C$ in $H^1(D)$ and $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(D)$ imply that

$$\lim_{n\to\infty} \int_{\Omega_n} |\nabla v_n|^2 = \int_\Omega |\nabla u_0|^2.$$

Taking $n \to \infty$ in the following inequality

$$\frac{\int_{\Omega_n} |\nabla v_n|^2}{\int_{\partial\Omega_n} v_n^2} \geq \sigma_k(\Omega_n) > \sigma_k(\Omega) + \varepsilon$$

we obtain

$$\max_{u\in S_k} \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2} < \sigma_k(\Omega) + \varepsilon \leq \limsup_{n\to\infty} \frac{\int_{\Omega_n} |\nabla v_n|^2}{\int_{\partial\Omega_n} v_n^2} \leq \frac{\int_\Omega |\nabla u_0|^2}{\int_{\partial\Omega} u_0^2}.$$

This is a contradiction, since $u_0 \in S_k$. □

The hypothesis that $\operatorname{Per}(\Omega_n) \to \operatorname{Per}(\Omega)$ was crucial in the proof of part (B) of the above theorem, and cannot be discarded. To justify this fact, we propose the following counterexample.

Example 3.6. Denote by S the unit square and by S_n the unit square where we have added a saw-tooth shape with 2^n sides on the upper side of S. For example, we can take S_1 to be S with a right isosceles triangle glued to S. S_2 can be obtained by cutting a square of length $\sqrt{2}/4$ from the top of the "tooth" of S_1. S_3 can be obtained from S_2 by cutting squares of side $\sqrt{2}/8$ from the top of each tooth of S_2. This procedure constructs inductively the sets S_n. Note that the sets S_n satisfy a uniform cone condition.

Furthermore, all the shapes S_n have the same perimeter, equal to $3 + \sqrt{2}$, thus $\operatorname{Per}(S_n) \to 2 + \sqrt{2} > 4 = \operatorname{Per}(S)$. We will show that the Steklov spectrum of S_n does not converge to the Steklov spectrum of S.

Proof: In the proof we will denote by T the edge of the square S to which the saw-tooth is glued, and B the other three edges of the square S. We denote by g_n the function whose graph represents the sawtooth in an orthogonal system of coordinates where the horizontal axis is directed by T. Note that in this case $|g_n'(x)| = 1$ for almost every $x \in T$. Denote by T_n the graph of g_n on T.

Let $u \in H^1(S)$ be an eigenfunction of S, corresponding to $\sigma_1(S)$. Since S is a Lipschitz domain, u can be extended to $H^1(\mathbb{R}^2)$, and then take the restrictions of u to S_n as test functions in the definition of $\sigma_1(S_n)$.

To do this, we need to make these restrictions admissible by modifying them with a constant in order to have the orthogonality to a constant function on S_n. We define $u_n = u - c_n$ such that

$$0 = \int_{\partial S_n} u_n = \int_{\partial S_n} u - c_n \operatorname{Per}(S_n).$$

This implies $c_n = \frac{1}{\operatorname{Per}(S_n)} \int_{\partial S_n} u$.

With the above notations we have

$$\begin{aligned}\int_{T_n} u &= \int_T u(x, g_n(x))\sqrt{1 + |g_n'(x)|^2} dx \\ &= \sqrt{2}\int_T u(x,0)dx + \sqrt{2}\int_T \int_0^{g_n(x)} \frac{\partial u}{\partial y}(x,y) dy dx.\end{aligned}$$

Using techniques similar to the ones involved in the proof of Proposition 3.2, we find that

$$\int_{T_n} u \to \sqrt{2}\int_T u \text{ as } n \to \infty.$$

In the same way, we can prove that

$$\int_{T_n} u^2 \to \sqrt{2}\int_T u^2 \text{ as } n \to \infty.$$

We evaluate

$$\begin{aligned}\int_{\partial S_n} (u-c_n)^2 &= \int_{\partial S_n} u^2 - c_n^2 \operatorname{Per}(S_n)\\ &= \int_B u^2 + \int_{T_n} u^2 - \frac{(\int_B u + \int_{T_n} u)^2}{3+\sqrt{2}}\end{aligned}$$

and we see that for $n \to \infty$ we have

$$\begin{aligned}\lim_{n\to\infty} \int_{\partial S_n} u_n^2 &= \int_B u^2 + \sqrt{2}\int_T u^2 - \frac{(\sqrt{2}-1)^2}{3+\sqrt{2}}\left(\int_T u\right)^2.\\ &= \int_{\partial S} u^2 + (\sqrt{2}-1)\int_T u^2 - \frac{(\sqrt{2}-1)^2}{3+\sqrt{2}}\left(\int_T u\right)^2\\ &> \int_{\partial S} u^2,\end{aligned}$$

by the Cauchy-Schwarz inequality. The equality could take place only if u is constant zero on T, but if this happens for every side of the square, then u is zero on the whole S, which is a contradiction.

Thus

$$\sigma_1(S) = \frac{\int_S |\nabla u|^2}{\int_{\partial S} u^2} > \lim_{n\to\infty} \frac{\int_{S_n} |\nabla u_n|^2}{\int_{\partial S_n} u_n^2} \geq \liminf_{n\to\infty} \sigma_1(S_n).$$

Therefore the sequence of first Steklov eigenvalues of S_n does not converge to the first Steklov eigenvalue of S. □

There exist examples in the literature which illustrate the fact that the ε-cone condition is also essential. Girouard and Polterovich consider in [8] one such examples. It consists of taking Ω_ε being two disks of radius 1 connected by a thin tube of length ε and width ε^3. In the limit, these connected disks converge to Ω which is formed of two tangent disks. Obviously, such sets do not satisfy a uniform cone condition. We have $\operatorname{Per}(\Omega_\varepsilon) \to \operatorname{Per}(\Omega)$, but the Steklov eigenvalues of Ω_ε converge to zero.

4. EXISTENCE RESULTS FOR THE OPTIMIZATION OF FUNCTIONALS OF THE STEKLOV SPECTRUM

In this sections we will present some consequences of the facts proved in the previous sections. We will be able to establish some existence results for the problem of maximizing the Steklov eigenvalue of Ω under different constraints.

Theorem 4.1. Suppose D is a bounded, open set in $\mathbb{R}^d$. Denote by $\mathcal{O}_\varepsilon$ the class of open subsets of D which satisfy an ε-cone property and have unit volume. Then the problem

$$\max_{\Omega\in\mathcal{O}_\varepsilon} \sigma_k(\Omega)$$

has a solution.

Proof: Take (Ω_n) a maximizing sequence. The Hausdorff convergence is compact, $\mathcal{O}_\varepsilon$ is closed under this convergence and therefore there exists an open set $\Omega \in \mathcal{O}_\varepsilon$ such that up to taking a subsequence and relabelling, we have $\Omega_n \xrightarrow{H^c} \Omega$. Proposition 2.3 or the estimate (1.1) implies that there exists an upper bound for $\operatorname{Per}(\Omega_n)$. The compactness properties of the perimeter (see for example [12, Theorem 12.26]) imply that there exists a subsequence denoted again (Ω_n) such that (Ω_n) converges to Ω in the sense of characteristic functions and furthermore, $\lim_{n\to\infty} \operatorname{Per}(\Omega_n) \geq \operatorname{Per}(\Omega)$ and applying Theorem 3.5 (A) we deduce that

$$\limsup_{n\to\infty} \sigma_k(\Omega_n) \leq \sigma_k(\Omega).$$

The fact that (Ω_n) is a maximizing sequence coupled with the above inequality proves that Ω is the set which maximizes $\sigma_k(\Omega)$ in the class $\mathcal{O}_\varepsilon$. □

Note that convex sets Ω satisfy a ε-cone condition, with ε depending on the radius of a ball contained in Ω, as well as of the box D containing Ω. We would like to give a general existence result for the maximization of $\sigma_k(\Omega)$ in the family of the convex sets. In order to apply the results of the previous section, we would need a bounding box for Ω. The result given below proves that a maximizing sequence for $\sigma_k(\Omega)$ is always confined in a bounded open set D.

Proposition 4.2. Suppose that (Ω_n) is a sequence of open, convex sets with unit volume, which satisfy the property that $\text{diam}(\Omega_n) \to \infty$. Then $\sigma_k(\Omega_n) \to 0$.

Proof: This result is a consequence of the bound (4.1) proved in [5], which states that if we denote by $I(\Omega) = \text{Per}(\Omega)/|\Omega|^{\frac{d-1}{d}}$ then

$$\sigma_k(\Omega) \leq c_d k^{\frac{2}{d}} \frac{|\Omega|^{\frac{d-2}{d}}}{\text{Per}(\Omega)}. \tag{4.1}$$

Indeed, we could consider a diameter of length M and make a Steiner symmetrization in the direction of the diameter. There exists a section ω orthogonal to the diameter which maximizes $\mathcal{H}^{n-1}(\omega)$. The fact that Ω has unit volume implies $\mathcal{H}^{n-1}(\omega) \geq 1/M$. Consider the cone C generated by ω and the considered diameter. This cone is contained in Ω, and by convexity, the perimeter of Ω is bounded from below by the perimeter of the cone C. Using techniques similar to those in our proof presented below, we can see that the $\text{Per}(C) \geq cM^{\frac{1}{d-1}}$, where c is a dimensional constant. This, together with (4.1) implies that $\sigma_k(\Omega) \to 0$ as $M \to \infty$.

In the case of convex sets it is possible to give a direct proof, which we present below. This proof avoids the technical measure theory result used in [5], in order to prove (4.1).

Let Ω be an open, convex set of $\mathbb{R}^d$, having unit volume. Denote by M its diameter, and denote X_0X_k one of its diameters. In order to make the proof easier to read, we divide it into several parts.

Part 1. Bound from below of the volume of a region. We call a *cap* of Ω the part of Ω contained in a half-space determined by a hyperplane α orthogonal to the diameter X_0X_k. We call *region* of Ω a subset of Ω contained between two hyperplanes α, β which are orthogonal to X_0X_k.

Let's start by giving a lower bound for the volume of a cap. Denote $Y = \alpha \cap X_0X_k$ and the length X_0Y by L. Denote Ω^- and Ω^+ the caps of Ω determined by α, which contain X_0 and X_k, respectively. Denote C^- the cone with vertex X_0 and base $\Omega \cap \alpha$. Denote also with C^+ the cone which is the dilated of C^- with center X_0 and a factor M/L. The convexity of Ω implies

$$C^- \subset \Omega^- \text{ and } C^+ \setminus C^- \supset \Omega^+.$$

Therefore we have

$$\frac{|\Omega^-|}{|\Omega^+|} \geq \frac{|C^-|}{|C^+| - |C^-|} = \frac{L^d}{M^d - L^d},$$

which, in turn, implies $|\Omega^-| \geq L^d/M^d|\Omega|$.

If instead of a cap, we consider a region, we can apply two times the above bound and find a similar lower bound. Denote Ω^- the part of Ω contained in the half-space determined by γ which contains X_0, Ω^+ the part of Ω contained in the half-space determined by β which contains X_k and Ω_0 the region determined by α and β. Denote also $A = \alpha \cap X_0X_k,\ B = \beta \cap X_0X_k$.

Using the bound on a cap, we have

$$|\Omega_0| \geq \frac{AB^d}{AX_k^d}|\Omega_0 \cup \Omega^+|,$$

and

$$|\Omega^+ \cup \Omega_0| \geq \frac{AX_k^d}{X_0X_k^d}|\Omega|.$$

Combining the two bounds, we arrive at

$$|\Omega_0| \geq \frac{L^d}{M^d}|\Omega|,$$

where we have denoted the length of AB by L.

Part 2. Bound from below of the perimeter of a region. Suppose we have a region Ω_0 of width L, like in the previous section. In the following, we will denote by c_d a constant which depends only on the dimension of the space. We perform a Steiner-symmetrization of this region with respect to the direction AB, which we denote Ω_0^*. For an introduction to Steiner symmetrization see [11, Chapter 6] or [3, Chapter 6]. It is known that performing a Steiner symmetrization preserves the volume, preserves the convexity and decreases the perimeter. Thus, as a first consequence, $\mathrm{Per}(\Omega_0^*) \le \mathrm{Per}(\Omega_0)$. Another property of the Steiner symmetrized set Ω_0^* is that all slices with a hyperplane orthogonal to AB are $d-1$-dimensional balls. Among these balls, there is one, denoted ω, having radius r_0, which has the maximal $\mathcal{H}^{d-1}$ measure. Denote $a = d(A,\omega), b = d(B,\omega)$. Obviously, we have $a + b = L$. Since

$$|\Omega_0^*| \ge \frac{L^d}{M^d},$$

we deduce that $\mathcal{H}^{d-1}(\omega) \ge \frac{L^{d-1}}{M^d}$, which gives us a lower bound $r \ge c_d \frac{L}{M^{\frac{d}{d-1}}}$.

We denote $\omega_1 = \alpha \cap \Omega, \omega_2 = \beta \cap \Omega$. The fact that Ω_0 is convex, and its $d-1$-dimensional slices orthogonal to AB are disks, means that the truncated cones determined by $T_1 = (\omega, \omega_1)$ and $T_2 = (\omega, \omega_2)$ are contained in Ω.

We know from [3, Lema 2.2.2] that since $T_1 \cup T_1 \subset \Omega_0^*$ and $T_1 \cup T_2, \Omega_0^*$ are convex, we have $\mathrm{Per}(T_1 \cup T_1) \le \mathrm{Per}(\Omega_0^*)$. If we denote by R the region of $\mathbb{R}^d$ situated between the hyperplanes α, β, then $\mathrm{Per}(T_1 \cup T_2, R) \le \mathrm{Per}(\Omega_0^*, R)$. This inequality is true because the part of the perimeters of Ω_0^* and $T_1 \cup T_2$ which is contained in ∂R is the same for both sets.

All we need in order to conclude, is to bound from below the lateral area of a truncated cone. If we denote by r_1, r the two radii of ω_1, ω, then we have two cases. If $r_1 = r$ then T_1 is a cylinder and the lateral area of T_1 is equal to $a\mathcal{H}^{d-2}(\omega) = c_d a r^{d-2}$. If $r_1 < r$ then the lateral area is given by

$$\int_{\omega \setminus \mathrm{proj}_\omega \omega_1} \sqrt{1 + \frac{a^2}{(r-r_1)^2}} \ge c_d a \frac{r^{d-1} - r_1^{d-1}}{r - r_1} \ge c_d a r^{d-2}.$$

Thus the lateral area of $T_1 \cup T_2$ is bounded below by

$$\mathrm{Per}(T_1 \cup T_2, R) \ge c_d L r^{d-2}.$$

Combining all the above estimates, we arrive at

$$\mathrm{Per}(\Omega_0) \ge c_d \frac{L^{d-1}}{M^{\frac{d(d-2)}{d-1}}}.$$

Thus for a region Ω_0 of Ω with width $L = \alpha M$ we have

$$\mathrm{Per}(\Omega_0) \ge c_d \alpha^{d-1} M^{\frac{1}{d-1}}.$$

Part 3. Upper bound on the Steklov spectrum

For $k \ge 1$ divide the diameter X_0X_k into k equal parts using points X_i, and use orthogonal hyperplanes α_i through X_i to divide Ω into k subsets of width M/k (in the direction of X_0X_k). We define k functions $(u_i) \subset H^1(\Omega)$ such that u_i is supported in region i. We choose them to depend only on the distance from the bounding hyperplanes. One choice is the following:

- u_i starts from 0 on α_{i-1} and increases with gradient 1 until it reaches 1.
- u_i is constant for a while.
- u_i descends with gradient 1 until it reaches -1.
- u_i is constant for a while.
- u_i increases with gradient 1 until it reaches 0.

A schematic picture can be found in Figure 1. Furthermore, we can translate the part where u_i grows from -1 to 1 so that $\int_{\partial\Omega} u_i = 0$. With this construction we have the following bound

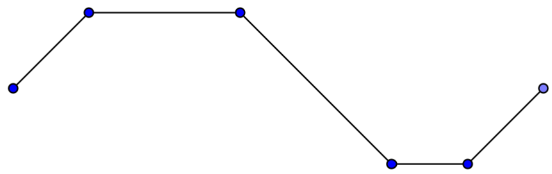

FIGURE 1. Form of the function u_i in the direction of the diameter

on the Rayleigh quotient corresponding to u_i:

$$\frac{\int_\Omega |\nabla u_i|^2}{\int_{\partial\Omega} u_i^2} \leq \frac{1}{\mathcal{H}^{d-1}(\partial\Omega \cap \{u_i = \pm 1\})}$$

Using the bounds obtained in the previous section, we have

$$\mathcal{H}^{d-1}(\partial\Omega \cap \{u_i = 1\}) \geq c_d \alpha_1^{d-1} (M/k)^{\frac{1}{d-1}}$$
$$\mathcal{H}^{d-1}(\partial\Omega \cap \{u_i = -1\}) \geq c_d \alpha_2^{d-1} (M/k)^{\frac{1}{d-1}}$$

where $\alpha_1 + \alpha_2 \geq 1 - \frac{4k}{M}$. Thus

$$\mathcal{H}^{d-1}(\partial\Omega \cap \{u_i = 1\}) + \mathcal{H}^{d-1}(\partial\Omega \cap \{u_i = -1\}) \geq c_d(\alpha_1 + \alpha_2)^{d-1} (M/k)^{\frac{1}{d-1}}.$$

These bounds allow us to conclude that as $M \to \infty$ we have

$$\frac{\int_\Omega |\nabla u_i|^2}{\int_{\partial\Omega} u_i^2} \leq c_d \frac{k^{\frac{1}{d-1}}}{(1 - 4k/M)^{d-1} M^{\frac{1}{d-1}}} \xrightarrow{M\to\infty} 0.$$

As a consequence, we have the bound

$$\sigma_k(\Omega) \leq \max \frac{\int_\Omega |\nabla \sum a_i u_i|^2}{\int_{\partial\Omega} (\sum a_i u_i)^2} \leq \max \frac{\int_\Omega |\nabla u_i|^2}{\int_{\partial\Omega} u_i^2},$$

where we have used the fact that the functions u_i have disjoint support in Ω. This means that

$$\sigma_k(\Omega) \to 0 \text{ as } M \to \infty.$$

□

Using the previous result, we can deduce the existence of a maximizer for the k-th Steklov eigenvalue in the class of convex sets.

Corollary 4.3. The problem

$$\max_{|\Omega|=1} \sigma_k(\Omega)$$

has a solution in the class of convex sets.

Proof: Take (Ω_n) a sequence of sets with measure 1 such that $\sigma_k(\Omega_n) \to \sup_{|\Omega|=1} \sigma_k(\Omega)$. If (Ω_n) contains a subsequence such that $\text{diam}(\Omega_n) \to \infty$, then by Theorem 4.2, $\sigma_k(\Omega_n)$ would have a subsequence converging to zero. This is impossible, since (Ω_n) is a maximizing sequence. Thus the diameters of (Ω_n) are bounded from above, and therefore we can assume that all the sets Ω_n are contained in a bounded open set D.

The by the compactness of Hausdorff convergence, there exists a subsequence denoted (Ω_n) such that $\Omega_n \xrightarrow{H^c} \Omega$. The properties of the Hausdorff convergence imply that Ω is also convex and contains a compact ball B (see [11, Chapter 2]). Proposition 2.2.15 in [11] proves that for n large enough, we must have $B \subset \Omega_n$. Proposition 2.4.4 in [11] allows us to say that for n large enough, the sets Ω_n and the set Ω satisfy a uniform cone condition. Thus, we can apply Theorem 3.5 to conclude that

$$\limsup_{n\to\infty} \sigma_k(\Omega_n) \leq \sigma_k(\Omega).$$

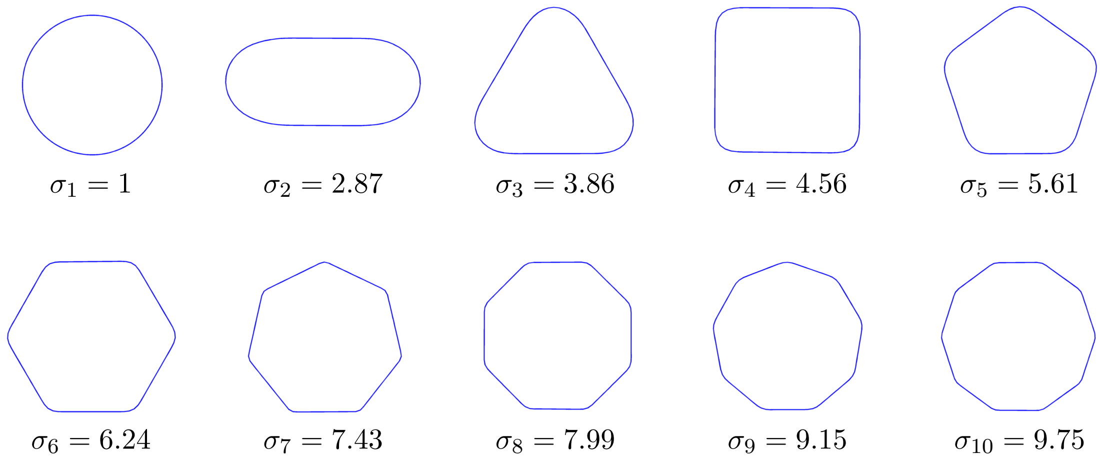

FIGURE 2. Convex shapes with unit area which give highest k-th Steklov eigenvalue in our numerical observations

The Hausdorff convergence implies the convergence of characteristic functions in $L^1(D)$, which, in turn implies that $|\Omega| = \lim_{n\to\infty} |\Omega_n| = 1$. Thus Ω maximizes $\sigma_k(\Omega)$ among convex sets of the same measure. $\square$

Remark 4.4. The treatment of the perimeter constraint, in the case of convex sets, is also straightforward, since we can apply Theorem 3.2 directly, for a maximizing sequence.

Corollary 4.5. In the following, we consider $\mathcal{A}$ to be the class of ε-cone sets contained in a bounded open set D, or the class of convex sets, having unit volume.

(A) If $F : \mathbb{R}^k \to \mathbb{R}$ is upper semi-continuous and increasing in every variable, then the problem
$$\max_{\Omega\in\mathcal{A}} F(\sigma_1(\Omega), ..., \sigma_k(\Omega)).$$
has a solution.

(A) If $G : \mathbb{R}^k \to \mathbb{R}$ is lower semi-continuous and increasing in every variable, then the problem
$$\min_{\Omega\in\mathcal{A}} G(1/\sigma_1(\Omega), ..., 1/\sigma_k(\Omega)).$$
has a solution.

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