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Dynamic risk measuring: discrete time in a context of uncertainty, and continuous time on a probability space

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Abstract

We characterize time-consistent dynamic risk measures.

In discrete time in context of uncertainty, we canonically associate a class of probability measures to any dynamic risk measure when the filtration comes from a process bounded at each time. Dynamic risk measures are conditional risk measures on a bigger space.

In continuous time, we characterize time consistency, studying composition of conditional risk measures. Using sufficient conditions for time consistency, and BMO martingales, we construct new families of time-consistent dynamic risk measures. Some are continuous generalizing those coming from BSDE. Others are with jumps.

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Introduction

In recent years there has been an increasing interest in methods defining the risk of a financial position. Artzner et al [1] have introduced the concept of coherent risk measures on a probability space. More recently Föllmer and Schied [17], [18] and [19], have addressed a more general issue, defining the notion of monetary measure of risk, not necessarily coherent and in a more general context, that of uncertainty where no probability measure is given a priori.

Several authors have then extended the notion of monetary risk measures to a conditional or dynamic setting.

In a continuous time dynamic setting F. Delbaen [8] has fully characterized the coherent dynamic risk measures and in particular proved that the time consitency for coherent dynamic risk measures is equivalent to the condition of m-stability. Other works concerning a time continuous dynamic setting are usually based on the Backward Stochastic Differential Equations (B.S.D.E.) approach called also conditional "g"-expectation. Important works along these lines are by Peng [24] and [25], Coquet et al. [7], Rosazza Gianin [20], and Barrieu and El Karoui [3].

There are several approaches in a discrete time setting. All these approaches assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in I\!N}, P)$ is given in advance and therefore that the probability P is known. For these approaches see for example F. Riedel [26] and Cheredito et al [6] where the notion of time-consistency is studied.

In Bion-Nadal [5] and Detlefsen and Scandolo [11] a new notion of conditional risk measures is introduced. The case of continuous convex conditional risk measures defined on probability spaces is studied by Detlefsen and Scandolo [11]. In [5] we have studied the more general case of convex conditional risk measures not necessarily continuous and in a context of uncertainty. Indeed, as pointed out by Avellaneda [2], it is important to deal with situations where the probability measure (and even its class) is not fixed in advance.

In this paper we study the dynamic risk measures $(\rho_{s,t})_{s\leq t}$ both in discrete time and in continuous time.

In section 1 starting from the conditional risk measures and their representation in terms of a set of probabilities and of a penalty function as in [5], we study the composition rule of conditional risk measures. We characterize the relation $\rho_{1,3} = \rho_{1,2}(-\rho_{2,3})$ in terms of a stability property of the set of probability measures (similar to the m stability property of F. Delbaen [8]) and of a cocycle condition on the penalty function.

In section 2 we study dynamic risk measures on a measurable filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in I\!N})$ in a discrete time setting. We do it in a context of uncertainty (i.e. whithout fixing in advance a probability). When the filtration is the natural filtration of a $I\!R^l$ valued process bounded at each time, we associate canonically a class of probability measures to any dynamic risk measure continuous from below. We prove also that the dynamic risk measures in discrete time are equivalent to conditional risk measures on a bigger space.

In section 3 we study dynamic risk measures in a continuous time setting, on a probability space. Using the results of section 1, we characterize the time consistency ($\forall r \leq s \leq t \;\; \rho_{r,t} = \rho_{r,s}(-\rho_{s,t})$) by necessary and sufficient conditions on the set of probabilities (stability) and on the minimal penalty functions (cocycle condition). When the dynamic risk measure is constructed from a family of probability measures and a penalty function (in general there is no reason for this penalty function to be the minimal one), we get sufficient conditions for the dynamic risk measure to be time-consistent: stability of the set of probability measures, locality and cocycle condition on the penalty function.

Using these sufficient conditions, we construct new classes of explicit examples of time-consistent dynamic risk measures:

Starting with any finite family of BMO orthogonal continuous martingales we construct classes of time-consistent dynamic risk measures. In the particular case of independent Brownian motions stopped in T we obtain that way the dynamic risk measures coming from B.S.D.E.

We are also able to construct similar classes of time-consistent dynamic risk measures starting from BMO martingales which are no more continuous (in that case we have a condition on the BMO norms). We thus get dynamic risk measures with jumps.

1 Conditional risk measures

1.1 Some recalls on convex conditional risk measures

In this part we recall some important notions and results of [5]. Let Ω be a set. Consider a σ-algebra $\mathcal{G}$. We denote $\mathcal{E}_{\mathcal{G}}$ the set of all bounded real valued $(\Omega, \mathcal{G})$ measurable maps. We consider that the set of all financial positions $\mathcal{X}$ is $\mathcal{E}_{\mathcal{G}}$. $\mathcal{X}$ is then a Banach space.

We consider a sub σ-algebra $\mathcal{F}$ of $\mathcal{G}$ and we assume that a probability measure P is given on $(\Omega, \mathcal{F})$.

From a financial point of view, the sub σ-algebra $\mathcal{F}$ can represent the partial information accessible for an investor and it is natural to assume that there is a well known probability on $(\Omega, \mathcal{F})$.

The definition of a risk measure conditional to a probability space is as follows [5]:

Definition 1 *A mapping*

$$\rho_{\mathcal{F}} : \mathcal{X} \rightarrow L^{\infty}(\Omega, \mathcal{F}, \mathcal{P})$$

is a normalized risk measure conditional to the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ *if it satisfies the following conditions:*

i) monotonicity: for all X,$Y \in \mathcal{X}$ *if* $X \leq Y$ *then* $\rho_{\mathcal{F}}(Y) \leq \rho_{\mathcal{F}}(X)$ P *a.s.*

ii) translation invariance: for all $Y \in \mathcal{E}_{\mathcal{F}}$*, for all* $X \in \mathcal{X}$,

$$\rho_{\mathcal{F}}(X+Y) = \rho_{\mathcal{F}}(X) - Y \;\; P \; a.s.$$

iii) multiplicative invariance: for all $X \in \mathcal{X}$*, for all* $\mathcal{A} \in \mathcal{F}$,

$$\rho_{\mathcal{F}}(X 1_{\mathcal{A}}) = 1_{\mathcal{A}} \rho_{\mathcal{F}}(X) \;\; P \; a.s.$$

Definition 2 *i) A risk measure conditional to a probability space* $(\Omega, \mathcal{F}, \mathcal{P})$ *is a mapping*

$$\rho_{\mathcal{F}} : \mathcal{X} \rightarrow L^{\infty}(\Omega, \mathcal{F}, \mathcal{P})$$

such that $\rho_{\mathcal{F}} - \rho_{\mathcal{F}}(0)$ *is a normalized conditional risk measure*

ii) A convex conditional risk measure is continuous from below if:

For every increasing sequence X_n *of elements of* $\mathcal{X}$ *converging to* X*, the decreasing sequence* $\rho_{\mathcal{F}}(X_n)$ *converges to* $\rho_{\mathcal{F}}(X)$ P *a.s..*

To each risk measure $\rho_{\mathcal{F}}$ conditional to the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ we associate its acceptance set:

$$\mathcal{A}_{\rho_{\mathcal{F}}} = \{X \in \mathcal{X} \; / \; \rho_{\mathcal{F}}(X) \leq 0 \;\; P \; a.s.\}$$

The most important results on conditional risk measures are the theorems of representation:

We recall here the theorem of representation of convex risk measures, continuous from below, conditional to a probability space. This representation theorem will be crucial for the study of dynamic risk measures.

Theorem 1 *Let* $\rho_{\mathcal{F}}$ *be a convex risk measure defined on* $(\Omega, \mathcal{G})$ *conditional to the probability space* $(\Omega, \mathcal{F}, P)$.

Assume that $\rho_{\mathcal{F}}$ *is continuous from below then for all* $X \in \mathcal{X}$

$$\rho_{\mathcal{F}}(X) = ess \; max_{Q \in \mathcal{M}}((E_Q(-X|\mathcal{F}) - \alpha^m(Q)) \;\; (I)$$

where $\alpha^m(Q) = ess\ sup_{\{Y \in \mathcal{A}_{\rho_\mathcal{F}}\}}(E_Q(-Y|\mathcal{F})$ *and* $\mathcal{M}$ *is the set of probability measures on* $(\Omega, \mathcal{G})$ *whose restriction to* $\mathcal{F}$ *is equal to* P *and such that for each* $Q \in \mathcal{M}$, $\alpha^m(Q)$ *is essentially bounded.*

In the following we refer to this representation as the usual representation.

Remark 1 *When we assume that a probability* P *is given on* $(\Omega, \mathcal{G})$ *we define a conditional risk measure on* $L^\infty(\Omega, \mathcal{G}, P)$ *as a risk measure* $\rho_\mathcal{F}$ *conditional to the probability space* $(\Omega, \mathcal{F}, P)$ *such that*

$$\rho_\mathcal{F}(X) = \rho_\mathcal{F}(Y) \;\; P\ a.s. \;\; if \;\; X = Y \;\; P\ a.s.$$

In that case in the theorem of representation the set $\mathcal{M}$ *is a set of probability measures on* $(\Omega, \mathcal{G})$ *absolutely continuous with respect to* P *whose restriction to* $\mathcal{F}$ *is equal to* P.

Remark 2 *In case of complete uncertainty (no probability is given even on* $\mathcal{F}$*) a risk measure on* $(\Omega, \mathcal{G})$ *conditional to* $(\Omega, \mathcal{F})$ *is a mapping from* $\mathcal{E}_\mathcal{G}$ *to* $\mathcal{E}_\mathcal{F}$ *satisfying the same equalities and inequality as in definition 1 but exactly instead of P.a.s.*

For the various theorems of representation (for example when the risk measures are no more continuous from below), we refer to [5].

1.2 Properties of the penalty function

Definition 3 *Consider a function* α *defined on a set* $\mathcal{M}$ *of probability measures on* $(\Omega, \mathcal{G})$ *with values into* $L^\infty(\Omega, \mathcal{F}, P)$.

We say that α *is local if* $\forall A \in \mathcal{F}\ \forall (Q_1, Q_2) \in \mathcal{M}^2$.

If $E_{Q_1}(X1_A|\mathcal{F}) = E_{Q_2}(X1_A|\mathcal{F})$ *P.a.s.* $\forall X \in \mathcal{E}_\mathcal{G}$, *then* $1_A\alpha(Q_1) = 1_A\alpha(Q_2)$ *P.a.s.*

Proposition 1 *i) Consider a risk measure on* $(\Omega, \mathcal{G})$ *conditional to* $(\Omega, \mathcal{F}, P)$. *The penalty function* α^m *is local.*

ii) For every probability measure Q *on* $(\Omega, \mathcal{G})$, $\alpha^m(Q)$ *is (P.a.s.) the limit of an increasing sequence* $f_n = E_Q(-X_n|\mathcal{F})$ *for some* $X_n \in \mathcal{A}_{\rho_\mathcal{F}}$

Proof:

i) $\alpha^m(Q) = Pess\ sup_{\{Y \in \mathcal{A}_{\rho_\mathcal{F}}\}}(E_Q(-Y|\mathcal{F}))$

$\forall A \in \mathcal{F}, 1_A\alpha^m(Q) = ess\ sup_{\{Y \in \mathcal{A}_{\rho_\mathcal{F}}\}}(E_Q(-Y1_A|\mathcal{F}))$ The local property of α^m follows.

ii) It is enough to prove that for Q fixed, the set $\{E_Q(-X|\mathcal{F}); X \in \mathcal{A}_{\rho_{\mathcal{F}}}\}$ is a lattice upward directed.

Let $(Y, Z) \in (\mathcal{A}_{\rho_{\mathcal{F}}})^2$ Let $B = \{\omega \in \Omega \ / \ E_Q(-Y|\mathcal{F})(\omega) > (E_Q(-Z|\mathcal{F})(\omega)\}$. From the bifurcation property of $\mathcal{A}_{\rho_{\mathcal{F}}}$, it follows that $X = Y1_B + Z1_{(\Omega-B)}$ is in $\mathcal{A}_{\rho_{\mathcal{F}}}$ and $E_Q(-X|\mathcal{F}) = ess\ sup(E_Q(-Y|\mathcal{F}), E_Q(-Z|\mathcal{F}))$.

Q.e.d.

1.3 Composition of conditional risk measures

Consider three σ-algebras $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$ on a space Ω.

Proposition 2 *Assume that $\rho_{2,3}$ is a risk measure on $(\Omega, \mathcal{F}_3)$ conditional to $(\Omega, \mathcal{F}_2)$ and $\rho_{1,2}$ a risk measure on $(\Omega, \mathcal{F}_2)$ conditional to $(\Omega, \mathcal{F}_1)$*

Then $\rho(X) = \rho_{1,2}(-\rho_{2,3}(X))$ defines a risk measure on $(\Omega, \mathcal{F}_3)$ conditional to $(\Omega, \mathcal{F}_1)$.

There is no difficulty in the verification of this proposition. We also have the same proposition for conditional risk measures on probability spaces.

Now there is a natural question:

Given three risk measures $(\rho_{i,j})_{(1\leq i<j\leq 3)}$ on $(\Omega, \mathcal{F}_j)$ conditional to $(\Omega, \mathcal{F}_i)$, how can we characterize the equality $\rho_{1,3}(X) = \rho_{1,2}(-\rho_{2,3}(X))$ $\forall X$ either in terms of acceptance sets or using the penalty functions?

The following theorem gives the answer to this question in the case where a probability measure P is given on $\mathcal{F}_3$ and assuming that the conditional risk measures are continuous from below. This theorem will be crucial for the study of dynamic risk measures (in continuous time).

Theorem 2 *Consider $\rho_{i,j}$ convex risk measures continuous from below on $(\Omega, \mathcal{F}_j, P)$ conditional to $(\Omega, \mathcal{F}_i, P)$. Consider the usual representation*

$\rho_{i,j}(X) = ess\ max_{Q\in\mathcal{M}_{i,j}}(E_Q(-X|\mathcal{F}_i) - \alpha^m_{i,j}(Q))$ Denote $\tilde{\mathcal{M}}_{i,j}$ the set of all probability measures Q on $\mathcal{F}_j$ absolutely continuous with respect to P whose restriction to $\mathcal{F}_i$ is equal to P. $\mathcal{M}_{i,j} = \{Q \in \tilde{\mathcal{M}}_{i,j} \ / \ \alpha^m_{i,j}(Q) \in L^\infty(\Omega, \mathcal{F}_i, P)\}$.

$\mathcal{A}_{i,j}$ is the acceptance set of $\rho_{i,j}$.

The following properties are equivalent:

i)$\rho_{1,3}(X) = \rho_{1,2}(-\rho_{2,3}(X))$ P.a.s. $\forall X \in \mathcal{E}_{\mathcal{F}_3}$ (set of bounded $\mathcal{F}_3$ measurable maps)

ii) $\mathcal{A}_{1,3} = \mathcal{A}_{1,2} + \mathcal{A}_{2,3}$

iii) $(\mathcal{M}_{i,j})_{1\leq i<j\leq 3}$ satisfy the following stability property:

$\forall Q \in \mathcal{M}_{2,3}$, $\forall R \in \mathcal{M}_{1,2}$ there exists $S \in \mathcal{M}_{1,3}$ such that

$$\forall f \in \mathcal{E}_{\mathcal{F}_3},\ E_S(f|\mathcal{F}_1) = E_R(E_Q(f|\mathcal{F}_2)|\mathcal{F}_1)\ \ P.a.s.\quad (II)$$

and the penalty function α^m satisfies the cocycle condition:

$\forall S \in \mathcal{M}_{1,3}$ $\forall R \in \mathcal{M}_{1,2}$ $\forall Q \in \tilde{\mathcal{M}}_{2,3}$ *satisfying the relation* (II)*,* $\alpha^m_{2,3}(Q)$ *is* R *integrable and* $\alpha^m_{1,3}(S) = E_R(\alpha^m_{2,3}(Q)|\mathcal{F}_1) + \alpha^m_{1,2}(R)$ *P.a.s..*

Proof of the theorem:

- i) implies ii)

let $X \in \mathcal{A}_{1,3}$ $\rho_{1,3}(X) \leq 0$ Denote $Z = X + \rho_{2,3}(X)$. By translation invariance, $\rho_{2,3}(Z) = 0$ so $Z \in \mathcal{A}_{2,3}$ $\rho_{1,2}(X - Z) = \rho_{1,2}(-\rho_{2,3}(X)) \leq 0$ So $X - Z \in \mathcal{A}_{1,2}$. Hence $\mathcal{A}_{1,3} \subset \mathcal{A}_{1,2} + \mathcal{A}_{2,3}$.

Conversely Let $Y \in \mathcal{A}_{1,2},\ Z \in \mathcal{A}_{2,3}$. $\rho_{1,3}(Y + Z) = \rho_{1,2}(-\rho_{2,3}(Z) + Y)$.

As $Z \in \mathcal{A}_{2,3}$, $-\rho_{2,3}(Z) + Y \geq Y$ and by hereditary property of $\mathcal{A}_{1,2}$ [5], $-\rho_{2,3}(Z) + Y \in \mathcal{A}_{1,2}$ and hence, $Y + Z \in \mathcal{A}_{1,3}$.

Thus ii) is proved.

- ii) implies iii)

Let $Q \in \mathcal{M}_{2,3}$ and $R \in \mathcal{M}_{1,2}$ $Q|_{\mathcal{F}_2} = P$, $R << P$ and $R|_{\mathcal{F}_1} = P$. Define the probability measure S on $(\Omega, \mathcal{F}_3)$ by $S(A) = E_R(E_Q(1_A|\mathcal{F}_2))$. S is absolutely continuous with respect to P and its restriction to $\mathcal{F}_1$ is equal to P and S satisfies (II)

For every $X \in \mathcal{A}_{1,3},\ X = Y + Z$ $(Y \in \mathcal{A}_{1,2},\ Z \in \mathcal{A}_{2,3})$

$E_S(-X|\mathcal{F}_1) = E_R(-Y|\mathcal{F}_1) + E_R(E_Q(-Z|\mathcal{F}_2)|\mathcal{F}_1)$

so $E_S(-X|\mathcal{F}_1) \leq \alpha^m_{1,2}(R) + E_R(\alpha^m_{2,3}(Q)|\mathcal{F}_1)$.

This proves that $\alpha^m_{1,3}(S)$ is P essentially bounded, so $S \in \mathcal{M}_{1,3}$ and that $\alpha^m_{1,3}(S) \leq E_R(\alpha^m_{2,3}(Q)|\mathcal{F}_1) + \alpha^m_{1,2}(R)$. Consider now $S \in \mathcal{M}_{1,3}$, $R \in \mathcal{M}_{1,2}$, $Q \in \tilde{\mathcal{M}}_{2,3}$ satisfying the relation (II), as above we get that $\alpha^m_{1,3}(S) \leq E_R(\alpha^m_{2,3}(Q)|\mathcal{F}_1) + \alpha^m_{1,2}(R)$.

Prove now the converse inequality. Let $Y \in \mathcal{A}_{1,2}$ and $Z \in \mathcal{A}_{2,3}$.

$E_R(-Y|\mathcal{F}_1) + E_R(E_Q(-Z|\mathcal{F}_2)|\mathcal{F}_1) = E_S(-Y - Z|\mathcal{F}_1)$ it follows that

$\forall Z \in \mathcal{A}_{2,3}$ $(E_R(E_Q(-Z|\mathcal{F}_2)|\mathcal{F}_1)) + \alpha^m_{1,2}(R) \leq \alpha^m_{1,3}(S)$.

From proposition 1, $\alpha^m_{2,3}(Q)$ is the limit of an increasing sequence

$f_n = E_Q(-Z_n|\mathcal{F}_2)$ for some $Z_n \in \mathcal{A}_{2,3}$ bounded from below by $-||f_1||_\infty$. Then $E_R(\alpha^m_{2,3}(Q))$ is the limit of the increasing sequence $E_R(E_Q(-Z_n))$ As $\mathcal{A}_{2,3} \subset \mathcal{A}_{1,3}$, we get $E_R(\alpha^m_{2,3}(Q)) \leq E_R(\alpha^m_{1,3}(S))$. Now as $\alpha^m_{2,3}(Q)$ is R integrable, it follows from [10] chapitre II that $E_R(\alpha^m_{2,3}(Q)|\mathcal{F}_1)$ is the limit a.s. of $E_R(E_Q(-Z_n|\mathcal{F}_2)|\mathcal{F}_1))$. Then $E_R(\alpha^m_{2,3}(Q)|\mathcal{F}_1) + \alpha^m_{1,2}(R) \leq \alpha^m_{1,3}(S)$. Q.e.d.

- iii) implies i)

Let $X \in \mathcal{E}_{\mathcal{F}_3}$. From the theorem of representation, there is a probability measure $R \in \mathcal{M}_{1,2}$ and a probability measure $Q \in \mathcal{M}_{2,3}$ such that

$\rho_{1,2}(-\rho_{2,3}(X)) = E_R(\rho_{2,3}(X)|\mathcal{F}_1) - \alpha^m_{1,2}(R)$

$= E_R(E_Q(-X|\mathcal{F}_2) - \alpha^m_{2,3}(Q)|\mathcal{F}_1) - \alpha^m_{1,2}(R)$

Applying hypothesis iii) we get the existence of a probability measure $S \in \mathcal{M}_{1,3}$ such that $\rho_{1,2}(-\rho_{2,3}(X)) = E_S(-X|\mathcal{F}_1) - \alpha^m_{1,3}(S)$.

From the representation of $\rho_{1,3}$ it follows that $\rho_{1,2}(-\rho_{2,3}(X)) \leq \rho_{1,3}(X)$.

We have to prove the converse inequality.

From theorem of representation applied to $\rho_{1,3}$, we get $R \in \mathcal{M}_{1,3}$ such that $\rho_{1,3}(X) = E_R(-X|\mathcal{F}_1) - \alpha_{1,3}(R)$.

Let $A = \{\omega \in \Omega \ / \ (E_P(\frac{dR}{dP}|\mathcal{F}_2)(\omega) > 0\}$ Define now the probability measure Q on $(\Omega, \mathcal{F}_3)$ by

$Q(B) = E_P(((\frac{(\frac{dR}{dP})}{E_P(\frac{dR}{dP}|\mathcal{F}_2)})1_A + 1_{(\Omega-A)})1_B)$.

Q is absolutely continuous with respect to P and the restriction of Q to $\mathcal{F}_2$ is equal to P i.e. $Q \in \tilde{\mathcal{M}}_{2,3}$. For every f bounded $\mathcal{F}_3$ measurable,

$E_R(E_Q(f|\mathcal{F}_2)|\mathcal{F}_1) = E_P((E_P(\frac{dR}{dP}|\mathcal{F}_2)(\frac{(\frac{dR}{dP})}{E_P(\frac{dR}{dP}|\mathcal{F}_2)})1_A + 1_{(\Omega-A)})f|\mathcal{F}_1)$

$= E_P(\frac{dR}{dP})1_A f|\mathcal{F}_1) = E_P(\frac{dR}{dP})f|\mathcal{F}_1)$. Indeed, $(\frac{dR}{dP})1_{(\Omega-A)} \geq 0$ and its conditional expectation with respect to $\mathcal{F}_2$ is 0 so it is 0 $P.a.s.$

So $E_R(E_Q(f|\mathcal{F}_2)|\mathcal{F}_1) = E_R(f|\mathcal{F}_1) \ \forall f \in \mathcal{E}_{\mathcal{F}_3}$.

From hypothesis iii), $\alpha^m_{1,3}(R) = E_R(\alpha^m_{2,3}(Q)|\mathcal{F}_1) + \alpha^m_{1,2}(R)$

and then $\rho_{1,3}(X) \leq \rho_{1,2}(-\rho_{2,3}(X)) \ \forall X \in \mathcal{E}_{\mathcal{F}_3}$.

So i) is proved.

Remark 3 *1) The equivalence of i) and ii) is also proved by Cheridito et al [6].*

2) The condition iii) is a generalization of the m-stability property of Delbaen [8]. Here there is a cocycle condition on the penalty (in the case of [8] the risk measures where coherent so the penalty was equal to 0).

The preceding theorem cannot be generalized to the case where the penalty function α is not equal to α^m; however we can prove that there are sufficient conditions on the penalty functions $(\alpha)_{i,j}$ in order to have the equality: $\rho_{1,3}(X) = \rho_{1,2}(-\rho_{2,3}(X)) \ P.a.s. \forall X \in \mathcal{E}_{\mathcal{F}_3}$.

Consider three σ-algebras $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$ on a space Ω. Consider a probability measure P on $(\Omega, \mathcal{F}_3)$.

Consider a set $\mathcal{M}$ of probability measures on $\mathcal{F}_3$ equivalent to P.

Theorem 3 *Assume that $\mathcal{M}$ satisfies the 2 stability properties:*

i) m-stability:

For every $Q \in \mathcal{M}$, for every $R \in \mathcal{M}$, there is $S \in \mathcal{M}$ such that

$$\forall f \in \mathcal{E}_{\mathcal{F}_3}, \ E_S(f|\mathcal{F}_1) = E_R(E_Q(f|\mathcal{F}_2)|\mathcal{F}_1) \ P.a.s.$$

ii) stability by bifurcation:

$\forall (Q_1, Q_2) \in \mathcal{M}^2$, $\forall A \in \mathcal{F}_i$ ($i = 1$ or $i = 2$), there is $Q \in \mathcal{M}$ such that

$$E_Q(f|\mathcal{F}_i) = E_{Q_1}(f1_A|\mathcal{F}_i) + E_{Q_2}(f1_{(\Omega-A)}|\mathcal{F}_i) \ P.a.s.$$

iii) Assume that the penalty function α is such that for all $1 \leq i < j \leq 3$ $ess\ sup_{Q\in\mathcal{M}}(-\alpha_{i,j}(Q))$ is bounded P a.s. Assume that the penalty function is local and satisfies the cocycle condition:

if $(Q, R, S) \in (\mathcal{M})^3$ are such that

$$\forall f \in \mathcal{E}_{\mathcal{F}_3},\ E_S(f|\mathcal{F}_1) = E_R(E_Q(f|\mathcal{F}_2)|\mathcal{F}_1) \ P.a.s.$$

then $\alpha_{1,3}(S) = E_R(\alpha_{2,3}(Q)|\mathcal{F}_1) + \alpha_{1,2}(R)$ P.a.s..

Then the risk measures $\rho_{i,j}$ on $(\Omega, \mathcal{F}_j, P)$ conditional to $(\Omega, \mathcal{F}_i, P)$ defined by

$$\rho_{i,j}(X) = Pess\ sup_{Q\in\mathcal{M}}\{E_Q(-X|\mathcal{F}_i)) - \alpha_{i,j}(Q)\}$$

satisfy the composition rule:

$$\rho_{1,3}(X) = \rho_{1,2}(-\rho_{2,3}(X)) \ P.a.s.\ \forall X \in \mathcal{E}_{\mathcal{F}_3}.$$

Proof:

As $ess\ sup_{Q\in\mathcal{M}}(-\alpha_{i,j}(Q))$ is bounded , $\rho_{i,j}$ is a well defined conditional risk measure. We want to adapt the proof of iii) implies i) in the preceding theorem.

The new difficulty here is that the *ess sup* is no more essentially attained.

- First using the stability by bifurcation of $\mathcal{M}$, we prove that for X fixed, $\{E_Q(-X|\mathcal{F}_i) - \alpha_{i,j}(Q)\ /Q \in \mathcal{M}\}$ is a lattice upward directed.

Indeed for every $(Q_1, Q_2) \in (\mathcal{M})^2$ consider

$A = \{\omega \in \Omega\ /\ E_{Q_1}(-X|\mathcal{F}_i)(\omega) - \alpha_{i,j}(Q_1) > E_{Q_2}(-X|\mathcal{F}_i)(\omega) - \alpha_{i,j}(Q_2)\}$

From bifurcation property, there is $Q \in \mathcal{M}$ such that

$\forall f\ E_Q(-f|\mathcal{F}_i) = E_{Q_1}(-f1_A|\mathcal{F}_i) + E_{Q_2}(-f1_{(\Omega-A)}|\mathcal{F}_i)$

From the local property of $\alpha_{i,j}$, $1_A\alpha_{i,j}(Q) = 1_A\alpha_{i,j}(Q_1)$ and

$1_{\Omega-A}\alpha_{i,j}(Q) = 1_{\Omega-A}\alpha_{i,j}(Q_2)$ So

$E_Q(-X|\mathcal{F}_i) - \alpha_{i,j}(Q) = max(E_{Q_1}(-X|\mathcal{F}_i) - \alpha_{i,j}(Q_1), E_{Q_2}(-X|\mathcal{F}_i) - \alpha_{i,j}(Q_2))$

- So for X fixed, there is a sequence $R_n \in \mathcal{M}$ such that $\rho_{1,2}(-\rho_{2,3}(X))$ is the increasing limit of $E_{R_n}(\rho_{2,3}(X)|\mathcal{F}_1) - \alpha_{1,2}(R_n)$, and a sequence $Q_k \in \mathcal{M}$ such that $\rho_{2,3}(X)$ is the increasing limit of $E_{Q_k}(-X|\mathcal{F}_2) - \alpha_{2,3}(Q_k)$.

As $\rho_{2,3}(X)$ is essentially bounded, for all n, $E_{R_n}(\rho_{2,3}(X)|\mathcal{F}_1) - \alpha_{1,2}(R_n)$ is a.s. the increasing limit of $E_{R_n}(E_{Q_k}(-X|\mathcal{F}_2) - \alpha_{2,3}(Q_k)|\mathcal{F}_1) - \alpha_{1,2}(R_n)$.

Using the m stability of $\mathcal{M}$ and the cocycle condition, this proves the inequality:

$\rho_{1,2}(-\rho_{2,3}(X)) \leq \rho_{1,3}(X)$ *P.a.s.*

Conversely, for every $X \in \mathcal{E}_{\mathcal{F}_3}$, there is a sequence R_n of probability measures in $\mathcal{M}$ such that $\rho_{1,3}(X)$ is the increasing limit of $E_{R_n}(-X|\mathcal{F}_1) - \alpha_{1,3}(R_n)$.

$E_{R_n}(-X|\mathcal{F}_1) - \alpha_{1,3}(R_n) = E_{R_n}(E_{R_n}(-X|\mathcal{F}_2)|\mathcal{F}_1) - E_{R_n}(\alpha_{2,3}(R_n)|\mathcal{F}_1) - \alpha_{1,2}(R_n)$.

It follows that $\rho_{1,3}(X) \leq \rho_{1,2}(-\rho_{2,3}(X))$ *P.a.s.*

Q.e.d.

From the proof of this theorem, we also obtain the following result:

Proposition 3 *Consider a set $\mathcal{M}_{i,j}$ of probability measures on $\mathcal{F}_j$ equivalent to P whose restriction to $\mathcal{F}_i$ is equal to P. Consider a penalty function $\alpha_{i,j}$ which assigns to each element of $\mathcal{M}_{i,j}$ an element of $L^\infty(\Omega, \mathcal{F}_i, P)$. Assume that ess $sup_{Q\in\mathcal{M}_{i,j}}(-\alpha_{i,j}(Q))$ is essentially bounded.*

Assume that the $\mathcal{M}_{i,j}$ satisfy the following stability conditions: For every $Q \in \mathcal{M}_{2,3}$, for every $R \in \mathcal{M}_{1,2}$, there is $S \in \mathcal{M}_{1,3}$ such that

$$\forall f \in \mathcal{E}_{\mathcal{F}_3},\ E_S(f|\mathcal{F}_1) = E_R(E_Q(f|\mathcal{F}_2)|\mathcal{F}_1)\ P.a.s.\ (II)$$

and also for all $S \in \mathcal{M}_{1,3}$, there are $Q \in \mathcal{M}_{2,3}$ and $R \in \mathcal{M}_{1,2}$ such that (II) is satisfied.

Assume that every $\mathcal{M}_{i,j}$ satisfies the stability by bifurcation with respect to $\mathcal{F}_i$. Assume that the penalty function is local and satisfies the following cocycle condition: if $(Q, R, S) \in (\mathcal{M}_{2,3} \times \mathcal{M}_{1,2} \times \mathcal{M}_{1,3})$ are such that

$$\forall f \in \mathcal{E}_{\mathcal{F}_3},\ E_S(f|\mathcal{F}_1) = E_R(E_Q(f|\mathcal{F}_2)|\mathcal{F}_1)\ P.a.s.$$

then $\alpha_{1,3}(S) = E_R(\alpha_{2,3}(Q)|\mathcal{F}_1) + \alpha_{1,2}(R)$ P.a.s..

Then the $\rho_{i,j} = ess\ sup_{Q\in\mathcal{M}_{i,j}}\{E_Q(-X|\mathcal{F}_i)) - \alpha_{i,j}(Q)\}$ satisfy also the composition rule.

2 Discrete time dynamic risk measures in uncertain context

2.1 Probability measure associated to a dynamic risk measure

We consider a space Ω and a numerable increasing family of σ-algebras $\mathcal{F}_n$ on Ω such that $\mathcal{F}_0$ is the trivial σ-algebra ($\mathcal{F}_0 = \{\emptyset, \Omega\}$). Denote $\mathcal{F}$ the σ-

algebra generated by the $\mathcal{F}_n$. We don't assume that a probability measure is given a priori.

Definition 4 *A dynamic risk measure on* $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in I\!N})$ *is a family* $((\rho_{n,n+1})_{n \in I\!N})$ *where* $\rho_{n,n+1}$ *is a convex risk measure on* $(\Omega, \mathcal{F}_{n+1})$ *conditional to* $(\Omega, \mathcal{F}_n)$.

Proposition 4 *Let* $n < m$. *Consider a dynamic risk measure as in the preceding definition. Then the relation* $\rho_{n,m} = \rho_{n,n+1} o(-\rho_{n+1,n+2})..o(-\rho_{m-1,m})$ *defines a risk measure on* $(\Omega, \mathcal{F}_m)$ *conditional to* $(\Omega, \mathcal{F}_n)$. *The family* $(\rho_{n,m})$ *is time-consistent; i.e.* $\forall n < m < r$ $\rho_{n,r} = \rho_{n,m} o(-\rho_{m,r})$.

Remark 4 *This notion of time-consistency first appeared in the work of Peng [25].*

Theorem 4 *Let* $(\mathcal{F}_n)_{n \in I\!N*}$ *be the natural filtration of a finite family of real valued processes* $(X^j)_{1 \leq j \leq k}$ *such that for all* j, $X_0^j = 0$ *and for all* (j, n), X_n^j *is bounded.* $\mathcal{F}_0$ *is the trivial* σ*-algebra.*

To every dynamic risk measure $(\rho_{n,n+1})_{n \in I\!N}$ *such that for all* $n \in I\!N$, $\rho_{n,n+1}$ *is continuous from below, is canonically associated a probability measure* P *on* $\mathcal{F} = U\mathcal{F}_n$ *and a dynamic risk measure* $((\overline{\rho_{n,n+1}})_{n \in I\!N})$ *on the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in I\!N}, P)$ *such that, for* $A \in \mathcal{F}_{n+1}$,

$P(A) = 0$ *iff* $\rho_{n,n+1}(\lambda 1_A) = \rho_{n,n+1}(0)$ *P a.s.* $\forall \lambda \in I\!R$ *and* $\overline{\rho_{n,n+1}}(X) = \rho_{n,n+1}(X)$ *P.a.s.*

The equivalence class of P *is thus uniquely determined by* $(\rho_{n,n+1})_{n \in I\!N}$.

In order to prove this theorem, we prove first the following lemma.

Lemma 1 *Let* $\rho_{\mathcal{F}}$ *be a convex risk measure defined on* $(\Omega, \mathcal{G})$ *conditional to* $(\Omega, \mathcal{F}, P)$, *continuous from below. Let* $\tilde{\rho}_{\mathcal{F}} = \rho_{\mathcal{F}} - \rho_{\mathcal{F}}(0)$.

Consider any representation of $\tilde{\rho}_{\mathcal{F}}$ *of the kind*

$\tilde{\rho}_{\mathcal{F}}(X) = ess\ max_{Q \in \mathcal{M}}((E_Q(-X|\mathcal{F}) - \alpha(Q))$ *P a.s.*

where $\mathcal{M}$ *is a set of probability measures* Q *on* $(\Omega, \mathcal{G})$ *whose restriction to* $\mathcal{F}$ *is equal to* P *and such that* $\alpha(Q)$ *is bounded (P a.s.) (such representations exist from theorem 1).*

Then $(Q(A) = 0\ \forall Q \in \mathcal{M})$ *iff* $(\tilde{\rho}_{\mathcal{F}}(\lambda 1_A) = 0$ *P a.s.* $\forall \lambda \in I\!R)$.

Proof:

Assume that $Q(A) = 0\ \forall Q \in \mathcal{M}$. Then

$\tilde{\rho}_{\mathcal{F}}(\lambda 1_A) = ess\ max_{Q \in \mathcal{M}}((E_Q(-\lambda 1_A)|\mathcal{F}) - \alpha(Q))\ = \tilde{\rho}_{\mathcal{F}}(0) = 0$ *P.a.s.*

Conversely assume that $\tilde{\rho}_{\mathcal{F}}(\lambda 1_A) = 0$ *P a.s.* $\forall \lambda \in I\!R$.

If there is $Q \in \mathcal{M}$ such that $Q(A) > 0$. Then $E_Q(1_A|\mathcal{F}) \neq 0$ (and $E_Q(1_A|\mathcal{F}) \geq 0$). So there is $\epsilon > 0$ such that $P(B_\epsilon) > 0$ where $B_\epsilon = \{\omega \in \Omega / E_Q(1_A|\mathcal{F})(\omega) \geq \epsilon\}$ ($B_\epsilon \in \mathcal{F}$).

Let $\lambda < 0$. $E_Q(-\lambda 1_A 1_{B_\epsilon}|\mathcal{F}) - \alpha(Q)1_{B_\epsilon} \geq -\lambda\epsilon 1_{B_\epsilon} - \alpha(Q)1_{B_\epsilon}$

$\alpha(Q)1_{B_\epsilon}$ is bounded (P $a.s.$). So there is $\lambda < 0$ such that

$E_Q(-\lambda 1_A 1_{B_\epsilon}|\mathcal{F}) - \alpha(Q)1_{B_\epsilon} > 1_{B_\epsilon}$ P $a.s.$

It follows that $1_{B_\epsilon}\tilde{\rho}_{\mathcal{F}}(\lambda 1_A) \neq 0$ $P.a.s.$ and thus we get a contradiction.

So for every $Q \in \mathcal{M}$, $Q(A) = 0$.

Q.e.d.

Lemma 2 *Assume that $\mathcal{G}$ is the σ-algebra generated by the bounded application $X = (X_i)_{1 \leq i \leq l}$ on Ω with values into $(\mathbb{R}^l, \mathcal{B})$ ($\mathcal{B}$ being the Borelian σ-algebra). For every set of probability measures $\mathcal{Q}$ on $(\Omega, \mathcal{G})$, there is a probability measure $\tilde{Q}$ (unique up to equivalence) such that $\forall A \in \mathcal{G}$ $\tilde{Q}(A) = 0$ iff $\forall \mu \in \mathcal{Q}$ $\mu(A) = 0$.*

Denote $\mathcal{A}$ the norm closed unital Banach subalgebra of $\mathcal{E}_{\mathcal{G}}$ generated by the X_i. This algebra is separable. From Dunford et al [14] the unit ball of its dual is metric compact for the weak* topology. So the weak* closure of $\{(E_\mu)_{|\mathcal{A}}; \mu \in \mathcal{Q}\}$ is metric compact . It has a numerable dense subset $(E_{Q_j})_{j \in \mathbb{N}^*}$. $\tilde{Q} = \sum_{j \in \mathbb{N}^*} \frac{Q_j}{2^j}$. is a probability measure on $(\Omega, \mathcal{G})$. Consider $\mu \in \mathcal{Q}$ and $A \in \mathcal{G}$ such that $\mu(A) > 0$. By definition of $\mathcal{G}$, there is a Borelian $B \in \mathcal{B}$ such that $A = X^{-1}(B)$. Denote ν the image of μ by X. $\nu(B) = \mu(A) > 0$. ν is a Borel measure on $\mathbb{R}^l$ so it is regular. There is $f \geq 0$ continuous with compact support , $f \leq 1_B$ such that $\int f(t)d\nu(t) > 0$ i.e.$\int f(X(s))d\mu(s) > 0$. Now $f(X)$ is an element of $\mathcal{A}$ It follows that there is j such that $\int f(X(s))dQ_j(s) > 0$. So $Q_j(A) > 0$ and $\tilde{Q}(A) > 0$.

This proves an implication. The converse implication is trivial. The unicity of $\tilde{Q}$ up to equivalence follows easily.

Proof of the theorem

We prove by recursion on $n \in \mathbb{N}$ the existence of the probability P on $(\Omega, \mathcal{F}_n)$.

$\mathcal{F}_0 = \{\emptyset, \Omega\}$. So there is a unique probability on it.

Assume that we have proved the existence of a probability P on $(\Omega, \mathcal{F}_n)$ satisfying the required conditions. Consider the risk measure on $\mathcal{E}_{\mathcal{F}_{n+1}}$ conditional to $(\Omega, \mathcal{F}_n, P)$ defined by $\tilde{\overline{\rho}}_{n,n+1}(X) = \overline{\rho}_{n,n+1}(X) - \overline{\rho}_{n,n+1}(0)$ $P.a.s.$ Consider $\mathcal{Q}_{n+1}$ the set of probability measures on $(\Omega, \mathcal{F}_{n+1})$ whose restriction to $\mathcal{F}_n$ is equal to P associated to the usual representation of $\tilde{\overline{\rho}}_{n,n+1}$. Applying lemma 2 to $\mathcal{Q}_{n+1}$ and then lemma1 this gives a probability measure P_{n+1} on $(\Omega, \mathcal{F}_{n+1})$ satisfying the required conditions.

2.2 A discrete time dynamic risk measure as a single conditional risk measure

We prove now that a discrete time dynamic risk measure can be viewed as a single conditional risk measure on a bigger measurable space. Furthermore, the 2 points of view are equivalent.

We consider a space Ω and a numerable increasing family of σ-algebras $\mathcal{F}_n$ on Ω such that $\mathcal{F}_0$ is the trivial σ-algebra ($\mathcal{F}_0 = \{\emptyset, \Omega\}$). We don't assume that a probability measure is given a priori.

Denote now $\tilde{\Omega} = \Omega \times I\!N$ and $\tilde{\mathcal{F}}$ the σ-algebra generated by the sets $A_i \times \{i\}$ where $A_i \in \mathcal{F}_i$. Denote also $\tilde{\mathcal{F}}^s$ the shifted algebra generated by the sets $A_i \times \{i\}$ where $A_i \in \mathcal{F}_{i-1}$.

Proposition 5 *There is a canonical bijection between the dynamic risk measures $\rho_{n,n+1}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in I\!N})$ and the convex risk measures on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ conditional to $(\tilde{\Omega}, \tilde{\mathcal{F}}^s)$.*

Proof:

- Let $\rho_{n,n+1}$ a dynamic risk measure.

Define $\Psi((\rho_{n,n+1})) = \rho$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ by $\rho(f)(\omega, i) = \rho_{i-1,i}(f_i)(\omega)$ where $f_i(\omega) = f(\omega, i)$.

For every open set U in $I\!R$ $\rho(f)^{-1}(U) = U_{i \in I\!N}(\{\omega / \rho_{i-1,i}(f_i)(\omega) \in U\} \times \{i\} = U_{i \in I\!N} A_i \times \{i\}$ where A_i is $\mathcal{F}_{i-1}$ measurable. So $\rho(f)$ is $(\tilde{\Omega}, \tilde{\mathcal{F}}^s)$ measurable.

f is $\tilde{\mathcal{F}}^s$ measurable iff for each i, f_i is $\mathcal{F}_{i-1}$ measurable; so the translation invariance property of ρ follows from the translation invariance property of the $\rho_{i-1,i}$ for every i.

Monotonicity and convexity of ρ easily follow from the same properties of the $\rho_{i-1,i}$.

The multiplicative invariance property of ρ follows from the fact that each subset $\tilde{\mathcal{F}}$ (resp $\tilde{\mathcal{F}}^s$) measurable can be written $U_{i \in I\!N} A_i \times \{i\}$ where A_i is $\mathcal{F}_i$ (resp $\mathcal{F}_{i-1}$)measurable. So ρ is a convex risk measure conditional to $(\tilde{\Omega}, \tilde{\mathcal{F}}^s)$.

- Conversely consider a convex risk measure ρ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ conditional to $(\tilde{\Omega}, \tilde{\mathcal{F}}^s)$. To each application $\mathcal{F}_i$ measurable f associate $\tilde{f}$ defined on $\tilde{\Omega}$ by $\tilde{f}(\omega, j) = 0$ if $j \neq i$ and $\tilde{f}(\omega, i) = f(\omega)$.

$\rho_{i-1,i}(f) = (\rho(\tilde{f}))_i$ defines a convex risk measures on $(\Omega, \mathcal{F}_i)$ conditional to $(\Omega, \mathcal{F}_{i-1})$. The map Φ defined by $\Phi(\rho) = (\rho_{i-1,i})_{i \in I}$ is the converse of Ψ.

Corollary 1 *When a probability P is given on $\mathcal{F}$, define the probability $\tilde{P}$ on $\tilde{\mathcal{F}}$ by: $\tilde{P}(U_{i \in I\!N} A_i \times \{i\}) = \sum_{i \in I\!N} \frac{1}{2^{i+1}} P(A_i)$. There* *is a canonical bijection between the set of dynamic risk measures on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}} P)$ and the set of convex risk measures on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ conditional to $(\tilde{\Omega}, \tilde{\mathcal{F}}^s, \tilde{P})$.*

Corollary 2 *Assume that the σ-algebras $\mathcal{F}_n$ are as in theorem 4 To every convex risk measure ρ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ conditional to $(\tilde{\Omega}, \tilde{\mathcal{F}}^s)$ continuous from below is canonically associated a class P of probability measure on $(\Omega, \mathcal{F})$ such that ρ is a convex risk measure on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ conditional to $(\tilde{\Omega}, \tilde{\mathcal{F}}^s, \tilde{P})$. Considering such a conditional convex risk measure is then equivalent to considering a dynamic risk measure continuous from below, on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}} P)$.*

Proof:

It results from theorem 4, proposition 5 and the fact that the continuity from below of ρ is equivalent to the continuity from below of each $\rho_{n,n+1}$.

3 Continuous time dynamic risk measures on a probability space

3.1 Time consistency for continuous time dynamic risk measures

We consider a probability space $(\Omega, \mathcal{F}, P)$ and an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}+}$ of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = U_{t\in\mathbb{R}+}(\mathcal{F}_t)$.

Definition 5 *A dynamic risk measure on $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}+}, P)$ is a family $(\rho_{s,t})_{0\le s\le t}$ of convex risk measure on $(\Omega, \mathcal{F}_t, P)$ conditional to $(\Omega, \mathcal{F}_s, P)$.*

Definition 6 *A dynamic risk measure is time consistent if*

$$\forall r < s < t \ \rho_{r,t} = \rho_{r,s}(-\rho_{s,t})$$

For each $\rho_{s,t}$ continuous from below, consider the usual representation (I). Using the same notations as in theorem 2 ,we prove the following characterization of the time consistency:

Theorem 5 *Consider a dynamic risk measure $\rho_{s,t}$ continuous from below. For each $\rho_{s,t}$ consider its usual representation.*

The following properties are equivalent:

i)The dynamic risk measure $\rho_{s,t}$ is time consistent.

ii) The acceptance sets $\mathcal{A}_{s,t}$ satify the following additive property:

$$\forall r < s < t \;\; \mathcal{A}_{r,t} = \mathcal{A}_{r,s} + \mathcal{A}_{s,t}$$

iii) The sets of probability measures $(\mathcal{M}_{s,t})$ satisfy the following stability property:

$\forall r < s < t \;\; \forall Q \in \mathcal{M}_{s,t}, \; \forall R \in \mathcal{M}_{r,s}$ there exists $S \in \mathcal{M}_{r,t}$ such that

$$\forall f \in \mathcal{E}_{\mathcal{F}_t}, \; E_S(f|\mathcal{F}_r) = E_R(E_Q(f|\mathcal{F}_s)|\mathcal{F}_r) \; P.a.s.$$

and the penalty function satisfies the cocycle condition:

$\forall S \in \mathcal{M}_{r,t} \; \forall R \in \mathcal{M}_{r,s} \; \forall Q \in \tilde{\mathcal{M}}_{s,t}$ satisfying the relation (II), $\alpha^m_{s,t}(Q)$ is R integrable and then $\alpha^m_{r,t}(S) = E_R(\alpha^m_{s,t}(Q)|\mathcal{F}_r) + \alpha^m_{r,s}(R)$ P.a.s.

Proof:

This theorem follows from the theorem 2 of composition for the conditional risk measures.

Corollary 3 *Consider $(g_{s,t})_{0 \le s < t}$ a family of stictly positive bounded $\mathcal{F}_s$-measurable functions such that $ln(g_{s,t})$ is essentially bounded. Consider the entropic dynamic risk measure defined as follows:*

Let $0 \le s \le t$. For every $X \in \mathcal{E}_{\mathcal{F}_t}$

$\rho_{s,t}(X) = ess \inf\{Y \in \mathcal{E}_{\mathcal{F}_s} \;/\; E(e^{[-\alpha(X+Y)]}|\mathcal{F}_s) \le g_{s,t}\}$

$= \frac{1}{\alpha}[lnE(e^{-\alpha X})|\mathcal{F}_s) - ln(g_{s,t})\}.$

Then $\rho_{s,t}(X) = ess\, max_{Q \in \mathcal{M}_{s,t}}(E_Q(-X|\mathcal{F}_s) - \alpha^m_{s,t}(Q))$

with $\alpha^m_{s,t}(Q) = \frac{1}{\alpha}(E_P(ln(\frac{dQ}{dP})\frac{dQ}{dP}|\mathcal{F}_s) - ln(g_{s,t}))$.

The entropic dynamic risk measure is time consistent if and only if the functions $g_{s,t}$ are $\mathcal{F}_0$ measurable and satisfy the relation $\forall 0 \le r \le s \le t \;\; ln(g_{r,t}) = ln(g_{r,s}) + ln(g_{s,t})$ a.s. In particular if we assume that there is a strictly positive real valued continuous function h such that $\forall (s,t) \; g_{s,t} = h(t-s)$ then the associated dynamic risk measure is time-consistent if and only if there is a real number λ such that $g_{s,t} = e^{\lambda(t-s)}$.

Proof:

The study of the conditional risk measure associated to a loss function and the computation of the penalty function is done in details in [5], section 5.

In particular in the case of the loss function $l(x) = e^{\alpha x}$, this gives the conditional entropic risk measure and the penalty function is $\alpha^m_{s,t}(Q) = \frac{1}{\alpha}(E_P(ln(\frac{dQ}{dP})\frac{dQ}{dP}|\mathcal{F}_s) - ln(g_{s,t}))$.

$\forall r < s < t \;\; \forall Q \in \mathcal{M}_{s,t}, \; \forall R \in \mathcal{M}_{r,s}$ There is a probability measure $S \in \mathcal{M}_{r,t}$ such that $\forall f \in \mathcal{E}_{\mathcal{F}_t}, \; E_S(f|\mathcal{F}_r) = E_R(E_Q(f|\mathcal{F}_s)|\mathcal{F}_r)$

Its Radon Nikodym derivative is $\frac{dS}{dP} = \frac{dQ}{dP}\frac{dR}{dP}$, and then
$E_R(\alpha^m_{s,t}(Q)|\mathcal{F}_r) + \alpha^m_{r,s}(R) = \frac{1}{\alpha}[E_P(\frac{dR}{dP}(E_P(\frac{dQ}{dP}ln(\frac{dQ}{dP})|\mathcal{F}_s)|\mathcal{F}_r)$
$+(E_P(\frac{dR}{dP}ln(\frac{dR}{dP})|\mathcal{F}_r) - E_P(\frac{dR}{dP}ln(g_{s,t})|\mathcal{F}_r) - ln(g_{r,s})]$.

As $\frac{dR}{dP}ln(\frac{dR}{dP})$ is $\mathcal{F}_s$ measurable, as $E(\frac{dQ}{dP}|\mathcal{F}_s) = 1$ and $\frac{dS}{dP} = \frac{dQ}{dP}\frac{dR}{dP}$, it follows that

$E_R(\alpha^m_{s,t}(Q)|\mathcal{F}_r)+\alpha^m_{r,s}(R) = \alpha^m_{r,t}(S)+\frac{1}{\alpha}[ln(g_{r,t})-ln(g_{r,s})-E_P(\frac{dR}{dP}ln(g_{s,t})|\mathcal{F}_r)]$.
Hence the dynamic risk measure is time consistent if and only if

$[ln(g_{r,t}) - ln(g_{r,s}) - [E_P(\frac{dR}{dP}ln(g_{s,t})|\mathcal{F}_r) = 0$ $P.a.s.$ for every probability measure $R \in \mathcal{M}_{r,s}$ It follows that $ln(g_{s,t})$ has to be $\mathcal{F}_r$ measurable for every r so it is $\mathcal{F}_0$ measurable and then the risk measure associated to the family of $\mathcal{F}_0$) measurable maps $g_{s,t}$ is time consistent if and only if $\forall 0 \leq r \leq s \leq t$ $ln(g_{r,t}) = ln(g_{r,s}) + ln(g_{s,t})$ $a.s.$.

The end of the proof is just the application of a classical result.

Q.e.d.

The dynamic entropic risk measure is also considered in [3] and in [11] in both cases only when $g_{s,t} = 1$ $\forall(s,t)$ and in the second paper only in a discrete time setting. In both cases the time-consistent property is verified.

Corollary 4 *Let* $p > 1$ *Consider the loss function* $l(x) = \frac{x^p}{p}$ $if x \geq 0$
$l(x) = 0$ *else.*
Consider $(g_{s,t})_{0\leq s<t}$ *a family of positive bounded* $\mathcal{F}_s$*-measurable map.*
The associated dynamic risk measure is defined as follows:
Let $0 \leq s \leq t$. *For every* $X \in \mathcal{E}_{\mathcal{F}_t}$
$\rho_{s,t}(X) = ess\ \inf\{Y \in \mathcal{E}_{\mathcal{F}_s}\ /\ E(l(-X-Y)|\mathcal{F}_s) \leq g_{s,t}\ Pa.s.\}$
Then the penalty function is
$\alpha^m_{s,t}(Q) = (pg_{s,t})^{\frac{1}{p}}E_P[(\frac{dQ}{dP})^q|\mathcal{F}]^{\frac{1}{q}}$ *(q is the conjugate of p)*
And this dynamic risk measure is not consistent if $g_{s,t} \neq 0$ *P.a.s..*

Proof:

The computation of the penalty function is done in [5] section 5. The non consistency when $g_{s,t} \neq 0$ $P.a.s.$is then an easy consequence of the formula.

Strict monotonicity: This property was first introduced by S. Peng for the conditional g-expectations in the lectures "Applications of BSDE in finance" given at IHP in Paris in March 2005.

Definition 7 *We say that the dynamic risk measure is strictly monotone if* $\forall t\ \forall(X,Y) \in \mathcal{E}^2_{\mathcal{F}_t}$ *if* $X \geq Y$ *and* $\rho_{0,t}(X) = \rho_{0,t}(Y)$ *then* $X = Y$ *a.s.*

As in the case of conditional g expectations, if the dynamic risk measure is strictly monotone then the time consistency has only to be checked for $r = 0$. More precisely:

Proposition 6 *Assume that the normalized dynamic risk measure is strictly monotone. Assume that* $\forall 0 < s < t$ $\rho_{0,t} = \rho_{0,s}(-\rho_{s,t})$

Then the dynamic risk measure is time-consistent.

Proof: Let $0 < r < s < t$. Let $X \in \mathcal{E}_{\mathcal{F}_t}$. Denote $Y = -\rho_{r,t}(X)$ and $Z = -\rho_{r,s}(-\rho_{s,t}(X))$.Let $A \in \mathcal{F}_r$

$\rho_{0,r}(1_A Z) = \rho_{0,r}((-\rho_{r,s}(-\rho_{s,t}(1_A X)))) = \rho_{0,s}((-\rho_{s,t}(1_A X)))$

$= \rho_{0,t}(1_A X) = \rho_{0,r}((-\rho_{r,t}(1_A X))) = \rho_{0,r}(1_A Y)$.

Let $A = \{\omega \in \Omega / Y(\omega) > Z(\omega)\}$ A is $\mathcal{F}_r$ measurable. From the strict monotonicity it follows that A is a negligeable set. So $Y \leq Z$ *a.s.* The converse inequality is proved in the same way. So $Y = Z$ *a.s.*

The following theorem is very important for the construction of exemples of time-consistent dynamic risk measures.

Theorem 6 *Consider a family* $\mathcal{Q}$ *of probability measures on* $(\Omega, \mathcal{F})$ *all equivalent to* P*. For all* $0 \leq r < s < t$

Assume that $\mathcal{Q}$ *satisfies the 2 stability properties:*

i) m-stability:

For every $Q \in \mathcal{Q}$*, for every* $R \in \mathcal{Q}$*, there is* $S \in \mathcal{Q}$ *such that*

$$\forall f \in \mathcal{E}_{\mathcal{F}_t},\ E_S(f|\mathcal{F}_r) = E_R(E_Q(f|\mathcal{F}_s)|\mathcal{F}_r)\ P.a.s.$$

ii) stability by bifurcation:

$\forall (Q_1, Q_2) \in \mathcal{Q}^2$*,* $\forall A \in \mathcal{F}_r$ *there is* $Q \in \mathcal{Q}$ *such that* $\forall f \in \mathcal{F}_s$*,*

$$E_Q(f|\mathcal{F}_r) = E_{Q_1}(f 1_A|\mathcal{F}_r) + E_{Q_2}(f 1_{(\Omega - A)}|\mathcal{F}_r)\ P.a.s.$$

iii) Assume that the penalty function α *is such that for all* s, t*,* $ess\ sup_{Q \in \mathcal{Q}}(-\alpha_{s,t}(Q))$ *is essentially bounded. Assume that* α *is local and satisfies the cocycle condition: if* $(Q, R, S) \in (\mathcal{Q})^3$ *are such that*

$$\forall f \in \mathcal{E}_{\mathcal{F}_t},\ E_S(f|\mathcal{F}_r) = E_R(E_Q(f|\mathcal{F}_s)|\mathcal{F}_r)\ P.a.s.$$

then $\alpha_{r,t}(S) = E_R(\alpha_{s,t}(Q)|\mathcal{F}_r) + \alpha_{r,s}(R)$ *P.a.s.*

Then the dynamic risk measure $(\rho_{s,t})_{0 \leq r < s < t}$ *defined by*

$$\rho_{s,t}(X) = P ess\ sup_{Q \in \mathcal{Q}}\{E_Q(-X|\mathcal{F}_s)) - \alpha_{s,t}(Q)\}$$

is time-consistent.

This theorem is just an application of the corresponding theorem 3 for the composition of conditional risk measures.

To construct time-consistent dynamic risk measures, we will also use the following lemma:

Lemma 3 *To each family $\mathcal{Q}_1$ of probability measures all equivalent to a probability P we can associate a minimal set of probability measures $\mathcal{Q}$ both m-stable and stable by bifurcation. It is the set of probability measures Q such that there is a subdivision $0 = t_0 < t_1 < ...t_n$ and for each $i \in \{0, ...n\}$ there are disjoint $\mathcal{F}_{t_i}$ measurable sets $A_{i,j}$, $U_j A_{i,j} = \Omega$ and probability measures $Q_{i,j} \in \mathcal{Q}_1$ such that*

$$\frac{(\frac{dQ}{dP})_{t_{i+1}}}{(\frac{dQ}{dP})_{t_i}} = \sum_j \frac{(\frac{dQ_{i,j}}{dP})_{t_{i+1}}}{(\frac{dQ_{i,j}}{dP})_{t_i}} 1_{A_{i,j}} \;\; \textit{and}\; \forall t > t_n \;\; \frac{(\frac{dQ}{dP})_t}{(\frac{dQ}{dP})_{t_n}} = \sum_j \frac{(\frac{dQ_{n,j}}{dP})_t}{(\frac{dQ_{i,j}}{dP})_{t_n}} 1_{A_{n,j}} \quad (III)$$

where $(\frac{dQ}{dP})_t$ means $E(\frac{dQ}{dP}|\mathcal{F}_t)$

Proof: Denote $\tilde{\mathcal{Q}}$ the set of probability measures whose Radon Nikodym derivative satisfies (III). It is easy to verify that $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ and that $\tilde{\mathcal{Q}}$ is m-stable.

Furthermore if Q_1 and Q_2 are in $\tilde{\mathcal{Q}}$ and A is $\mathcal{F}_r$ measurable we can construct a new subdivision $(s_i)_{0 \leq i \leq m}$ containing r adapted to both Q_1 and Q_2. Consider Q such that $(\frac{dQ}{dP})_r = (\frac{dQ_1}{dP})_r$ and for $s > r$,

$\frac{(\frac{dQ}{dP})_s}{(\frac{dQ}{dP})_r} = \frac{(\frac{dQ_1}{dP})_s}{(\frac{dQ_1}{dP})_r} 1_A + \frac{(\frac{dQ_2}{dP})_s}{(\frac{dQ_2}{dP})_r} 1_{\Omega - A}$. Then $Q \in \tilde{\mathcal{Q}}$ and $\forall s > r$, $\forall f \in \mathcal{F}_s$,

$E_Q(f|\mathcal{F}_r) = E_{Q_1}(f 1_A|\mathcal{F}_r) + E_{Q_2}(f 1_{(\Omega - A)}|\mathcal{F}_r)$ $P.a.s.$ so $\tilde{\mathcal{Q}}$ is stable by bifurcation.

3.2 Dynamic risk measure associated to a family of BMO continuous martingales

For continuous BMO martingales we refer to the book of Kazamaki [21].

Consider a filtered complete probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0 \leq t \leq \infty})$ satisfying the usual hypothesis. Let $(M_t, \mathcal{F}_t)$ be a uniformly integrable martingale with $M_0 = 0$ For $1 \leq p < \infty$, let

$$||M||_{BMO_p} = \sup_S ||E[|M_\infty - M_S|^p|\mathcal{F}_S]^{\frac{1}{p}}||_\infty$$

the sup being taken over all stopping times S.

There is then a positive constant K_p such that for any uniformly integrable continuous martingale:

$$||M||_{BMO_1} \leq ||M||_{BMO_p} \leq K_p ||M||_{BMO_1} \quad (IV)$$

Recall the following definition of the BMO martingales.

A uniformly integrable continuous martingale M_t is a BMO continuous martingale if $||M||_{BMO_1} < \infty$.

It is proved in Kazamaki [21] theorem 2.3. that if M_t is a continuous BMO martingale, then $\mathcal{E}(M)$ is a uniformly integrable martingale (where $\mathcal{E}(M)_t = exp(M_t - \frac{1}{2} < M >_t)$.

Definition 8 *A family $\mathcal{M}$ of martingales on $(\Omega, \mathcal{F}, P)$ is stable if:*

$\forall\ 0 \leq s,\ \forall (M,N) \in \mathcal{M}^2,\ \forall A \in \mathcal{F}_s,\quad (\tilde{M})_t$ *defined by* $(\tilde{M})_t = (N_t - N_s)1_A + (M_t - M_s)1_{\Omega - A} + M_s$ *for* $s < t$ *and* $(\tilde{M})_t = M_t$ *for* $t \leq s$ *is a martingale in* $\mathcal{M}$.

Remark: The set of continuous martingales on $(\Omega, \mathcal{F}, P)$ is stable.

Lemma 4 *To each set $\mathcal{M}_1$ of martingales on $(\Omega, \mathcal{F}, P)$ is associated a minimal stable set of martingales $\mathcal{M}$ containing $\mathcal{M}_1$. It is the intersection of all the stable sets of martingales containing $\mathcal{M}_1$.*

$\mathcal{M}$ is the set of all martingales $\tilde{M}$ on $(\Omega, \mathcal{F}, P)$ for which there exists a subdivision $0 = t_0 < t_1 < ...t_n$ and for each $i \in \{0, ...n\}$ there are disjoint $\mathcal{F}_{t_i}$ measurable sets $A_{i,j}$ such that $U_j A_{i,j}$ and martingales $M_{i,j}$ in $\mathcal{M}_1$

such that $(\tilde{M})_{t_{i+1}} - (\tilde{M})_{t_i} = \sum_j ((M_{i,j})_{t_{i+1}} - (M_{i,j})_{t_i})1_{A_{i,j}}$

The proof of this lemma is the same as the proof of lemma 3.

notation

Let $\mathcal{M}$ be a stable set of continuous BMO martingales. For each $M \in \mathcal{M}$, denote (Q_M) the probability measure equivalent to P of Radon Nikodym derivative $\frac{dQ_M}{dP} = \mathcal{E}(M)$. Denote $\mathcal{Q}(\mathcal{M}) = \{(Q_M)/M \in \mathcal{M}\}$.

From lemmas 3 and 4 we deduce the following result:

Lemma 5 *Let $\mathcal{M}$ be a stable set of continuous BMO martingales. Then the set of probability measures $\mathcal{Q}(\mathcal{M})$ is both m-stable and stable by bifurcation.*

It is proved in Kazamaki [21] that the class $BMO(Q_M)$ is equal to the class $BMO(P)$. More precisely we can prove the following lemma:

Lemma 6 *For every $K > 0$ there exists $\tilde{K} > 0$ such that for every continuous BMO martingale M such that $||M||_{BMO_2(P)} \leq K$, for every X continuous BMO martingale, $||X||_{BMO_2(Q_M)} \leq \tilde{K} ||X||_{BMO_2(P)}$*

Proof:

From theorem 3.1. of Kazamaki [21] there is p_0 such that for every M such that $||M||_{BMO_2(P)} \leq K$, for every $p \leq p_0$, $\mathcal{E}(M)$ satisfies

$$(R_p) \;\; E[(\mathcal{E}(M))_\infty^p|\mathcal{F}_T] \leq C_p(\mathcal{E}(M))_T^p$$

for every stopping time T.

Apply now the conditional Hölder inequality (denote q the conjugate exponent of p).

$||X||_{BMO_1(Q_M)} \leq \sup_T ||(E((\frac{(\mathcal{E}(M))_\infty}{(\mathcal{E}(M))_T})^p|\mathcal{F}_T)||_\infty^{\frac{1}{p}} ||X||_{BMO_q(P)}$

Applying now the inequalities (IV) and R_p, we get

$||X||_{BMO_2(Q_M)} \leq K_2 K_q (C_p)^{\frac{1}{p}} ||X||_{BMO_2(P)}$

Q.e.d.

We are now able to construct dynamic risk measures using continuous BMO martingales. We will give several exemples.

Proposition 7 *Consider a stable family $\mathcal{M}$ of BMO continuous martingales. Define on $\mathcal{Q}(\mathcal{M})$ the penalty function α as follows:*

$$\forall 0 \leq s \leq t \;\; \alpha_{s,t}(Q_M) = E_{Q_M}(Z_t(M) - Z_s(M)|\mathcal{F}_s)$$

.

Assume that one of the following conditions is satisfied:

i) There is a positive bounded predictable process b_s such that

$$\forall M \in \mathcal{M} \;\; Z_t(M) = \int_0^t b_s d[M,M]_s$$

ii) There is a positive K such that for all $M \in \mathcal{M}$, $||M||_{BMO(1)} \leq K$. There is a bounded predictable process b_s such that

$$\forall M \in \mathcal{M} \;\; Z_t(M) = \int_0^t b_s d[M,M]_s$$

iii) There is a positive K such that for all $M \in \mathcal{M}$, $||M||_{BMO(1)} \leq K$. There is a bounded predictable process H such that

$$\forall M \in \mathcal{M} \;\; Z_t(M) = (H.M)_t$$

Then

$$\rho_{s,t}(X) = esssup_{M \in \mathcal{M}}(E_{Q_M}(-X|\mathcal{F}_s) - \alpha_{s,t}(Q_M))$$

defines a time-consistent dynamic risk measure on the filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0 \leq t \leq \infty})$.

Proof:

Let $0 \leq s \leq t$.

We verify first that $esssup_{M \in \mathcal{M}}(-\alpha_{s,t}(Q_M))$ is essentially bounded.

In cases i) and ii), The process b_s is bounded by C.

$\alpha_{s,t}(Q_M) = (E_{Q_M}(\int_s^t b_u d[M,M]_u|\mathcal{F}_s)$

$||\alpha_{s,t}(Q_M)||_\infty \leq C(E_{Q_M}(\int_s^t d[M,M]_u|\mathcal{F}_s)$. So $||\alpha_{s,t}(Q_M)||_\infty \leq C||M||^2_{BMO_2(Q_M)}$

Applying now the preceding lemma, it follows that for each M,$||\alpha_{s,t}(Q_M)||_\infty$ is finite, and in case ii) that there is $\tilde{K}$ such that $||esssup_{M \in \mathcal{M}}(-\alpha_{s,t}(Q_M))||_\infty \leq \tilde{K}$

In case i), $\alpha_{s,t}(Q_M) \geq 0$ a.s. and $\alpha_{s,t}(Q_0) = 0$ so $esssup_{M \in \mathcal{M}}(-\alpha_{s,t}(Q_M)) = 0$

In case iii) H is bounded by B, $H.M$ is BMO and $||H.M||_{BMO_2} \leq B||M||_{BMO_2} \leq KB$

So it follows from the inequality (IV) and the preceding lemma that there is K' such that

$$\forall M \in \mathcal{M} \ ||\alpha_{s,t}(Q_M)||_\infty \leq ||H.M||_{BMO_1(Q_M)} \leq K'$$

So $||esssup_{M \in \mathcal{M}}(-\alpha_{s,t}(Q_M))||_\infty \leq K'$

It remains to prove that the penalty function is local and satisfies the cocycle condition.

-α is local:

Let $(M_1, M_2) \in \mathcal{M}^2$ let $A \in \mathcal{F}_s$ such that $\forall X \in \mathcal{F}_t \ \ E_{Q_{M_1}}(X1_A|\mathcal{F}_s) = E_{Q_{M_2}}(X1_A|\mathcal{F}_s)$

It follows that $(\frac{(\mathcal{E}(M_1))_t}{(\mathcal{E}(M_1))_s})1_A = (\frac{(\mathcal{E}(M_2))_t}{(\mathcal{E}(M_2))_s})1_A$ and so for every $s \leq u \leq t$

$1_A((M_1)_u - (M_1)_s - \frac{1}{2}[M_1, M_1]_s^u) = 1_A((M_2)_u - (M_2)_s - \frac{1}{2}[M_2, M_2]_s^u)$

From the unicity in the Doob Meyer decomposition, it follows that $1_A((M_1)_u - (M_1)_s) = 1_A(M_2)_u - (M_2)_s)$ and then in each of the cases of the proposition, $1_A\alpha_{s,t}(Q_{M_1}) = 1_A\alpha_{s,t}(Q_{M_2})$. So α is local.

- cocycle condition:

Let $0 \leq r < s < t$, let $(M, N, R) \in \mathcal{M}^3$ be such that

$$\forall f \in \mathcal{E}_{\mathcal{F}_t}, \ E_{Q_M}(f|\mathcal{F}_r) = E_{Q_R}(E_{Q_N}(f|\mathcal{F}_s)|\mathcal{F}_r)$$

i.e. $(\frac{(\mathcal{E}(M))_t}{(\mathcal{E}(M))_r}) = (\frac{(\mathcal{E}(N))_t}{(\mathcal{E}(N))_s})(\frac{(\mathcal{E}(R))_s}{(\mathcal{E}(R))_r})$. Then

$E_{Q_R}(\alpha_{s,t}(Q_N)|\mathcal{F}_r)+\alpha_{r,s}(Q_R) = E(\frac{(\mathcal{E}(R))_s}{(\mathcal{E}(R))_r}(\frac{(\mathcal{E}(N))_t}{(\mathcal{E}(N))_s}(Z_t(N)-Z_s(N)|\mathcal{F}_s)|\mathcal{F}_r))$
$+E(\frac{(\mathcal{E}(R))_s}{(\mathcal{E}(R))_r}(Z_s(R)-Z_r(R))|\mathcal{F}_r)$

Denote $Y_{r,s} = \frac{(\mathcal{E}(R))_s}{(\mathcal{E}(R))_r}(Z_s(R)-Z_r(R))$ It is $\mathcal{F}_s$ measurable and $\mathcal{E}(N)$ is a martingale so

$E((\frac{(\mathcal{E}(N))_t}{(\mathcal{E}(N))_s}Y_{r,s}|\mathcal{F}_r) = E(E((\frac{(\mathcal{E}(N))_t}{(\mathcal{E}(N))_s}|\mathcal{F}_s)Y_{r,s}|\mathcal{F}_r)) = E(Y_{r,s}|\mathcal{F}_r)$

It follows that

$E_{Q_R}(\alpha_{s,t}(Q_N)|\mathcal{F}_r) + \alpha_{r,s}(Q_R)$
$= E((\frac{(\mathcal{E}(M))_t}{(\mathcal{E}(M))_r})(Z_t(N) - Z_s(N) + Z_s(R) - Z_r(R))|\mathcal{F}_r)$.

As in the proof of the locality of α, $M_u - M_s = N_u - N_s$ $\forall s \leq u \leq t$ and $M_v - M_r = R_v - R_r$ $\forall r \leq v \leq s$. And then in each of the cases i) ii) and iii) of the proposition, $Z_t(N) - Z_s(N) + Z_s(R) - Z_r(R) = Z_t(M) - Z_r(M)$ So α satisfies the cocycle condition.

Hence from the theorem 6, $\rho_{s,t}$ is in each case a time-consistent dynamic risk measure.

Q.e.d.

We give now an exemple of a stable family of continuous BMO martingales of BMO norm uniformly bounded.

Lemma 7 *Consider $(N_i)_{1\leq i\leq k}$ strongly orthogonal continuous martingales. Consider $(\phi_i)_{1\leq i\leq k}$ non negative predictable processes such that $\forall i$, $\phi_i.N_i$ is a continuous BMO martingale. The set $\mathcal{M}$ of continuous BMO martingales of the form the $\sum_{1\leq i\leq k} H_i.N_i$ where H_i is a locally bounded predictable process such that $|H_i| \leq \phi_i$ a.s. is a stable set of continuous BMO martingales with norm BMO uniformly bounded.*

Proof:

It is easy to verify that this set is stable; and for every $M \in \mathcal{M}$, $||M||^2_{BMO_2} \leq \sum_{1\leq i\leq k} ||\phi_i.N_i||^2_{BMO_2}$

Q.e.d.

Now we can construct in this context another family of exemples of time-consistent dynamic risk measures.

Proposition 8 *Consider a family of strongly orthogonal continuous martingales $(N_i)_{1\leq i\leq k}$. Consider a non negative locally bounded predictable ϕ such that for all i, $\phi.N_i$ is BMO. As in the preceding lemma consider the stable set $\mathcal{M}$ of BMO martingales $\sum_{1\leq i\leq k} H_i.N_i$ with predictable H_i such that $|H_i| \leq \phi$ a.s. Consider as in the proposition 7, the penalty function α defined on $\mathcal{Q}(\mathcal{M})$ as follows:*

$$\forall 0 \leq s \leq t \;\; \alpha_{s,t}(Q_M) = E_{Q_M}(Z_t(M) - Z_s(M)|\mathcal{F}_s).$$

Assume now that one of the following conditions is satisfied:

i) There are Borel functions $b_i(s, x_1, x_2, ..., x_k)$ *with quadratic growth in* (x_i)*; i.e. there is a constant* $K > 0$ *such that* $|b_i(s,x)| \leq K(\phi^2 + \sum_{1 \leq i \leq k} |x_i|^2)$ *such that for* $\forall M \in \mathcal{M}$ *,*

$$Z_t(\sum_{1 \leq i \leq k} H_i.N_i) = \sum_{1 \leq i \leq k} \int_0^t b_i(s, H_1, H_2, ...H_k) d[N_i, N_i]_s$$

ii) There are Borel functions $a_i(s, x_1, x_2, ..., x_k)$ *with linear growth in* (x_i)*; i.e. there is a constant* $K > 0$ *such that* $|a_i(s, x_1, x_2, ..., x_k)| \leq K(\phi + \sup_{1 \leq i \leq k} |x_i|$ *such that for* $\forall M \in \mathcal{M}$ *,*

$$Z_t(\sum_{1 \leq i \leq k} H_i.N_i) = (\sum_{1 \leq i \leq k} a_i(s, H_1, H_2, ...H_k).N_i)_t$$

Then $\rho_{s,t}(X) = esssup_{M \in \mathcal{M}}(E_{Q_M}(-X|\mathcal{F}_s) - \alpha_{s,t}(Q_M))$ *defines a time-consistent dynamic risk measure.*

Proof:

The processes $a_i(s, H_1(s), H_2(s), ...H_k(s))$ and $b_i(s, H_1(s), H_2(s), ...H_k(s))$ are locally bounded predictable.

The proof of this proposition is similar to the proof of the preceding one.

In case i), we get that there is a constant C such that for every $M = \sum_{1 \leq i \leq k} H_i.N_i$,

$||\alpha_{s,t}(Q_M)||_\infty \leq C(\sum_i (||\phi.N_i||^2_{BMO_2(Q_M)}$

In case ii) we get that $||\alpha_{s,t}(Q_M)||_\infty \leq C \sum_{1 \leq i \leq k} ||(\phi.N_i)||_{BMO_1(Q_M)}$.

As in the proof of case iii) of proposition 7, using the lemma 6 it follows that $esssup_{M \in \mathcal{M}}(-\alpha_{s,t}(Q_M))$ is essentially bounded. The end of the proof is as that of proposition 7

Proposition 9 *Consider a family of strongly orthogonal continuous martingales* $(N_i)_{1 \leq i \leq k}$*. The set* $\mathcal{M}$ *of all the martingales of the form* $M = \sum_{1 \leq i \leq k} H_i.N_i$ *such that* $H_i.N_i$ *is BMO for all i is stable.*
Consider $b_i(s, x_1, x_2, ..., x_k)$ *such that there is a non negative predictable process* ψ_i *such that* $|b_i(s, x_1, x_2, ..., x_k)| \leq k(\psi_i)^2 + \sum_{1 \leq i \leq k} |x_i|^2$*. Assume that* $\psi_i.N_i$ *is BMO.*

Assume that $\rho_{s,t}(X) = ess\ max_{M \in \mathcal{M}}((E_{Q_M}(-X|\mathcal{F}_s) - \alpha_{s,t}(Q_M))$ *P a.s. where*

$\forall 0 \leq s \leq t \ \ \alpha_{s,t}(Q_M) = E_{Q_M}(Z_t(M) - Z_s(M)|\mathcal{F}_s)$ *with*

$$Z_t(\sum_{1 \leq i \leq k} H_i.N_i) = \sum_{1 \leq i \leq k} \int_0^t b_i(s, H_1, H_2, ...H_k) d[N_i, N_i]_s$$

Then the dynamic risk measure is time-consistent.

Proof:

As $b_i(s, x_1, x_2, ..., x_k)$ is of quadratic growth, from the preceding proposition, for each M $||\alpha_{s,t}(Q_M)||_\infty$ is finite for each M. Furthermore there is $\tilde{M}$ in $\mathcal{M}$ such that

$essmax_{M\in\mathcal{M}(-\alpha_{s,t}(Q_M))=-\alpha_{s,t}(Q_{\tilde{M}})}$ and hence it is essentially bounded.

The rest of the proof is the same as that of the preceding proposition.

Remark 5 *This exemple generalizes the dynamic risk measures obtained from the BSDE.*

Indeed consider $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $\mathcal{F}_t$ is the augmented filtration of a d dimensional Brownian motion. When the driver $g(t, z)$ satisfies $g(t, 0) = 0$ and is continuous and convex(in z), and satisfies the condition of quadratic growth the associated BSDE

$-dy_t = g(t, z_t)dt - z_t^* dB_t$

$y_T = X$

has a solution which gives rise to a dynamic risk measure. $\rho_{s,T}(-X) = y_s$ P. Barrieu and N. El Karoui [3] section 7.3 have computed the representation associated to this dynamic risk measure, assuming that g is strongly convex.

$\rho_{s,t}(X) = essmax_{M\in\mathcal{M}}(E_{Q_M}(-X|\mathcal{F}_s) - \alpha_{s,t}(Q_M))$

where $\mathcal{M}$ is the set of martingales of the form $M = \sum_{1\leq i\leq k} H_i.N_i$ such that $H_i.N_i$ is BMO for all i. And the N_i are independent Brownian motions. The penalty function is $\alpha_{s,t}(Q_M) = E_{Q_M}(Z_t(M) - Z_s(M)|\mathcal{F}_s)$ with

$Z_t(\sum_{1\leq i\leq k} H_i.N_i) = \sum_{1\leq i\leq k} \int_0^t G(s, H_1, H_2, ...H_k)ds$

where G has quadratic growth. So the dynamic risk measures obtained as solutions of BSDE are particular cases of the dynamic risk measures obtained in the preceding proposition when the reference orthogonal martingales are independent Brownian motions.

In that sense the dynamic risk measures constructed as in the preceding propositions can be viewed as generalizations of the BSDE when we start with general continuous uniformly integrable martingales; and no more with Brownian motions.

Now we will give more exemples constructed in the same manner using as before BMO martingales. But now we do not assume continuity of the martingales. We consider cadlag BMO martingales. Jumps are allowed.

3.3 Dynamic risk measure associated to a family of BMO martingales with jumps

The references for the general right continuous BMO martingales are the papers of C.Doléans-Dade and P.A.Meyer [12] and [13]

Consider a filtered complete probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\leq t\leq\infty})$ satisfying the usual hypothesis. Let $(M_t, \mathcal{F}_t)_{0\leq t\leq T}$ be a (uniformly integrable) square integrable cadlag martingale with $M_0 = 0$. M is in BMO if

$$||M||_{BMO} = sup_S ||E[[M,M]_{S-}^{\infty}|\mathcal{F}_S]||_{\infty}^{\frac{1}{2}}$$

is finite (where $[M,M]_{S-}^{\infty}$ means $[M,M]_{\infty} - [M,M]_{S-}$).

Denote $\mathcal{E}(M)$ the unique solution of the stochastic integral

$Z_t = 1 + \int_0^t Z_{s-}dM_s$. It is well known that

$\mathcal{E}(M)_t = exp(M_t - \frac{1}{2} < M^c, M^c >_t)\Pi_{s\leq t}(1 + \Delta M_s)e^{-\Delta M_s}$.

Recall now the following result which is included in the proof of theorem 1 of C.Dolans-Dade and P.A.Meyer [12].

Let M a BMO martingale ($M_0 = 0$) such that $||M||_{BMO} < \frac{1}{8}$, then $\mathcal{E}(M)$ is a strictly positive process which is a martingale.

In order to construct time-consistent dynamic risk measures we will make use of the two following lemmas:

Lemma 8 *Assume that M is a BMO martingale. Then for every T,*
$E(([M,M]_{T-}^{\infty})^2|\mathcal{F}_T) \leq 2(||M||_{BMO})^4$.

Proof:

We apply the theorem 23 of the chapter V of [22] to the increasing process $[M,M]_t$, the constant positive random variable $(||M||_{BMO})^2$ and the continuous increasing function $\phi(x) = 2x$.

Thus we get $E(([M,M]_{\infty})^2|\mathcal{F}_0) \leq 2E([M,M]_{\infty}(||M||_{BMO})^2$. We end the proof as in the proof of the lemma 1 in [12] applying the preceding result to the martingale $M'_t = M_{T+t} - M_{T^-}$, and the σ-algebras $\mathcal{F}'_t = \mathcal{F}_{T+t}$.

Lemma 9 *Assume that M is a BMO martingale with $|M||_{BMO} \leq K < \frac{1}{16}$. Then $E((\frac{\mathcal{E}(M)_{\infty}}{\mathcal{E}(M)_{T-}})^2|\mathcal{F}_T) \leq \frac{1}{1-16K} < \infty$.*

Proof:

$(\frac{\mathcal{E}(M)_{\infty}}{\mathcal{E}(M)_{T-}})^2 = exp2[(M_{\infty} - M_{T^-}) - (< M^c, M^c >_{\infty} - < M^c, M^c >_{T^-})]$
$(\Pi_{s\geq T}(1 + \Delta M_s)e^{-\Delta M_s})^2$.

The jumps of M are bounded by $||M||_{BMO} < 1$.

As in the proof of theorem 1 in [12], it follows from the inequality $e^x \geq 1 + x$ that each factor of the preceding infinite product is between 0 and 1. It follows that $(\frac{\mathcal{E}(M)_{\infty}}{\mathcal{E}(M)_{T-}})^2 \leq \exp 2|M_{\infty} - M_{T^-}|$

Consider now as in the end of the proof of the preceding lemma the martingale $M'_t = M_{T+t} - M_{T^-}$, the σ-algebras $\mathcal{F}'_t = \mathcal{F}_{T+t}$.

Let $M'^* = \sup_t |M'_t|$. It follows then from the John Nirenberg inequality that $E(exp(2M'^*)|\mathcal{F}'_0) < \frac{1}{1-16||M||_{BMO}}$.

Q.e.d.

We are now able to construct dynamic risk measures associated to stable sets of BMO martingales with BMO norms bounded by a constant $K < \frac{1}{16}$.

Proposition 10 *Consider a stable family $\mathcal{M}$ of BMO cadlag martingales (with $M_0 = 0$). Assume that there is a $0 < K < \frac{1}{16}$ such that for every $M \in \mathcal{M}$, $||M||_{BMO} \leq K$. For every M denote Q_M the probability measure such that $\frac{dQ_M}{dP} = \mathcal{E}(M)$.*

Define on $\mathcal{Q}(\mathcal{M})$ the penalty function α as follows:

$\forall 0 \leq s \leq t \;\; \alpha_{s,t}(Q_M) = E_{Q_M}(Z_t(M) - Z_s(M)|\mathcal{F}_s)$.

Assume that one of the following conditions is satisfied:

i) There is a bounded predictable process b_s

such that $\forall M \in \mathcal{M} \;\; Z_t(M) = \int_0^t b_s d[M, M]_s$

ii) There is a bounded predictable process H such that $\forall M \in \mathcal{M} \;\; Z_t(M) = (H.M)_t$

Then $\rho_{s,t}(X) = esssup_{M\in\mathcal{M}}(E_{Q_M}(-X|\mathcal{F}_s) - \alpha_{s,t}(Q_M))$ defines a time-consistent dynamic risk measure.

Proof:

As $||M||_{BMO} < K$ and $K < \frac{1}{8}$, it follows that $\mathcal{E}(M)$ is a strictly positive martingale and so Q_M is a probability measure equivalent to P.

In case i) as the process b_s is bounded by C,

$||\alpha_{s,t}(Q_M)||_\infty \leq C||(E_{Q_M}([M, M]_s^t|\mathcal{F}_s)||_\infty$

Applying the conditional Hölder inequality and the two preceding lemma, we get that for every $M \in \mathcal{M}$

$$||\alpha_{s,t}(Q_M)||_\infty^2 \leq C^2||E((\frac{\mathcal{E}(M)_t}{\mathcal{E}(M)_s})^2|\mathcal{F}_s)||_\infty||E(([M, M]_s^\infty)^2|\mathcal{F}_s)||_\infty \leq \frac{4C^2}{1-16K}K^4$$

and so $||esssup_{M\in\mathcal{M}}(-\alpha_{s,t}(Q_M))||_\infty$ is finite.

In case ii), the proof is similar.

The proof of the locality and the cocycle condition of α is the same as in the case of continuous BMO martingales.

We give now another exemple in the case where the family $\mathcal{M}$ is constructed from a family of orthogonal BMO martingales.

Proposition 11 *Consider a family of strongly orthogonal cadlag martingales* $(N_i)_{1\leq i\leq k}$ *(with* $(N_i)_0 = 0$*). Consider a locally bounded non negative predictable* ϕ *such that for all* i, $\phi.N_i$ *is BMO and such that:* $(\sum_{1\leq i\leq k}(||\phi.N_i||_{BMO})^2)^{\frac{1}{2}} < \frac{1}{16}$. *Consider the stable set* $\mathcal{M}$ *of BMO martingales* $\sum_{1\leq i\leq k} H_i.N_i$ *with* $|H_i| \leq \phi$ *a.s. Consider the penalty function* α *defined on* $\mathcal{Q}(\mathcal{M})$ *as in proposition 8.*

Then $\rho_{s,t}(X) = esssup_{M\in\mathcal{M}}(E_{Q_M}(-X|\mathcal{F}_s) - \alpha_{s,t}(Q_M))$ *defines a time consistent dynamic risk measure.*

The proof of this proposition is similar to that of the preceding one.

Conclusion

Making use of the conditional risk measures and their dual representation obtained in [5], we have studied the dynamic risk measures both in discrete time and in continuous time.

In discrete time (section 2), the study is done in a context of uncertainty (without fixing in advance a probability measure), which is very relevant for the study of financial markets. The main result is that when the filtration $\mathcal{F}_n$ is the natural filtration of a $I\!R^l$ valued process bounded at each time, we can associate canonically a class of probability measures to any dynamic risk measure continuous from below.

We prove also that a dynamic risk measure (which is sequence of conditional risk measures) can be viewed as a single conditional risk measure on a bigger space.

In continuous time (section 3) we have fully characterized the time-consistency of a dynamic risk measure on a filtered probability space, in terms of its dual representation: stability of the set of probability measures and cocycle condition of the minimal penalty function. We have also proved that a dynamic risk measure defined from any stable set of probability measures and any penalty function (which is not assumed to be the minimal one) is time-consistent if the penalty function is local and satisfies the cocycle condition.

This allows us to construct new families of time-consistent dynamic risk measures using BMO martingales. The examples obtained generalize the dynamic risk measures which are obtained as solutions of BSDE. Using BMO right continuous martingales, we also construct time-consistent dynamic risk measures which can have jumps. These various examples will be very usefull for dynamic pricing and hedging in incomplete markets. This will be the subject of a future work.

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