STABILITY OF INTERFACES AND STOCHASTIC DYNAMICS IN THE REGIME OF PARTIAL WETTING.

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Abstract. The goal of this paper is twofold. First, assuming strict convexity of the surface tension, we derive a stability property with respect to the Hausdorff distance of a coarse grained representation of the interface between the two pure phases of the Ising model. This improves the $\mathbb{L}^1$ description of phase segregation.

Using this result and an additional assumption on mixing properties of the underlying FK measures, we are then able to derive bounds on the decay of the spectral gap of the Glauber dynamics in dimensions larger or equal to three. These bounds are related to previous results by Martinelli [Ma] in the two-dimensional case. Our assumptions can be easily verified for low enough temperatures and, presumably, hold true in the whole of the phase coexistence region.

1. Introduction

During the last decade, a series of studies enabled to derive rigorously the occurrence of phase segregation starting from a model with microscopic interactions. The phase separation phenomenon has been established for a fairly general class of models, but the correspondence between the microscopic models and the equilibrium crystal shapes (solution of the Wulff variational problem) is extremely loose. Thus, important questions remain and a complete theory of phase coexistence is far from being achieved.

A thorough description of the phase coexistence phenomena should include a characterization of the structure of the interface (thickness, fluctuation, detailed structure ...) as well as an understanding of the relaxation of the system to the pure phases away from the interface. So far such complete program has been achieved only in the case of two dimensional nearest neighbor Ising model [DKS, DH, ISc, Pf, PV, BCK]. The strategy developed in this context, relies on the one dimensional structure of the interface; this enabled to derive not only the Wulff construction, but as well quantitative statements on the microscopic configurations: existence of a unique large droplet, localization of the interface wrt the Hausdorff distance ...

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For systems in three or more dimensions the interface is a more complicated geometrical object and a different approach of phase coexistence, the $\mathbb{L}^1$-theory, was initiated in order to bypass the complexity of the microscopic configurations.

In this new framework, a weaker characterization of the phase segregation is obtained in terms of local averages of the magnetization. In this way, the occurrence of macroscopic equilibrium crystals whose shapes are solutions of a variational problem can be predicted, but unlike the two dimensional case, nothing can be inferred on the interface. In fact one can not even conclude from these results that the equilibrium crystal contains only a pure phase: as the statements are formulated in terms of averages and interfaces are understood only in $\mathbb{L}_1$-sense, one could not rule out the situation when equilibrium crystals contain minority phase impurities or even are made of a collection of small crystals glued together .

The first step is to propose a relevant interpretation of the interface. Let us, as an example, consider a three dimensional Ising model with Dobrushin boundary conditions, i.e. mixed boundary conditions which enforce an horizontal interface. In this case, the interface can be unambiguously defined as the unique open contour in the system. At low temperature, the interface is a rigid two dimensional hyperplane with some protuberances attached to it (e.g. one dimensional filaments). The statistic of these excitations is known and the open contour which forms this interface is localized wrt the Hausdorff distance. On the one hand, as the temperature increases above the roughening temperature the interface is expected to be macroscopically flat but with some logarithmic fluctuations. However, as the temperature approaches the critical temperature, the behavior of the microscopic contour becomes irregular and in particular one dimensional filaments are conjectured to percolate through the whole system [ABL]. Thus a microscopic representation of the interface is then irrelevant since the microscopic contour might be completely delocalized (see [CePi] for a discussion on this phenomenon). The way out is to renormalize the system at a proper mesoscopic scale for which the interface becomes regular. This is characteristic of the physicists heuristics which says that the complex microscopic configurations can be reduced to an effective interface model and should share the same properties on a suitable mesoscopic scale.

As mentioned previously, the $\mathbb{L}^1$-theory sheds little light on the statistical properties of random interfaces. The goal of this paper is to show that, nevertheless, on a mesoscopic level some smoothness properties of the interface are restored. Though much more modest than the heuristic picture described above, our results show that the low dimensional excitations of the coarse grained interface disappear and we recover a macroscopic stability with respect to the Hausdorff distance of the random interface.

The exact statement of this stability result is given in Subsection 3.1 along with some comments on the implications for the statistical Hausdorff stability of higher dimensional mesoscopic Wulff shapes.

The second part of this paper deals with dynamical properties, we derive bounds on the logarithmic asymptotics of the spectral gap of the Glauber dynamics. Such asymptotics are non trivial whenever the energy level of the bottleneck between two pure, though possibly metastable, phases is strictly higher than the free energy of each of the respective phases.

For a two dimensional Ising model with free boundary conditions, Martinelli derived in [Ma] the precise logarithmic asymptotics of the spectral gap. He proved that the dominant time scale to reach one equilibrium state starting from the other one is the creation of an interface; once created the interface moves in a much shorter amount of time until the other equilibrium state is reached. Martinelli devised very ingenious techniques in order to control the occurrence of the interface and its motion, in particular the dynamical estimates were reduced to some statements on the equilibrium measure. The analysis of [Ma] has been based on very specific facts about the Hausdorff stability of the 2D nearest neighbor Ising model interfaces, on the closely related exponential mixing properties of finite volume pure state and on exact surface order large deviation asymptotics for the magnetization inside the phase coexistence region. In higher dimensions, we are going to use the large deviation estimates of the $\mathbb{L}^1$-theory and the Hausdorff stability of the random interface on the macroscopic scale in order to extend the results of [Ma].

The upper and lower bounds derived in higher dimensions on the spectral gap are expressed in terms of variational principles for which (contrary to the two-dimensional) the solution is not explicit. For this reason, we cannot prove that the lower and upper bound coincide as it was shown in the two dimensional case [Ma]. A more complete discussion on the interplay between the metastability and the wetting is postponed to Subsection 3.2.

Apart from being dependent on the validity of Pisztora coarse graining (c.f. Subsection 2.2) our proof of the interface stability relies on strict convexity of surface tension. The analysis of the spectral gap asymptotics for the Glauber dynamics requires an additional assumption on exponential mixing properties of the underlying FK measures. Both assumptions are described and discussed in Subsection 2.5 and are expected to hold for a wide range of sub-critical temperatures.

While completing this paper, we learnt about the recent work by N. Sugimine [Su1, Su2] on upper and lower bounds for spectral gap for the three dimensional low temperature Ising model with mixed $+/\emptyset$ boundary conditions.

2. Notations and Assumptions

2.1. **The microscopic model.** We consider the nearest neighbor ferromagnetic Ising model in dimension $d \geqslant 2$. For any domain $\Delta \subset \mathbb{Z}^d$ and boundary conditions η outside Δ, the Gibbs measure on $\{\pm 1\}^\Delta$ at inverse temperature β will be denoted by $\mu^\eta_{\beta,\Delta}$. Thus, given $\sigma \in \{\pm 1\}^\Delta$,

$$\mu^\eta_{\beta,\Delta}(\sigma) \;=\; \frac{1}{\mathbf{Z}^\eta_{\beta,\Delta}} \exp\{\,-\beta \mathbf{H}^\eta_\Delta(\sigma)\}\,,$$

where the nearest neighbour Hamiltonian $\mathbf{H}^{\eta}_{\Delta}$ is given by

$$\mathbf{H}^{\eta}_{\Delta}(\sigma) \;=\; -\frac{1}{2}\sum_{i\sim j\in\Delta}\sigma_i\sigma_j \;-\; \sum_{\substack{i\in\Delta\\ j\in\mathbb{Z}^d\setminus\Delta}}\eta_j\sigma_i .$$

There exists a critical value β_c such for any $\beta > \beta_c$ a phase transition, characterized by symmetry breaking, occurs. Throughout the paper, we always consider an inverse temperature β for which the system is in a phase coexistence regime and, we denote by $m^*(\beta)$ the spontaneous magnetization in the $+$ phase.

It will be convenient to work with an alternative representation of the microscopic system, namely the FK representation. Given $\Delta \subset \mathbb{Z}^d$ let $\mathcal{E}^d_{\Delta}$ be the set of bonds, i.e. the pairs of nearest neighbor vertices in $\mathbb{Z}^d$, with at least one end-point belonging to Δ. The percolation configuration ξ on $\mathcal{E}^d_{\Delta}$, or with an abuse of notation on Δ, is an element $\xi \in \Xi_{\Delta} \overset{\Delta}{=} \{0,1\}^{\mathcal{E}^d_{\Delta}}$. We shall suppress the domain sub-index and write Ξ whenever $\Delta = \mathbb{Z}^d$.

Given $\xi \in \Xi$ and a bond $b \in \mathcal{E}^d$, we say that b is open if $\xi(b) = 1$. Two sites of $\mathbb{Z}^d$ are said to be connected if one can be reached from another via a chain of open bonds. Thus, each $\xi \in \Xi$ splits $\mathbb{Z}^d$ into the disjoint union of maximal connected components, which are called the open clusters of Ξ. Given a finite subset $B \subset \mathbb{Z}^d$ we use $c_B(\xi)$ to denote the number of different open finite clusters of ξ which have a non-empty intersection with B.

Below we give a general definition of FK measures which are related to the finite volume spin Gibbs states on $\Delta \subset \mathbb{Z}^d$. We use a provisional notation intended to illustrate the connection between the Gibbs states and the FK measures. A more precise notation will be introduced later for particular cases which show up in the main body of the paper.

The set of bonds connecting Δ to Δ^c will be denoted by $\mathcal{E}_{\partial\Delta}$. The boundary conditions are specified by a frozen percolation configuration $\pi \in \Xi \setminus \Xi_{\Delta}$ and by the collection $\mathbf{p} \in [0, 1 - \mathrm{e}^{-2\beta}]^{\mathcal{E}_{\partial\Delta}}$, which describes the "activity" of the bounds on the boundary of Δ.

We write $\xi \vee \pi$ for the joint configuration in Ξ and define the finite volume FK measure on Δ with the boundary conditions π and $\mathbf{p}$ as:

$$\Phi^{\pi,\mathbf{p}}_{\Delta}(\xi) \;\overset{\Delta}{=}\; \frac{1}{\mathbf{Z}^{\pi}_{\Delta}}\left(\prod_b \big(1-p(b)\big)^{1-\xi_b}\big(p(b)\big)^{\xi_b}\right) 2^{c_{\Delta}(\xi\vee\pi)}(\xi)\,, \tag{2.1}$$

where, for a bond $b = (i\,,\,j)$ with $(i\,,\,j) \in \Delta$, we set $p(b) = p(\beta, b) = 1 - \exp(-2\beta)$; otherwise if b is in $\mathcal{E}_{\partial\Delta}$ the corresponding percolation probability $p(b) = \mathbf{p}(b)$. The two extreme configurations such that $\mathbf{p}(b) \equiv 1 - \exp(-2\beta)$ and $\pi \equiv 1$ and, respectively, $\mathbf{p}(b) \equiv 0$ or $\pi \equiv 0$ for all b in $\mathcal{E}_{\partial\Delta}$ lead to the FK measures with wired $\Phi^{\mathrm{w}}_{\Delta}$ or free $\Phi^{\mathrm{f}}_{\Delta}$ boundary conditions respectively. The intermediate values of $\mathbf{p}$ have a natural interpretation as a magnetic boundary field. This enables to represent the Gibbs measures for which the boundary conditions η have only non-negative components.

If η has only non negative coordinates, the Gibbs state $\mu^{\eta}_{\beta,\Delta}$ can be reconstructed as follows (see [ES]): First for any $b = (i\,,\,j) \in \mathcal{E}_{\partial\Delta}$ with $j \in \Delta^c$ set $\mathbf{p}(b) = p(\beta,\eta,b) = 1 - \mathrm{e}^{-2\eta_j\beta}$. Next sample a bond configuration $\xi \in \Xi_\Delta$ from the FK measure $\Phi^{w,\mathbf{p}}_{\beta,\Delta}$, and paint with 1 all the clusters of ξ connected to the regions of the boundary where $\eta_i > 0$, whereas all the remaining clusters of ξ are to be painted into ± 1 with probability $1/2$ each. The corresponding joint bond-spin probability measure is denoted by $\mathbb{P}^{\eta}_{\beta,\Delta}$.

$$\mathbb{P}^{\eta}_{\beta,\Delta}(\sigma,\xi) = P^{\eta,\xi}_{\Delta}(\sigma)\ \Phi^{\pi,\mathbf{p}}_{\beta,\Delta}(\xi)\,,$$

where $P^{\eta,\xi}_{\Delta}$ denotes the painting measure. The Gibbs state $\mu^{\eta}_{\beta,\Delta}$ is then recovered as the σ-marginal of $\mathbb{P}^{\eta}_{\beta,\Delta}$.

The representation for more general boundary conditions which correspond to sign changing η will be discussed later.

2.2. **Scales and coarse graining.** All scales are binary. The running microscopic scale will be $N = 2^n$ and the associated renormalization scale $K = 2^k$. We shall work either with fixed finite scales K or else we shall explicitly relate K to N as $K = N^a$, where the fixed positive number $a = a(n)$ (the dependence on n is only in order to be compatible with the binary notation) satisfies

$$0 < a_1 \leqslant a \leqslant a_2 < 1/d. \tag{2.2}$$

All our computations go through if instead of $K = N^a$ we choose the mesoscopic scale $K = C\log N$ for C large enough.

We introduce now the mesoscopic partitions of $\mathbb{D}_N = \{1,\dots,N\}^d$. At each fixed mesoscopic scale $K = 2^k$ we split the microscopic domain $\mathbb{D}_N$ into the disjoint union of shifts of the mesoscopic box $\mathbb{B}_K \stackrel{\Delta}{=} \{-\frac{1}{2}K+1,\dots,\frac{1}{2}K\}^d$. These shifted boxes are centered at the lattice points from the rescaled set $\mathbb{D}_{N,K} \stackrel{\Delta}{=} K\left(\mathbb{D}_{N/K} - (1/2,\dots,1/2)\right)$:

$$\mathbb{D}_N \ = \bigvee_{i\in\mathbb{D}_{N,K}} \mathbb{B}_K(i), \tag{2.3}$$

where $\mathbb{B}_K(i) \stackrel{\Delta}{=} i + \mathbb{B}_K = (i+\mathbb{B}_K)\cap\mathbb{Z}^d$.

As explained in the introduction a key tool to understand the interface behavior is a renormalization procedure. In this paper we will use a coarse graining implemented by Pizstora [Pi] by means of the FK representation. We recall below the main features of this coarse graining.

First of all we shall set up the notion of good box on the K-scale which characterizes a local equilibrium in a pure phase

Definition 2.1. *Let us say that a K-box $\mathbb{B}_K(i) \subset \mathbb{D}_N$, centered in i, is good with respect to the percolation configuration $\xi \in \Xi$ if*
(1) There exists a crossing cluster $C^* = C^*(\mathbb{B}_{2K}(i))$ connected to all the faces of the inner vertex boundary of the 2K-box $\mathbb{B}_{2K}(i)$.
(2) Any FK-connected cluster of vertices of $\mathbb{B}_{2K}(i)$ which has a diameter larger than $K/10$ is necessarily connected to C^*.

Fundamental techniques developed by Pizstora in [Pi] imply that there exists $\mathcal{B}$ a subset of $]\beta_c, \infty[$ for which the following holds: for any $\beta \in \mathcal{B}$, there is a constant $c > 0$, such that for all scales $K \geqslant K_0$ large enough (and, in particular, for our basic scale $K = N^a$),

$$\inf_{\pi} \Phi^{\pi,\mathbf{p}}_{\mathbb{B}_{2K}(i)} \left(\xi \text{ is a good configuration in } \mathbb{B}_K(i) \right) \geqslant 1 - \exp(-cK)\,, \tag{2.4}$$

uniformly in the boundary conditions π, boundary bond activities $\mathbf{p}$ and in $i \in \mathbb{D}_{N,K}$.

The important point is that the set $\mathcal{B}$ is defined in a non perturbative way (see [Pi]). In particular, it is conjectured to coincide with $]\beta_c, \infty[$.

2.3. **Equilibrium setting.** In equilibrium, our result on the localization concerns primarily interfaces imposed by mixed boundary conditions. We also investigate the consequences of the localization on the structure of the mesoscopic droplet when two phases coexist. We define below the two corresponding frameworks.

2.3.1. *Pure boundary conditions.* The Gibbs measure on the set $\mathbb{D}_N = \{1, \dots, N\}^d$ with $+$ boundary conditions will be denoted by μ_N^+ and the corresponding FK measure by Φ_N^{w}.

An important quantity to study phase coexistence is the surface tension which we now introduce. Let $\vec{n} \in \mathbb{S}^{d-1}$ be a unit normal and assume for the definiteness that $(\vec{n}, \vec{e}_d) > 1/\sqrt{d}$. Let Z_N^+ and $Z_N^\pm(\vec{n})$ be the partition functions on $\{-N, \dots, N\}^d$ with respectively "+" and mixed boundary conditions, the latter being defined by $\sigma_i = \mathrm{sign}((\vec{n}, i))$, with $\mathrm{sign}(0) = 1$. The bulk surface tension in the direction orthogonal to $\vec{n}$ is

$$\tau(\vec{n}) \;\triangleq\; \lim_{N \to \infty} -\frac{(\vec{n}, \vec{e}_d)}{(2N)^{d-1}} \log \frac{Z_N^\pm(\vec{n})}{Z_N^+} \tag{2.5}$$

The equilibrium crystal shape of volume $a > 0$ is the Wulff shape $\mathcal{K}_a$ defined by

$$\mathcal{K}_a = \left(\frac{a}{|\mathcal{K}|} \right)^{1/d} \mathcal{K}, \quad \text{where} \quad \mathcal{K} = \bigcap_{\vec{n}} \left\{ x \in \mathbb{R}^d, \quad (x \cdot \vec{n}) \leqslant \tau(\vec{n}) \right\}. \tag{2.6}$$

For our purpose, it will be more convenient to recall the phase coexistence Theorem directly in the FK coarse grained setting. For any configuration ξ in $\Xi_{\mathbb{D}_N}$, we partition the set $\mathbb{D}_{N,K}$ into 3 sets (not necessarily connected)

$$\mathbb{D}_{N,K} = C_N^+(\xi) \bigvee C_N^0(\xi) \bigvee C_N^-(\xi)\,, \tag{2.7}$$

where $C_N^+(\xi)$ denotes the set of good boxes $\mathbb{B}_K$ for which the unique crossing cluster (see Definition 2.1) is connected to the boundary of $\mathbb{D}_N$, $C_N^-(\xi)$ denotes the set of good boxes $\mathbb{B}_K$ for which the crossing cluster is *not* connected to the boundary of $\mathbb{D}_N$ and $C_N^0(\xi)$ the boxes which are not good.

The phase coexistence will be imposed by a volume constraint at the mesoscopic level defined by

$$\mathbb{V}_{N,a} = \left\{ \xi; \quad \left| C_N^-(\xi) \right| \geqslant a \frac{N^d}{K^d} \right\}.$$

This is the set of configurations for which there is a density at least $a > 0$ of good blocks detached from the boundary (i.e. of the $-$ phase in the spin language).

The $\mathbb{L}^1$-approach (see [CePi, BIV1]) implies that for any $\beta \in \mathcal{B}$ there is a sequence (δ_N) vanishing to 0 such that

$$\lim_{N\to\infty} \Phi_N^{\mathrm{w}} \left(\bigcup_{i \in \mathbb{D}_{N,K}} \left\{ \xi; \quad \left| \left(\mathbb{Z}^d \cap (i + \mathcal{K}_{a(N/K)^d})\right) \Delta C_N^- \right| < \delta_N \frac{N^d}{K^d} \right\} \Big| \ \mathbb{V}_{N,a} \right) = 1 \,,$$

where Δ denotes the symmetric difference of the sets in $\mathbb{D}_{N,K}$ and $|\cdot|$ the cardinal of a set.

2.3.2. *Mixed boundary conditions.* Let $\mathbb{L}^d$ be the lattice half-space $\{i \in \mathbb{Z}^d \ : \ i_d > 0\}$. The exteriour boundary of $\mathbb{D}_N = \{1, \dots N\}^d$ in $\mathbb{L}^d$ will be denoted by $\partial^{\mathrm{ext}}\mathbb{D}_N$. The bottom face of $\mathbb{D}_N$ is denoted by $\partial_{\mathrm{b}}^{\mathrm{int}}\mathbb{D}_N = \{1, \dots, N\}^{d-1} \times \{1\}$. We consider mixed boundary conditions equal to 1 in $\partial^{\mathrm{ext}}\mathbb{D}_N$ and to -1 outside $\mathbb{L}^d$. The corresponding Ising measure will be denoted $\mu_N^{\pm}$ accordingly.

The FK representation of the mixed boundary conditions requires some care. Consider the graph $\left(\mathbb{L}^d, \mathcal{L}^d\right)$, where the edge set $\mathcal{L}^d$ consists of all (unoriented) pairs of nearest neighbour vertices $(i,j) \subset \mathbb{L}^d$. Let $\mathcal{L}_N^d$ denote the set of bonds of $\mathcal{L}^d$ which have a non-empty intersection with $\mathbb{D}_N$.

It happens to be convenient to augment the graph $(\mathbb{L}^d, \mathcal{L}^d)$ with a ghost site $\mathfrak{g}$ connected to all the sites in the bottom layer $\partial_{\mathrm{b}}^{\mathrm{int}}\mathbb{D}_N$. In this way the edge set for the model is given by

$$\mathcal{L}_{N,\pm}^d \ \stackrel{\Delta}{=} \ \mathcal{L}_N^d \bigcup \left\{ (i, \mathfrak{g}) \ \middle| \ i \in \partial_{\mathrm{b}}^{\mathrm{int}}\mathbb{D}_N \right\} .$$

The sample space for finite volume FK states on $\mathbb{D}_N$ is given by

$$\Xi_{N,\pm} \stackrel{\Delta}{=} \{0,1\}^{\mathcal{L}_{N,\pm}^d} .$$

Define the FK percolation event $\mathfrak{J}_N \subset \Xi_{N,\pm}$ as

$$\mathfrak{J}_N \ \stackrel{\Delta}{=} \ \left\{ \xi \in \Xi_{N,\pm} \ \middle| \ \mathfrak{g} \not\leftrightarrow \partial^{\mathrm{ext}}\mathbb{D}_N \right\} , \tag{2.8}$$

and set

$$\Phi_N^{\pm}(\,\cdot\,) = \Phi_N^{\mathrm{w}}\left(\,\cdot\,\middle|\,\mathfrak{J}_N\right) .$$

2.4. **Dynamical setting: boundary fields.** In the second part of the paper, we are going to study the slow relaxation of the Glauber dynamics which occurs when magnetic fields are applied on the faces of $\mathbb{D}_N$. Let $\mathbf{h} = (h_1, \dots, h_{2d})$ be a vector with non negative coordinates. The Gibbs measure with each boundary magnetic field h_i applied on the i^{th}-face of the cube $\mathbb{D}_N$ is denoted by $\mu_N^{\mathbf{h}}$. In this way free boundary conditions correspond to $\mathbf{h} = (0, \dots, 0)$, whereas pure + boundary conditions correspond to $\mathbf{h} = (1, \dots, 1)$. As we shall explain below our results on the relaxation speed are non-trivial only when the boundary magnetic fields $h_1, \dots, h_{2d}$ are in the partial wetting regime, which is the case for example for free boundary conditions, but not for the pure + ones.

In a metastable regime, the rescaled evolution of the system can be described by an energy landscape which is related to equilibrium thermodynamic quantities. Therefore we first proceed in recalling the basic framework of equilibrium phase coexistence (we refer to [BIV1] for a detailed review). A heuristic discussion of the interplay between the equilibrium properties and the dynamics is postponed to subsection 3.2. The basic thermodynamic quantities in this context are the bulk surface tension (2.5) and the boundary free energy.

The influence of a magnetic field $h \in \mathbb{R}$ applied along the boundary leads to a specific surface energy. Consider the partition functions $Z_N^{-,h}$ and $Z_N^{+,h}$ with "$-$" and "+" boundary conditions on $\mathbb{L}^d \setminus \mathbb{D}_N$ and h outside $\mathbb{L}^d$. The boundary free energy Δ_h is defined as the difference between the interfacial free energies of the coexisting phases:

$$\Delta_h \triangleq \lim_{N\to\infty} \frac{1}{N^{d-1}} \log \frac{Z_N^{+,h}}{Z_N^{-,h}}. \tag{2.9}$$

We refer the reader to [FP1] and [FP2] for a detailed study of the boundary surface tension as well as related phenomena.

On the macroscopic level the equilibrium phase coexistence is governed by a variational principle involving the bulk surface tension and the wall free energies Δ_{h_i}. As it has been realized in [ABCP] the appropriate macroscopic setting is that of the functions of bounded variations and we shall repeatedly refer to [EG] for the necessary background.

The microscopic system is embedded in the continuous domain $\widehat{\mathbb{D}} = [0,1]^d$. A macroscopic distribution of phases in $\widehat{\mathbb{D}}$ is represented by a signed indicator function u taking values $\{\pm 1\}$ according to the local equilibrium. Let $\mathcal{O}$ be an open smooth neighborhood of $\widehat{\mathbb{D}}$. For *non negative* boundary fields, it is enough to consider functions u taking values in the set of bounded variation functions $\mathrm{BV}(\mathcal{O}, \{\pm 1\})$ and equal to 1 outside $\widehat{\mathbb{D}}$.

Let $\mathcal{P}_i$ be the i^{th}-face of the cube $\widehat{\mathbb{D}}$. The boundary of $\widehat{\mathbb{D}}$ is denoted by $\mathcal{P} = \cup_i \mathcal{P}_i$. The interfacial energy associated to u is defined by

$$\mathcal{W}_{\mathbf{h}}(u) = \int_{\partial^* u \setminus \mathcal{P}} \tau(\vec{n}_x) d\mathcal{H}_x^{(d-1)} + \sum_{i=1}^{2d} \Delta_{h_i} \int_{\partial^* u \cap \mathcal{P}_i} d\mathcal{H}_x^{(d-1)}, \tag{2.10}$$

where $\partial^* u$ is the reduced boundary [EG] of $\{u = -1\}$. In the particular case of boundary magnetic field acting only on one of the faces of $\mathbb{D}_N$ the probability of observing spin configurations which are close (in the $\mathbb{L}^1$ sense) to some macroscopic configuration u was proven to decay exponentially fast with the order $N^{d-1}\mathcal{W}_{\mathbf{h}}(u)$ (see [BIV2]).

Finally, the optimal interfacial energy under a volume constraint is defined as

$$\mathcal{F}^{\mathbf{h}}(m) = \inf_{u \in \mathrm{BV}} \left\{ \mathcal{W}_{\mathbf{h}}(u) \quad \Big| \quad \int_{\widehat{\mathbb{D}}} u(x)\, dx = \frac{m}{m^*} \right\}. \tag{2.11}$$

2.5. **The assumptions.** There are two main assumptions. The first one is of geometric nature and will play a crucial role in the localization of the interface. The second assumption is a mixing property for the FK measure and will be only used in the estimation of the spectral gap.

2.5.1. *Strict convexity of the surface tension.* Recall that a d-dimensional simplex is the convex envelop $S = S(u^1, \dots, u^{d+1})$ of $(d+1)$ points $u^1, \dots u^{d+1} \in \mathbb{R}^d$ in general position. The latter means that S has a non-empty interiour. Given such a d-dimensional simplex S let $F_1, \dots, F_{d+1}$ be its faces and $\vec{n}_1, \dots, \vec{n}_{d+1}$ the corresponding outer normals. By the Gauss-Green theorem [EG],

$$\sum_{k=1}^{d+1} |F_k| \vec{n}_k = 0. \tag{2.12}$$

Given an axis direction $\vec{e}_d$ let us say that a simplex $S = S(u^1, \dots, u^{d+1})$ is $\vec{e}_d$-oriented if:

(i) $u^1, \dots, u^d \in \{x \in \mathbb{R}^d : x_d = 0\}$,
(ii) $u^{d+1} \in \{x \in \mathbb{R}^d : x_d > 0\}$.

We shall always number the faces of $\vec{e}_d$-oriented simplices S in such a way that $F_1 \subset \{x \in \mathbb{R}^d : x_d = 0\}$ or, equivalently, $\vec{n}_1 = -\vec{e}_d$. Thus, for a given $\vec{e}_d$-oriented simplex S, (2.12) yields a representation of $\vec{e}_d$ as a non-trivial linear combination

$$|F_1| \vec{e}_d = \sum_{k=2}^{d+1} |F_k| \vec{n}_k.$$

We say that the surface tension τ_β is strictly convex at $\vec{e}_d$ if the following strict inequality

$$|F_1| \tau(\vec{e}_d) < \sum_{k=2}^{d+1} |F_k| \tau(\vec{n}_k) \tag{2.13}$$

holds for any $\vec{e}_d$-oriented simplex $S = S(u^1, \dots, u^{d+1})$. In [DS2] (2.13) is called Strong Simplex Inequality. It is shown to be equivalent to the following fact (Lemma 3.5 in [DS2]): Assume that $\vec{v}_1, \dots, \vec{v}_d$ are d vectors in general position, and assume that $\vec{e}_d$ lies in the interiour of the positive cone spanned by $\vec{v}_1, \dots, \vec{v}_d$, that is there exist positive numbers $\lambda_1, \dots, \lambda_d$, such that $\vec{e}_d = \sum_{k=1}^d \lambda_k \vec{v}_k$, then,

$$\tau(\vec{e}_d) < \sum_{k=1}^{d} \lambda_k \tau(\vec{v}_k). \tag{2.14}$$

Assumption (SC): τ is strictly convex at $\vec{e}_d$.

This assumption is of course true at low enough temperatures when the Wulff shape exhibits a flat facet in the $\vec{e}_d$ direction. Presumably it is true for every $\beta > \beta_c$ since, at least on the heuristic level, kinks on the boundary of the Wulff shape would correspond to pathological large interface fluctuations. We could have avoided this assumption by simply considering a direction which is orthogonal to smooth portions of the Wulff shape. This would not mend the situation and we prefer this assumption for the notational simplicity and in order to stress the existing flaws in the theory.

This assumption **(SC)** is related to the stability properties of the associated variational problem. In Section 5, we are going to consider more general boundary conditions which would lead to a different variational problem. We proceed now in discussing this new framework and show how assumption **(SC)** enables to control the stability of the new variational problem. This will be useful only for Section 5, thus the reader is invited to skip this discussion on a preliminary run-through.

We consider the macroscopic domain $\widehat{\mathbb{D}} = [0,1]^d$. The faces of $\widehat{\mathbb{D}}$ are denoted as follows; the top face $\partial_t^{\text{ext}}\widehat{\mathbb{D}} = \{x_d = 1\}$, the bottom face $\partial_b^{\text{int}}\widehat{\mathbb{D}} = \{x_d = 0\}$ and the remainder which are the side faces $\partial_s^{\text{ext}}\widehat{\mathbb{D}}$. A boundary magnetic field equal to 1 is applied on $\partial_t^{\text{ext}}\widehat{\mathbb{D}}$, a boundary magnetic field equal to -1 on $\partial_b^{\text{int}}\widehat{\mathbb{D}}$ and an $\varepsilon > 0$ boundary field on the sides. This last field leads to a boundary surface tension denoted by Δ_ε. We refer to the subsection 5.1, for the explicit microscopic definition. We are going to check that this modification of the boundary fields has no impact on the stability of the variational problem.

Define the modified Wulff shape

$$\mathcal{K}^\varepsilon \stackrel{\Delta}{=} \mathcal{K} \cap \{x \,:\, |x_i| \leqslant \Delta_\varepsilon \quad \forall\, i = 1, \dots, d-1\}, \tag{2.15}$$

and let τ^ε be the support function of $\mathcal{K}^\varepsilon$.

Let $\mathcal{O}$ be an open smooth neighborhood of $\widehat{\mathbb{D}} = [0,1]^d$. We consider the boundary condition $g \in \mathrm{BV}\left(\mathcal{O} \setminus \widehat{\mathbb{D}}, \{\pm 1\}\right)$ specified by

$$g(x) = \begin{cases} 1, & \text{if } x_d > 0\,, \\ -1, & \text{if } x_d \leqslant 0\,. \end{cases}$$

Given a ± 1-valued function u on $\widehat{\mathbb{D}}$ define

$$u \vee g(x) \;=\; \begin{cases} u(x) & \text{if } x \in \mathrm{int}\widehat{\mathbb{D}}\,, \\ g(x) & \text{if } x \in \mathcal{O} \setminus \widehat{\mathbb{D}}\,. \end{cases} \tag{2.16}$$

It is well known [EG] that $u \vee g \in \mathrm{BV}(\mathcal{O}, \{\pm 1\})$ whenever the phase function $u \in \mathrm{BV}(\mathrm{int}\widehat{\mathbb{D}}, \{\pm 1\})$. For any v in $\mathrm{BV}(\mathcal{O}, \{\pm 1\})$, there exists a generalized notion of the boundary of $\{v = -1\}$ called reduced boundary [EG] and denoted by $\partial^* v$. If $\{v = -1\}$ is a regular set, $\partial^* v$ coincides with the usual boundary ∂v. Given a phase function $u \in \mathrm{BV}(\mathrm{int}\widehat{\mathbb{D}}, \{\pm 1\})$ we use $\partial_g^* u$ to denote the reduced boundary of u in the presence of the b.c. g:

$$\partial_g^* u \;=\; \partial^*(u \vee g) \cap \widehat{\mathbb{D}} \;=\; \partial^*(u \vee g) \setminus \partial^* g. \tag{2.17}$$

Finally define the functional $\widehat{\mathcal{W}}_\varepsilon(\cdot|g)$ on $\mathrm{BV}(\mathrm{int}\widehat{\mathbb{D}}, \{\pm 1\})$:

$$\widehat{\mathcal{W}}_\varepsilon\left(u \,\middle|\, g\right) \;=\; \int_{\partial_g^* u} \tau^\varepsilon(\vec{n}_x) d\mathcal{H}_x^{(d-1)},$$

where τ^ε is the support function of $\mathcal{K}^\varepsilon$ (see (2.15)).

Proposition 2.1. *Assume that* **(SC)** *holds, that is τ is assumed to be strictly convex at $\vec{e}_d$. Then, $u = \mathbb{1}(\cdot)$ is the unique minimum of $\widehat{\mathcal{W}}_\varepsilon(\cdot|g)$ on* $\mathrm{BV}(\mathrm{int}\widehat{\mathbb{D}}, \{\pm 1\})$.

The proof of Proposition 2.1 is relegated to the Appendix.

Corollary 2.1. *In the notation above define the functional $\mathcal{W}_\varepsilon(\cdot|g)$ on* $\mathrm{BV}(\mathrm{int}\widehat{\mathbb{D}}, \{\pm 1\})$ *via*

$$\mathcal{W}_\varepsilon\left(u \,\middle|\, g\right) \;=\; \int_{\partial_g^* u \setminus \partial_s \widehat{\mathbb{D}}} \tau(\vec{n}_x) d\mathcal{H}_x^{(d-1)} \;+\; \Delta_\varepsilon \int_{\partial_g^* u \cap \partial_s \widehat{\mathbb{D}}} d\mathcal{H}_x^{(d-1)}.$$

Then $u = \mathbb{1}(\cdot)$ is the stable minimum of $\mathcal{W}_\varepsilon(\cdot|g)$ in the following sense: For every $\nu > 0$ there exists $c_2 = c_2(\nu) > 0$ such that

$$\mathcal{W}_\varepsilon\left(u \,\middle|\, g\right) \leqslant \mathcal{W}_\varepsilon\left(\mathbb{1} \,\middle|\, g\right) + c_2(\nu) \implies \|u - \mathbb{1}\|_1 \leqslant \nu. \tag{2.18}$$

Proof. Proposition 2.1 and standard compactness considerations imply that the functional $\widehat{\mathcal{W}}_\varepsilon\left(\cdot \middle| g\right)$ is stable in the above sense. On the other hand

$$\widehat{\mathcal{W}}_\varepsilon\left(\cdot \middle| g\right) \leqslant \mathcal{W}_\varepsilon\left(\cdot \middle| g\right),$$

and, of course, both functionals attain the same value on $u = \mathbb{1}$. In particular, any function u which satisfies the left hand side of (2.18) automatically satisfies the very same inequality with $\widehat{\mathcal{W}}_\varepsilon\left(\cdot \middle| g\right)$ instead of $\mathcal{W}_\varepsilon\left(\cdot \middle| g\right)$. □

2.5.2. *Mixing property.* The localization of the interface will be derived on a coarse grained level. Throughout the paper we will assume that β belongs to $\mathcal{B}$ so that Pisztora's coarse graining holds.

The analysis of the dynamics will require an assumption on the exponential mixing of a pure phase. It is well known (see [Gri]) that for all β (except possibly for a countable number) there is no phase transition in the FK representation, i.e. that the limiting FK measures $\Phi^{\mathrm{f}}_{\mathbb{Z}^d}$ and $\Phi^{\mathrm{w}}_{\mathbb{Z}^d}$ coincide. We will need an enhanced property of uniqueness and will suppose that the boundary effect vanishes exponentially fast.

We introduce $\Lambda_{N,M} = \{-N, \ldots, N\}^{d-1} \times \{-M, \ldots, M\}$ and consider two types of FK measures on this set with different boundary conditions. We denote by $\Phi^{\mathrm{w,f,f}}_{\Lambda_{N,M}}$ (resp $\Phi^{\mathrm{w,f,w}}_{\Lambda_{N,M}}$) the measure with wired boundary conditions on the face $\{x_d = M\}$ (resp on the faces $\{x_d = \pm M\}$) and free elsewhere.

Definition 2.2. *Let $\mathcal{B}_1$ be the subset of $\mathcal{B}$ containing the inverse temperatures β for which there exists $c_1 = c_1(\beta), c_2 = c_2(\beta) > 0$ such that*

$$\forall (N, M), \forall b \in \Xi_{\Lambda_{N,M/2}}, \qquad \left| \Phi^{\mathrm{w,f,f}}_{\Lambda_{N,M}}(\xi_b) - \Phi^{\mathrm{w,f,w}}_{\Lambda_{N,M}}(\xi_b) \right| \leqslant c_2 \exp\big(- c_1 M \big).$$

Assumption (MP): We will suppose that $\beta \in \mathcal{B}_1$, i.e that the mixing property holds.

The previous assumption holds for β large enough. This can be easily derived along the lines of the proof of Theorem 5.3 (c) of [Gri].

We conjecture that the mixing property should be valid on $]\beta_c, \infty[$. In fact **(MP)** can be related to the notion of strong mixing which was introduced in the context of Ising model by Dobrushin and Shlosman [DS3] (see also Martinelli and Olivieri [MO] for the regular strong mixing property). The counterpart of this notion for the FK model can be stated as follows: there exists $c_1(\beta), c_2(\beta) > 0$ such that for any cube Δ of $\mathbb{Z}^d$, any pair of boundary conditions π, π'

$$\forall b \in \Xi_\Delta, \qquad \left|\Phi_\Delta^{\pi}(\xi_b) - \Phi_\Delta^{\pi'}(\xi_b)\right| \leqslant c_2 \exp\big(- c_1 \operatorname{dist}(b, \pi \wedge \pi')\big) ,$$

where $\pi \wedge \pi'$ refers to the region where π and π' differ. This property implies **(MP)** and we conjecture that it holds for the parameters β for which the FK measure is unique in the thermodynamic limit.

Finally, we stress the fact that **(MP)** does not apply directly to the Ising model, nevertheless combined with the localization of the interface, it will have useful implications on the mixing of the spin system. This will be discussed in Section 5.

3. The results

Throughout the paper, the dimension d is fixed larger or equal to 3 and β belongs to $\mathcal{B}$, the domain of validity of Pisztora's coarse graining [Pi].

Theorem 3.3 and Theorem 3.5 hold for every $\beta \in \mathcal{B}$. Results on Hausdorff stability with respect to axis directions (Theorem 3.1 and Theorem 3.2) require an additional assumption **(SC)**, namely we need to assume that the surface tension τ_β is strictly convex at $\vec{e}_d$. These stability results play an important role in our proof of the lower bound on the spectral gap (Theorem 3.4) which also relies on the mixing property **(MP)**.

3.1. **The Hausdorff stability.** As we have already mentioned the conjectured percolation of minority spins at moderately low temperatures [ABL] suggests that microscopic interfaces are not the appropriate objects to describe stability properties of phase boundaries. In any case, however, the phases are characterized by the order parameter (spontaneous magnetization) $\pm m^*(\beta)$ in the sense that local spin averages, or local magnetization profiles, inside what is expected to be " + " or " − " phases should converge, as the averaging scale grows, to $m^*(\beta)$ or $-m^*(\beta)$ respectively. Our main stability result below is formulated in terms of phase boundaries induced by local magnetization profiles on large *finite* scales.

Consider the decomposition (2.3). Given a small number $\rho > 0$ let us define phase labels $\tilde{u}^{\rho}_{N,K} \in \{0, \pm 1\}^{\mathbb{D}_{N,K}}$ as follows:

$$\tilde{u}^{\rho}_{N,K}(i) \;=\; \begin{cases} 1\,, & \text{if } \left| \dfrac{1}{K^d} \displaystyle\sum_{j \in \mathbb{B}_K(i)} \sigma_j - m^*(\beta) \right| \;\leqslant\; \rho\,, \\ -1\,, & \text{if } \left| \dfrac{1}{K^d} \displaystyle\sum_{j \in \mathbb{B}_K(i)} \sigma_j + m^*(\beta) \right| \;\leqslant\; \rho\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.19}$$

Thus, $\tilde{u}^{\rho}_{N,K}$ uses the resolution ρ to label the proximity of the local magnetization profile to the order parameter $\pm m^*(\beta)$ on the renormalization scale $K = 2^k$. For the spin model on $\mathbb{D}_N$ with mixed boundary conditions described in Subsection 2.3.2 we shall (by abuse of notation) extend $\tilde{u}^{\rho}_{N,K}$ to the whole of $\mathbb{Z}^d_K \triangleq K\left(\mathbb{Z}^d - (1/2,\dots,1/2)\right)$ as follows:

$$\tilde{u}^{\rho}_{N,K}(i) = \begin{cases} 1 & \text{if } i \in \mathbb{Z}^d_K \setminus \mathbb{D}_{N,K} \text{ and } i_d > 0\,, \\ -1 & \text{if } i \in \mathbb{Z}^d_K \setminus \mathbb{D}_{N,K} \text{ and } i_d < 0\,. \end{cases}$$

It happens to be more convenient to work with the adjusted phase labels $u^{\rho}_{N,K}$: $u^{\rho}_{N,K}(i) = 1$ (respectively -1) if $\tilde{u}^{\rho}_{N,K}(i) = 1$ (respectively to -1) both at i and at all $*$-neighbours of i in $\mathbb{Z}^d_K$. Otherwise, $u^{\rho}_{N,K}(i)$ is set to be equal to zero. The advantage of such adjustment is that any nearest neighbour path of vertices of $\mathbb{D}_{N,K}$ which connects regions with different phase labels is forced to contain a site with zero phase label. Accordingly let us define the collection of phase boundaries induced by $u^{\rho}_{N,K}(i)$ as

$$\underline{\partial}^{\rho}_{N,K} \triangleq \{i \in \mathbb{D}_{N,K} \,:\, u^{\rho}_{N,K}(i) = 0\}.$$

The set $\underline{\partial}^{\rho}_{N,K}$ is in general disconnected and for fixed finite values of the renormalization scale K contains (for entropic reasons) many small components even in the case of pure boundary conditions. In the case of mixed boundary conditions, however $\underline{\partial}^{\rho}_{N,K}$ contains a unique unbounded connected component which we shall denote as $\partial^{\rho}_{N,K}$. By the construction $\partial^{\rho}_{N,K}$ contains an infinite flat double layer outside $\mathbb{D}_{N,K}$ and, in fact, all the non-trivial geometry of $\partial^{\rho}_{N,K}$ is confined to $\mathbb{D}_{N,K}$. Here is our Hausdorff stability result in terms of phase labels $u^{\rho}_{N,K}$:

Theorem 3.1. *Assume that $\beta \in \mathcal{B}$ and that the Assumption* **(SC)** *holds, that is the surface tension τ_β is strictly convex at $\vec{e}_d$. Then for any $\nu > 0$ and $\rho > 0$ there exists a finite scale $K_0 = K_0(\nu,\rho)$ and a positive constant $c = c(\nu,\rho)$ such that for every $K \geqslant K_0$ and for all N sufficiently large,*

$$\mu^{\pm}_N\left(\partial^{\rho}_{N,K} \cap \{i \,:\, i_d > \nu N\} \neq \emptyset\right) \leqslant \mathrm{e}^{-c(\nu,\rho)N}. \tag{3.20}$$

The above statement asserts that on large enough, though still finite, renormalization scales K the interface is macroscopically stable in the sense that the order of its fluctuations is smaller than the linear size of the system N. Since the fluctuations in question are expected to be of the $\log N$-size for moderately low temperatures and, at least for axis oriented interfaces, are known to be bounded for sufficiently low ones [DS1], the result is far from being optimal and, in a way, it illustrates limitations of the $\mathbb{L}_1$-approach.

The proof of Theorem 3.1 is based on the following result on the stability of the FK interfaces. In the sequel we employ the notation introduced in Subsections 2.2 and 2.3.

Theorem 3.2. *Assume that $\beta \in \mathcal{B}$ is such that* **(SC)** *holds. Let $K = N^a$, where a is chosen according to* (2.2). *Then for any $\nu > 0$,*

$$\lim_{N\to\infty} \Phi^{\pm}_N\left(\textit{There is a bad block in } \{i \,:\, i_d > \nu N\}\right) = 0\,. \tag{3.21}$$

More precisely, there exists $c_1 = c_1(\nu) > 0$, such that uniformly in N large enough,

$$\Phi_N^{\pm}\left(\mathfrak{g} \leftrightarrow \{i\,:\, i_d \geqslant \nu N\}\right) \;\leqslant\; \mathrm{e}^{-c_1(\nu)N}\,. \tag{3.22}$$

Notice that the second part of the statement implies that in the FK representation the interface is localized even on a microscopic level without any additional renormalization procedures.

Let us now prove Theorem 3.1 as a consequence of inequality (3.22).

Proof of Theorem 3.1. Let us decompose the percolation event

$$\mathcal{T}_{N,\nu} \stackrel{\Delta}{=} \{\xi \in \mathfrak{J}_N\,:\, \mathfrak{g} \not\leftrightarrow \{i\,:\, i_d \geqslant \nu N\}\}$$

according to the realization of the maximal connected component $\mathcal{C}(\mathfrak{g})$ of the ghost site $\mathfrak{g}$. By the very definition,

$$\mathcal{C}(\mathfrak{g}) \subseteq \mathbb{D}_N^{\nu} \stackrel{\Delta}{=} \mathbb{D}_N \cap \{i\,:\, i_d < \nu N\},$$

on $\mathcal{T}_{N,\nu}$. Consequently, for every percolation event A which depends only on the bonds connected to the upper sub-box $\mathbb{D}_N \setminus \mathbb{D}_N^{\nu}$, the following decoupling bound holds:

$$\Phi_N^{\pm}\left(A \,\middle|\, \mathcal{T}_{N,\nu}\right) \;\leqslant\; \max_{\pi} \Phi^{\pi}_{\mathbb{D}_N\setminus\mathbb{D}_N^{\nu}}(A). \tag{3.23}$$

We stress the fact that in this Theorem the coarse graining scale K is independent of N, unlike in (3.21). As in [BIV1, BIV2] define the joint spin-bond label $v^{\rho}_{N,K} = v^{\rho}_{N,K}[\sigma,\xi] \in \{0,1\}^{\mathbb{D}_{N,K}}$ via:

$$v^{\rho}_{N,K}(i) \;=\; \begin{cases} 1, & \text{if } |u^{\rho}_{N,K}| = 1 \text{ and } \mathbb{B}_{2K}(i) \text{ is } \xi \text{ good}\,,\\ 0, & \text{otherwise}\,. \end{cases} \tag{3.24}$$

Clearly,

$$\underline{\partial}^{\rho}_{N,K} \cap \mathbb{D}_{N,K} \;\subseteq\; \{i \in \mathbb{D}_{N,K}\,:\, v^{\rho}_{N,K}(i) = 0\}.$$

On the other hand, it follows from (3.23) that the distribution of the field $v^{\rho}_{N,K}$ on $\{0,1\}^{\mathbb{D}_{N,K}\setminus\mathbb{D}^{\nu}_{N,K}}$ under $\mathbb{P}^{\pm}_{\beta,\mathbb{D}_N}(\,\cdot\,|\mathcal{T}_{N,\nu})$ stochastically dominates the Bernoulli site percolation process $\mathbb{P}^{\mathrm{Bern}}_p$, where the probability $p = p(\beta,\rho,K)$ of a particular site to be occupied satisfies $\lim_{K\to\infty} p(\beta,\rho,K) = 1$, see Section 3.2 of [BIV2] for more details and references. As a result, (3.20) follows from the exponential decay of connectivities for the sub-critical site percolation once $(1-p)$ is sufficiently small (or, equivalently, once K is sufficiently large). □

We turn now to the case of the Wulff shape for which the phase coexistence is imposed in a more indirect way via a volume constraint. A straightforward modification of the techniques which we shall employ for the proof of Theorem 3.2 yields (see (2.7)):

Theorem 3.3. *Assume that $\beta \in \mathcal{B}$. Let $K = N^a$, where a is chosen according to (2.2). Then for any $\nu > 0$,*

$$\lim_{N\to\infty} \Phi_N^{\mathrm{w}}\left(\bigcup_{i\in\mathbb{D}_N}\left\{(i + \mathcal{K}_{a(1-\nu)(N/K)^d}) \subset \mathcal{C}_N^{-}\right\} \,\middle|\, \mathbb{V}_{N,a}\right) = 1\,.$$

Notice that we made no additional assumptions on the strict convexity of τ_β. Indeed, directions v at which τ_β is not strictly convex correspond to non-smooth portions of the boundary $\partial\mathcal{K}$ which have zero surface measure.

Theorem 3.3 implies that there is no percolation of the $+$ phase inside the interior of the $-$ droplet. In this way usual conclusion of the $\mathbb{L}_1$-theory is clearly upgraded. On the other hand we are not able to establish a complete statement on the Hausdorff localization, i.e. that there always exists i in $\mathbb{D}_N$ such that

$$(i + \mathcal{K}_{a(1-\nu)(N/K)^d}) \subset \mathcal{C}_N^- \subset (i + \mathcal{K}_{a(1+\nu)(N/K)^d}) \,.$$

This would imply that the interface between $\mathcal{C}_N^-$ and $\mathcal{C}_N^+$ is always localized close to the boundary of the Wulff shape. This limitation is due to our method of proof: we are able to prove that large protuberances of the interface are not statistically favorable and therefore can be chopped. However, the volume constraint $\mathbb{V}_{N,a}$ prevents us to control the percolation of the $-$ phase inside the $+$ phase because erasing a filament of $-$ blocks might be in conflict with the volume constraint.

3.2. **Spectral gap.** We study the relaxation of the Glauber dynamics for the Ising model in a finite domain with a boundary magnetic field. The metastable behavior of the dynamics will be related to the equilibrium wetting phenomenon which occurs for a certain range of the magnetic field.

The evolution of the system is given by the Glauber dynamics. The Dirichlet form associated to the dynamics is

$$\forall f \in \mathbb{L}^2(\mu_N^{\mathbf{h}}), \qquad \mathcal{E}_N^{\mathbf{h}}(f,f) = \sum_{x \in \mathbb{D}_N} \mu_N^{\mathbf{h}}\big(|f(\sigma^x) - f(\sigma)|^2\big)\,,$$

where σ^x is the spin configuration deduced from $\sigma \in \{\pm 1\}^{\mathbb{D}_N}$ by flipping the spin at site x. The reader is referred to the lecture notes by Martinelli [Ma] and Guionnet, Zegarlinski [GZ] for a precise definition and related results on the Glauber dynamics. In the phase transition regime, the two phases segregate and the relaxation of the system is related to the slow motion of the interfaces.

A convenient parameter to capture the signature of this slowing down is the spectral gap of the dynamics defined as follows

$$\mathrm{SG}(N, \mathbf{h}) = \inf_f \ \frac{\mathcal{E}_N^{\mathbf{h}}(f,f)}{\mu_N^{\mathbf{h}}\big(f - \mu_N^{\mathbf{h}}(f)\big)^2}\,. \tag{3.25}$$

We first consider the case where a positive magnetic field $\mathbf{h} = (0, \dots, 0, h_d, 0, \dots, 0)$ is applied only on the face $\{i \in \mathbb{D}_N \,:\, i_d = 1\}$ of the cube.

Theorem 3.4. *Let $\beta \in \mathcal{B}$ is such that both the strict convexity Assumption* **(SC)** *and the mixing Assumption* **(MP)** *hold. Then,*

$$\liminf_{N \to \infty} \frac{1}{N^{d-1}} \ \log \mathrm{SG}(N, \mathbf{h}) \geqslant \, - \tau(\vec{e}_d) + \Delta_{h_d}\,, \tag{3.26}$$

where the wall free energy corresponding to the field h_d is denoted by Δ_{h_d}.

For general fields $\mathbf{h} = (h_1, \dots, h_{2d})$ with non negative components, we introduce the functional

$$\mathcal{G}^{\mathbf{h}}(m) = \mathcal{F}^{\mathbf{h}}(m) - \sum_{i=1}^{2d} \Delta_{h_i}\,, \tag{3.27}$$

where $\mathcal{F}^{\mathbf{h}}$ was introduced in (2.11). We get

Theorem 3.5. *For any $\beta \in \mathcal{B}$, the following asymptotic hold*

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mathrm{SG}(N, \mathbf{h}) \leqslant - \sup_{m\in]-m^*,m^*[} \mathcal{G}^h(m)\,. \tag{3.28}$$

Remark 3.1. *Notice that the statement of Theorem 3.5 does not depend on Assumptions* **(SG)** *or* **(MP)**. *In* [Ma] *only the free boundary conditions have been considered, but, in view of the results of* [PV]*, the proof pertains to the case of boundary magnetic fields in the partially wetting regime. In dimension 2, Pisztora's coarse graining is not valid, but an alternative coarse graining for which the $\mathbb{L}^1$-approach holds has been devised in* [BoMa]. *In particular, the proof of Theorem 3.5 goes through in two dimensions. Furthermore, since in 2 dimensions the Wulff shape is always strictly convex the mixing property* **(MP)** *is known to be valid up to the critical temperature, the conclusion of Theorem 3.4 is also valid in dimension 2 for any $\beta > \beta_c$.*

The functional $\mathcal{G}^{\mathbf{h}}$ should be interpreted as an energy landscape parametrized by the averaged magnetization. The time for a configuration starting in the $-$ phase to relax to the $+$ phase provides an estimation of the spectral gap. This explains why the supremum is taken over the values of m in $]-m^*, m^*[$. The supremum of $\mathcal{G}^{\mathbf{h}}$ is related to the energy of the bottleneck and Theorem 3.5 asserts that if it is positive then the system has a metastable behavior and evolves extremely slowly. In this case the system has time to equilibrate and equilibrium parameters should be relevant as well to describe the influence of the boundary field on the dynamics. We expect that inequality (3.28) is, in fact, an equality. For appropriate choices of $\mathbf{h}$, an explicit upper bound can be obtained. In particular for nearest neighbor Ising model in two dimensions and $\mathbf{h} = (0, h_2, 0, 0)$, one can check that the RHS of (3.26) and the LHS of (3.28) coincide. In general, estimating the supremum of $\mathcal{G}^{\mathbf{h}}$ boils down to solve a difficult variational problem involving subtle boundary effects. For general values of $\mathbf{h} = (h_1, \dots, h_{2d})$, this seems to be out reach for the moment.

Moreover the situation is far from being understood even in the case of a cube with free boundary conditions : in dimension $d \geqslant 3$, the solutions of the isoperimetric problem (2.11) are not known for general volume constraints. In the particular case of isotropic surface tension and for a volume constraint which is the half of the volume of the cube, it was only recently proven by Barthe and Maurey [BaMa] that the solution is the half cube. We refer the reader to the review by Ros [Ro] for further discussions on this issue.

This explains why, contrary to the two-dimensional case, we cannot derive precise asymptotics of the logarithm of the spectral gap.

We turn now to a more physical interpretation of our results. The behavior of the dynamics is very sensitive to the boundary conditions. In dimension two, Martinelli showed that for free boundary conditions or, equivalently, in the case of zero boundary magnetic fields $\mathbf{h} \equiv 0$, $\log \mathrm{SG}(N, \mathbf{h})$ scales like $-N\tau(\vec{e}_d)$. Instead, when at least one side of the square has all + boundary conditions and the other sides free boundary conditions (i.e. $\mathbf{h} = (0, \ldots, 0, 1)$), then for any $\varepsilon > 0$ and N large enough

$$\log \mathrm{SG}(N, h) \geqslant -N^{1/2+\varepsilon} (\gg -N\tau(\vec{e}_d)) .$$

In this case the spectral gap is conjectured to decay polynomially fast. An appealing interpretation of this result would be to relate the dynamics of the Ising model to an effective model evolving in a one well potential (+ bc) or a two well potential (free bc): the positive magnetic field $\mathbf{h}$ enforces a unique ground state whereas the transitions between two symmetric wells on the energy landscape in the case of the free boundary conditions have to cross the saddle point whose height scales like the logarithm of the inverse spectral gap.

Following the seminal work of Martinelli, various other types of boundary conditions have been investigated to understand better the crossover between the two regimes. Alexander [A] showed that small (at least logarithmic) modifications of the boundary conditions at the corners of a two dimensional cube leads to drastic changes in the scaling of the spectral gap. Alexander and Yoshida [AY] investigated the influence of an alteration of the + boundary conditions by an arbitrary small density of spins. Roughly speaking, they showed that in two dimensions if the boundary conditions have an average magnetization less than 1, there exists some inverse temperature β_0 large enough above which the dynamics exhibits a metastable phase. Our result was originally motivated by [AY]; the magnetic field $h < 1$ can be interpreted as an effective boundary condition after averaging the spins. For simplicity, let us focus on the case $\mathbf{h} = (0, \ldots, 0, h_d, 0, \ldots, 0)$. Extrapolating the results of [AY] to this setting, one can state that in two dimensions and for any h_d in $[0, 1[$, there exists β large enough such that the spectral gap decays exponentially fast. Theorems 3.4 and 3.5 will enable us to interpret these results in a more qualitative way. In order to do so, we first recall some statements on the wetting transition.

It was derived in [FP2] that $\Delta_{h_d} \in [-\tau(\vec{e}_d), \tau(\vec{e}_d)]$. In fact there exists a critical value $h_c \leqslant 1$ such that if $h_d < h_c$ then $\Delta_{h_d} < \tau(\vec{e}_d)$. The critical value h_c characterizes the influence of the boundary field on the thermodynamic properties of the Gibbs measure. More precisely, one should also distinguish the partial drying regime ($0 \leqslant h_d < h_c$) from the partial wetting regime ($0 \geqslant h_d > -h_c$). We refer to Pfister, Velenik [PV] or to [BIV1] for further discussions on the equilibrium issues.

The previous Theorem shows that h_c is also related to the metastable behavior of the system and thus it also plays the role of a critical value in the dynamical setting. Nevertheless, for any $h_d > 0$, the Gibbs measure is unique in the thermodynamic limit. This confirms the fact that the dynamical properties cannot be deduced simply from the bulk properties, but that the metastability is related to surface properties (the picture of the effective magnetization evolving in a one well potential was too simplistic).

4. Hausdorff localization: Proof of Theorem 3.2

4.1. **FK phase labels.** We use the notation introduced in Subsections 2.2 and 2.3. Define the following dependent percolation process on $\mathbb{D}_{N,K}$:

$$X^{\mathrm{FK}}_{N,K}(i) \;=\; \begin{cases} \;\;1, & \text{if } \mathbb{B}_{2K}(i) \text{ is FK good and } C^*(\mathbb{B}_{2K}(i)) \text{ is connected to } \partial^{\mathrm{ext}}\mathbb{D}_N \\ -1, & \text{if } \mathbb{B}_{2K}(i) \text{ is FK good and } C^*(\mathbb{B}_{2K}(i)) \text{ is not connected to } \partial^{\mathrm{ext}}\mathbb{D}_N \\ \;\;0, & \text{if } \mathbb{B}_{2K}(i) \text{ is FK bad} \end{cases} \tag{4.29}$$

We recall the choice $K = N^a$ for some $a \in]0, 1/d[$.

Exactly as in [BIV1] the stability assumption implies:

Lemma 4.1. *For every $\alpha > 0$ there exists a positive constant $c_4 = c_4(\alpha)$, such that*

$$\Phi^{\pm}_N \left(\sum_{i \in \mathbb{D}_{N,K}} X^{\mathrm{FK}}_{N,K}(i) \leqslant (1-\alpha)\left(\frac{N}{K}\right)^d \right) \;\leqslant\; \exp\left(-c_4 \frac{N^{d-1}}{K^{d-1}}\right), \tag{4.30}$$

for all N large enough.

Recall that the total number of mesoscopic boxes in $\mathbb{D}_N$ is $\frac{N^d}{K^d}$.

4.2. **Logic of the proof.** Back to Theorem 3.2 we shall use Lemma 4.1 in the regime $0 < \alpha \ll \nu \ll 1$. We argue that on the event

$$\left\{ \xi \in \Xi_{N,\pm} \;:\; \sum_{i \in \mathbb{D}_{N,K}} \mathbb{I}_{\{X^{\mathrm{FK}}_{N,K}(i) \neq 1\}}(\xi) \leqslant \alpha \left(\frac{N}{K}\right)^d \right\} \tag{4.31}$$

νN-long fingers of the minority phase are improbable in the sense that one is always able to find a horizontal layer where such finger can be amputated at a substantial energetic cost. The argument is just a careful computation along the lines of the minimal section method introduced in [BBBP].

4.3. **Fingers and finger labels.** The excitations of the interface on the coarse grained level will be named *fingers*. We stress the fact that this terminology does not refer only to the low dimensional excitations.

Given a configuration $\xi \in \Xi_{N,\pm}$, let us define the associated finger $\mathbb{F}_{N,K} \subset \mathbb{D}_{N,K}$ as

$$\mathbb{F}_{N,K} \;=\; \left\{ i \in \mathbb{D}_{N,K} \;:\; X^{\mathrm{FK}}_{N,K}(i) = 0 \text{ or } -1 \right\}.$$

For every mesoscopic layer $l = 1, 2, \dots$ define the finger label

$$f^l_{N,K} = \#\{ i \;:\; i_d = (l - \frac{1}{2})K, \quad i \in \mathbb{F}_{N,K} \}.$$

With ν fixed set $R = [\nu N/K] + 1$. To simplify the notation we shall assume that $R = 2^r$. Given a collection

$$\vec{f}_{N,K} \;=\; \left(f^1_{N,K}, \dots, f^R_{N,K} \right)$$

of *strictly positive* finger labels we shall use $\mathcal{F}(\vec{f}_{N,K})$ to denote the set of all those percolation configurations $\xi \in \Xi_{N,\pm}$ which are compatible with $f^l_{N,K}$ in each of the mesoscopic layers $l = 1, 2, \ldots, R$. Evidently,

$$\{\xi \in \Xi_{N,\pm} \ : \ \mathfrak{g} \leftrightarrow \{i \, : \, i_d > \nu N\}\} \ \subseteq \ \bigvee_{\vec{f}_{N,K}} \mathcal{F}[\vec{f}_{N,K}], \tag{4.32}$$

where the disjoint union is, of course, over all strictly positive finger labels. A large fluctuation of the interface occurs when bad blocks percolate on a distance larger than νN.

4.4. **The target estimate on the probability of $\mathcal{F}(\vec{f}_{N,K})$.** Our proof of Theorem 3.2 relies on the following uniform upper bound:

Theorem 4.1. *There exists $c_5 = c_5(\beta) > 0$ such that*

$$\Phi^{\pm}_N \left(\mathcal{F}[\vec{f}_{N,K}] \right) \ \leqslant \ \mathrm{e}^{-c_5 \nu N} \tag{4.33}$$

uniformly in strictly positive finger labels $\vec{f}_{N,K}$ and in N sufficiently large.

Since the total number of different finger labels (recall the choice of scales $K = N^a$) is bounded above as

$$\left(\left(\frac{2N}{K} \right)^{(d-1)} \right)^R \ \leqslant \ \exp\left\{ c_5(d) \nu N^{1-a} \log N \right\}, \tag{4.34}$$

Theorem 3.2 instantly follows.

4.5. **Splitting of $\mathcal{L}_{N,\pm}$ with respect to l-th mesoscopic layer.** Given a mesoscopic layer $l = 1, 2, \ldots, R$ define the following mesoscopic sets:

$$\begin{aligned}
\mathbb{H}^{-,l}_{N,K} &\triangleq \{i \in \mathbb{D}_{N,K} \, : \, i_d \leqslant (l - \frac{1}{2})K\}, \\
\mathbb{H}^{l}_{N,K} &\triangleq \{i \in \mathbb{D}_{N,K} \, : \, i_d = (l - \frac{1}{2})K\}, \\
\mathbb{H}^{+,l}_{N,K} &\triangleq \{i \in \mathbb{D}_{N,K} \, : \, i_d > (l - \frac{1}{2})K\}.
\end{aligned}$$

Their microscopic counterparts are denoted by $\mathcal{H}^{-,l}_{N,K}, \mathcal{H}^{l}_{N,K}$ and, respectively, $\mathcal{H}^{+,l}_{N,K}$, where

$$\mathcal{H}^{-,l}_{N,K} = \bigcup_{i \in \mathbb{H}^{-,l}_{N,K}} \mathbb{B}_{2K}(i), \quad \mathcal{H}^{l}_{N,K} = \bigcup_{i \in \mathbb{H}^{l}_{N,K}} \mathbb{B}_{2K}(i) \text{ and } \mathcal{H}^{+,l}_{N,K} = \bigcup_{i \in \mathbb{H}^{+,l}_{N,K}} \mathbb{B}_{2K}(i).$$

Accordingly, we split the set of all edges $\mathcal{L}^d_{N,\pm}$ into the disjoint union

$$\mathcal{L}^d_{N,\pm} \ = \ \mathcal{E}^{-,l}_{N,K} \bigvee \mathcal{E}^{+,l}_{N,K},$$

where

$$\mathcal{E}^{-,l}_{N,K} \ \triangleq \ \left\{ (i \, , \, j) \ : \ \text{either } i \text{ or } j \text{ belong to } \ \mathcal{H}^{-,l}_{N,K} \cup \mathfrak{g} \right\},$$

and $\mathcal{E}^{+,l}_{N,K} = \mathcal{L}^d_{N,\pm} \setminus \mathcal{E}^{-,l}_{N,K}$.

The induced notation for the splitting of the percolation configurations $\xi \in \Xi_{N,\pm}$ is $\xi = \xi_-^l \vee \xi_+^l$ with

$$\xi_-^l \in \Xi_{N,K}^{-,l} \triangleq \{0,1\}^{\mathcal{E}_{N,K}^{-,l}} \quad \text{and} \quad \xi_+^l \in \Xi_{N,K}^{+,l} \triangleq \{0,1\}^{\mathcal{E}_{N,K}^{+,l}}.$$

Given a percolation configuration $\xi_-^l \in \Xi_{N,K}^{-,l}$, we use $\mathcal{C}_{N,K}^{-,l}(\mathfrak{g})$ to denote the connected cluster of the ghost site $\mathfrak{g}$ *inside* $\mathcal{H}_{N,K}^{-,l}$.

4.6. **The event $\mathcal{A}_l(\vec{f}_{N,K})$.** Let $\vec{f}_{N,K}$ be a collection of strictly positive finger labels. For each mesoscopic layer l, we define the associated event $\mathcal{A}_l(\vec{f}_{N,K}) \subset \Xi_{N,K}^{-,l}$:

$$\mathcal{A}_l(\vec{f}_{N,K}) \;=\; \left\{ \xi_-^l \in \Xi_{N,K}^{-,l} \;\middle|\; \begin{array}{l} (i)\;\; \mathfrak{g} \not\leftrightarrow \partial^{\mathrm{ext}}\mathbb{D}_N \textbf{ inside } \mathcal{H}_{N,K}^{-,l} \\ \\ (ii)\; \#\{\partial_t^{\mathrm{int}}\mathcal{H}_{N,K}^{-,l} \cap \mathcal{C}_{N,K}^{-,l}(\mathfrak{g})\} \leqslant f_{N,K}^l (2K)^{d-1} \end{array} \right\}$$

where

$$\partial_t^{\mathrm{int}}\mathcal{H}_{N,K}^{-,l} \triangleq \{u \in \mathcal{H}_{N,K}^{-,l} \;:\; u_d = (l+1/2)K\}$$

is the top layer of the box $\mathcal{H}_{N,K}^{-,l}$.

Notice that if $\xi = \xi_-^l \vee \xi_+^l \in \mathcal{F}(\vec{f}_{N,K})$ then, necessarily, $\xi_-^l \in \mathcal{A}_l(\vec{f}_{N,K})$. Indeed, the connected cluster $\mathcal{C}_{N,K}^{-,l}(\mathfrak{g})$ is capable of hitting the top layer $\partial_t^{\mathrm{int}}\mathcal{H}_{N,K}^{-,l}$ only from within the set

$$\bigcup_{i \in \mathbb{F}_{N,K} \cap \mathbb{H}_{N,K}^l} \mathbb{B}_{2K}(i).$$

4.7. **Surgery in l-th mesoscopic layer.** Given a percolation configuration $\xi_-^l \in \Xi_{N,K}^{-,l}$ define a new configuration $\bar{\xi}_-^l$ as

$$\bar{\xi}_-^l(b) \;=\; \begin{cases} 0, & \text{if } b = (u,v) \text{ with } u \in \partial_t^{\mathrm{int}}\mathcal{H}_{N,K}^{-,l} \cap \mathcal{C}_{N,K}^{-,l}(\mathfrak{g}) \\ & \text{and } v_d = u_d + 1, \\ \\ \xi_-^l(b), & \text{otherwise.} \end{cases} \tag{4.35}$$

In other words, the surgery $\xi_-^l \mapsto \bar{\xi}_-^l$ cuts off all the vertical bonds which emerge from those points of the top layer $\partial_t^{\mathrm{int}}\mathcal{H}_{N,K}^{-,l}$ which are ξ_-^l-connected (inside $\mathcal{H}_{N,K}^{-,l}$) to the ghost site $\mathfrak{g}$.

For every $l = 1, 2, \ldots, R$ and for each $\vec{f}_{N,K}$ the map $\xi_-^l \mapsto \bar{\xi}_-^l$ leaves $\mathcal{A}_l(\vec{f}_{N,K})$ invariant and is at most $2^{f_{N,K}^l (2K)^{d-1}}$ to 1 on the latter set.

Furthermore, given $\vec{f}_{N,K}$, a mesoscopic layer l and a percolation configuration $\xi_-^l \in \mathcal{A}_l(\vec{f}_{N,K})$, the concatenation $\xi = \bar{\xi}_-^l \vee \eta$ belongs to $\mathfrak{J}_N = \{\xi \in \Xi_{N,\pm} \;:\; \mathfrak{g} \not\leftrightarrow \partial\mathbb{D}_N\}$ for every $\eta \in \Xi_{N,K}^{+,l}$. In particular, given any event $\mathcal{B} \subset \Xi_{N,K}^{+,l}$ and any $\xi_-^l \in \mathcal{A}_l(\vec{f}_{N,K})$,

$$\Phi_N^{\pm}\left(\mathcal{B} \,\middle|\, \bar{\xi}_-^l\right) \;\leqslant\; \max_{\pi} \Phi_{\mathcal{E}_{N,K}^{+,l}}^{\pi}\left(\mathcal{B}\right), \tag{4.36}$$

where $\Phi_{\mathcal{E}_{N,K}^{+,l}}^{\pi}$ is the FK-measure on $\Xi_{N,K}^{+,l}$ with wired boundary conditions on the lateral sides and on the top of $\mathcal{H}_{N,K}^{+,l}$ and, respectively, with π boundary conditions on the bottom side of $\mathcal{H}^{+,l}_{N,K}$. Thus, the surgery operation (4.35) effectively decouples the percolation configuration ξ^l_+ on the edges $\mathcal{E}^{+,l}_{N,K}$ of the upper box $\mathcal{H}^{+,l}_{N,K}$ from the global constraint $\mathfrak{J}_N$.

4.8. **Upper bound in one mesoscopic layer.** Given a mesoscopic layer $\mathbb{H}^m_{N,K}$ and a percolation configuration $\xi \in \Xi_{N,\pm}$, let us define the site percolation process $Y^m_{N,K}$ on $\mathbb{H}^m_{N,K}$ as:

$$Y^m_{N,K}(i) \;=\; \begin{cases} \;\;1\,, & \text{if } \mathbb{B}_{2K}(i) \text{ is good and } C^* \leftrightarrow \partial^{\text{ext}}\mathbb{D}_N \textbf{ in } \mathcal{H}^m_{N,K}\,, \\ -1\,, & \text{if } \mathbb{B}_{2K}(i) \text{ is good and } C^* \not\leftrightarrow \partial^{\text{ext}}\mathbb{D}_N \textbf{ in } \mathcal{H}^m_{N,K}\,, \\ \;\;0\,, & \text{if } \mathbb{B}_{2K}(i) \text{ is bad}\,, \end{cases}$$

where C^* denotes the crossing cluster of $\mathbb{B}_{2K}(i)$ (see Subsection 2.2).

Clearly, the percolation process $(Y^m_{N,K})$ dominates the restriction of $(X^{\text{FK}}_{N,K})$ to $\mathbb{H}^m_{N,K}$

$$\sum_{i\in\mathbb{H}^m_{N,K}} \mathbb{I}_{\{Y^m_{N,K}(i)\neq 1\}} \;\geqslant\; \sum_{i\in\mathbb{H}^m_{N,K}} \mathbb{I}_{\{X^{\text{FK}}_{N,K}(i)\neq 1\}}. \tag{4.37}$$

Lemma 4.2. *There exists a positive constant $c_6 > 0$, such that*

$$\max_{\pi} \Phi^{\pi}_{\mathcal{E}^m_{N,K}} \left(\sum_{i\in\mathbb{H}^m_{N,K}} \mathbb{I}_{\{Y^m_{N,K}(i)\neq 1\}} \;\geqslant\; f \right) \;\leqslant\; \exp\left(-c_6 \min\left\{ fK, f^{\frac{d-2}{d-1}} K^{d-1} \right\} \right), \tag{4.38}$$

uniformly in positive integers f and N sufficiently large.

A section of a finger on the layer m leads to an exponential decay of order fK if it contains at least $\frac{f}{2}$ blocks with label 0 or of order $f^{\frac{d-2}{d-1}}K^{d-1}$ if it contains at least $\frac{f}{2}$ blocks with label -1. The claim of the lemma follows from (2.4) via a straightforward modification of the argument presented in the Appendix A in [BIV1].

We stress the fact that the scaling $K = N^a$ will be used only in the derivation of (4.38).

4.9. **Upper bound on $\mathcal{F}(\vec{f}_{N,K})$.** We shall consider only even mesoscopic layers $m = 2, 4, \ldots, R$. For every such m and every positive collection $\vec{f}_{N,K}$ of finger labels, define the event $\mathcal{B}_m$, which depends only on the percolation configuration inside the slab $\mathcal{H}^m_{N,K}$:

$$\mathcal{B}_m \;=\; \mathcal{B}_m(\vec{f}_{N,K}) \;=\; \{\xi \,:\, \sum_{i\in\mathbb{H}^m_{N,K}} \mathbb{I}_{\{Y^m_{N,K}(i)\neq 1\}} \;\geqslant\; f^m_{N,K}\}.$$

Given an even mesoscopic layer $l = 2, 4, \ldots, R$,

$$\mathcal{F}(\vec{f}_{N,K}) \;\subseteq\; \bigcap_{m=l+2}^{R} \mathcal{B}_m(\vec{f}_{N,K}) \bigcap \mathcal{A}_l(\vec{f}_{N,K}).$$

We will compute the balance between the energetic cost of chopping a finger at the section l and the gain of erasing this finger above the level l. Consequently, the estimates (4.36) and (4.38) imply:

$$
\begin{aligned}
\Phi_N^{\pm}\Big(\mathcal{F}(\vec{f}_{N,K})\Big) &\leqslant \Phi_N^{\pm}\left(\bigcap_{m=l+2}^{R}\mathcal{B}_m(\vec{f}_{N,K})\bigcap\mathcal{A}_l(\vec{f}_{N,K})\right)\\
&= \sum_{\xi_-^l\in\mathcal{A}_l}\Phi_N^{\pm}\left(\bigcap_{m=l+2}^{R}\mathcal{B}_m(\vec{f}_{N,K})\,;\,\xi_-^l\right)\\
&\leqslant \mathrm{e}^{c_7 f_{N,K}^l K^{d-1}}\sum_{\bar{\xi}_-^l\,:\,\xi_-^l\in\mathcal{A}_l}\Phi_N^{\pm}\left(\bigcap_{m=l+2}^{R}\mathcal{B}_m(\vec{f}_{N,K})\,;\bar{\xi}_-^l\right)\\
&\leqslant \mathrm{e}^{c_7 f_{N,K}^l K^{d-1}}\max_{\pi}\Phi^{\pi}_{\mathcal{E}_{N,K}^{+,l}}\left(\bigcap_{m=l+2}^{R}\mathcal{B}_m(\vec{f}_{N,K})\right)\\
&\leqslant \exp\left(c_7 f_{N,K}^l K^{d-1}\;-\;c_6\sum_{m=l+2}^{R}\min\left\{f_{N,K}^m K,(f_{N,K}^m)^{\frac{d-2}{d-1}}K^{d-1}\right\}\right).
\end{aligned}
\tag{4.39}
$$

4.10. **Proof of Theorem 4.1.** We claim that there exists a positive constant $c_8>0$, such that

$$
\min_{2\leqslant l\leqslant R/2}\left\{c_7 f_{N,K}^l K^{d-1}\;-\;c_6\sum_{m=l+2}^{R}\min\left\{f_{N,K}^m K,(f_{N,K}^m)^{\frac{d-2}{d-1}}K^{d-1}\right\}\right\}\;\leqslant\;-\,c_8 N\,,
\tag{4.40}
$$

uniformly over all strictly positive collections $\vec{f}_{N,K}$ of finger labels which comply with the volume bound (4.31), which we rewrite in terms of $\vec{f}_{N,K}$ as:

$$
\sum_{l=2}^{R} f_{N,K}^l\leqslant\alpha\left(\frac{N}{K}\right)^d.
\tag{4.41}
$$

In view of (4.30) a substitution of (4.40) to (4.39) yields the target bound (4.33).

Let us turn to the proof of (4.40). The percolation of bad blocks between the layers $R/2$ and R produces an energy cost of an order at least N. Therefore, it is enough to examine the layers below $R/2$ and to prove that

$$
\min_{2\leqslant l\leqslant R/2}\left\{f_{N,K}^l K^{d-1}\;-\;\frac{c_6}{c_8}\sum_{m=l+2}^{R/2}\min\left\{f_{N,K}^m K,(f_{N,K}^m)^{\frac{d-2}{d-1}}K^{d-1}\right\}\right\}\;\leqslant\;0\,.
\tag{4.42}
$$

Set $n = R/8$. Suppose that (4.42) is violated then either there exists a sequence of even numbers $2 \leqslant l_1 < l_2 < \cdots < l_n \leqslant R/2$ such that for every $i = 1, \ldots, n-1$;

$$K^{d-1} f_{N,K}^{l_i} > \frac{c_6}{c_7} \sum_{j=i+1}^{n} K f_{N,K}^{l_j}\,, \tag{4.43}$$

or there exists a sequence of even numbers $2 \leqslant m_1 < m_2 < \cdots < m_n \leqslant R/2$ such that for every $i = 1, \ldots, n-1$;

$$f_{N,K}^{l_i} > \frac{c_6}{c_7} \sum_{j=i+1}^{n} \left(f_{N,K}^{m_j}\right)^{\frac{d-2}{d-1}} + 1. \tag{4.44}$$

The constants c_6 and c_7 do not depend on the values of ν in $R = [\nu N/K] + 1$ and α in (4.41). We claim that under an appropriate choice of $0 < \nu \ll \alpha \ll 1$ (see (4.49) below) both (4.43) and (4.44) contradict the volume constraint in (4.41). The latter is a consequence of the following two elementary numeric lemmas:

Lemma 4.3. *Fix $\chi > 0$. Assume that a sequence of positive integers $a_1, \ldots, a_n$ satisfies:*

$$a_i \geqslant \chi \sum_{j=i+1}^{n} a_j \tag{4.45}$$

for every $i = 1, \ldots, n-1$. Then

$$\sum_{i=1}^{n} a_i \geqslant (1+\chi)^{n-1} - \frac{1-\chi}{\chi}. \tag{4.46}$$

Lemma 4.4. *Fix $\chi > 0$. Set $\gamma = (d-2)/(d-1)$. Assume that a sequence of positive integers $a_1, \ldots, a_n$ satisfies:*

$$a_i \geqslant \chi \sum_{j=i+1}^{n} a_j^{\gamma} + 1 \tag{4.47}$$

for every $i = 1, \ldots, n-1$. Then there exists $c_9 = c(\chi, d) > 0$, such that

$$\sum_{i=1}^{n} a_i \geqslant c_9 n^d. \tag{4.48}$$

The inequality (4.43) corresponds to the choice of $\chi = K^{2-d} c_6/c_7$ in (4.45) and, consequently, the estimate (4.46) yields in this case:

$$\sum_{l=2}^{R/2} f_{N,K}^{l} \geqslant \exp\left\{c_{10} \nu \frac{N}{K^{d-1}}\right\},$$

which, by the choice of $K = N^a$; $a < 1/d$, is clearly incompatible with (4.41).

On the other hand, the inequality (4.44) corresponds to the choice of $\chi = c_6/c_8$ in (4.47) and, consequently, the estimate (4.48) yields:

$$\sum_{l=2}^{R/2} f_{N,K}^{l} \geqslant c_{11} \left(\frac{\nu N}{K}\right)^d.$$

Thus, the volume constraint (4.41) is violated whenever

$$\nu^d > \frac{\alpha}{c_{12}}, \tag{4.49}$$

which defines the appropriated choice of α and ν.

4.11. **Proof of Lemma 4.3.** Consider the sequence $\bar{a}_0, \dots, \bar{a}_{n-1}$ given by:

$$\bar{a}_0 = 1 \quad \text{and} \quad \bar{a}_i = \chi \sum_{j=1}^{i-1} \bar{a}_j \quad \text{for } i > 1. \tag{4.50}$$

The system (4.50) is exactly solvable:

$$\bar{a}_i = \chi (1+\chi)^{i-1} \quad \text{for } i \geqslant 1.$$

A look at the conditions of Lemma 4.3 reveals that $a_i \geqslant \bar{a}_{n-i}$ for every $i = 1, \dots, n$. Hence (4.46).

4.12. **Proof of Lemma 4.4.** For some $c > 0$, consider the sequence $\bar{a}_0, \dots, \bar{a}_{n-1}$ given by

$$\bar{a}_i = (1 + ic)^{1/(1-\gamma)} = (1+ic)^{d-1}.$$

By convexity:

$$\begin{aligned} \bar{a}_{i+1} - \bar{a}_i &\leqslant (d-1)c\,(1 + (i+1)c)^{d-2} \\ &\leqslant (d-1)c(1+c)^{d-2}\,(1+ic)^{d-2} = (d-1)c(1+c)^{d-2}\bar{a}_i^{\gamma}. \end{aligned}$$

Let us choose $c = c(\chi, d)$ according to $(d-1)c(1+c)^{d-2} = \chi$. Then,

$$\bar{a}_i \leqslant \chi \sum_{j=0}^{i-1} \bar{a}_j^{\gamma} + 1,$$

for all $j = 1, \dots, n-1$. Comparing with (4.47) we readily infer that $(1+ic)^{d-1} = \bar{a}_i \leqslant a_{n-i}$ and, consequently, (4.48) follows with, for example,

$$c_9 = \frac{1}{dc} \min_{n>1} \frac{(1 + c(n-1))^d}{n^d}.$$

5. Exponential mixing

The analysis of the spectral gap will rely on two properties of the equilibrium measure: the localization of the interface and the mixing of the system to a pure phase away from the interface. These estimates will be used in Subsection 6.1 in a specific framework, slightly different from the one of Theorem 3.2.

In this Section we establish the technical estimates which will be necessary for the derivation of the lower bound of the spectral gap. First, we formulate the localization property in the appropriate setting (see Subsection 6.1). Then the mixing property **(MP)** of the FK measure (see Definition 2.2) is combined with the localization in order to derive a control on the Gibbs measure.

5.1. **A new setting.** Let $\delta = 2^{-m}$ be the relative height of the new domain

$$\mathbb{D}_N^\delta \triangleq \{i \in \mathbb{L}^d \,:\, 1 \leqslant i_k \leqslant N,\, k = 1, \ldots, d-1, \quad 0 \leqslant i_d \leqslant \delta N\}\,. \tag{5.51}$$

We use $\partial^{\text{ext}}\mathbb{D}_N^\delta$ to denote the exteriour boundary of $\mathbb{D}_N^\delta$ in $\mathbb{L}^d$. This boundary consists of two parts: $\partial^{\text{ext}}\mathbb{D}_N^\delta = \partial_s^{\text{ext}}\mathbb{D}_N^\delta \bigvee \partial_t^{\text{ext}}\mathbb{D}_N^\delta$, where t stands for *top* and s stands for *sides*. The bottom face of $\mathbb{D}_N^\delta$ is denoted by $\partial_b^{\text{int}}\mathbb{D}_N^\delta$.

We are going to established the Hausdorff stability of the interface when a negative (respectively positive) magnetic field is applied on $\partial_b^{\text{int}}\mathbb{D}_N^\delta$ (resp $\partial_t^{\text{ext}}\mathbb{D}_N^\delta$) and a small positive field $\varepsilon > 0$ is applied on the faces of $\partial_s^{\text{ext}}\mathbb{D}_N^\delta$. This amounts to consider the Hamiltonian on $\{\pm 1\}^{\mathbb{D}_N^\delta}$ which is given by

$$\begin{aligned} \mathbf{H}_{N,\varepsilon}(\sigma) \;\triangleq\; & -\frac{1}{2}\sum_{(i\,,j)\subset\mathbb{D}_N^\delta} \sigma_i\sigma_j \;-\; \sum_{\substack{(i\,,j)\\ i\in\mathbb{D}_N^\delta, j\in\partial_t^{\text{ext}}\mathbb{D}_N^\delta}} \sigma_i \;-\; \varepsilon \sum_{\substack{(i\,,j)\\ i\in\mathbb{D}_N^\delta, j\in\partial_s^{\text{ext}}\mathbb{D}_N^\delta}} \sigma_i \\ & + \sum_{i\in\partial_b^{\text{int}}\mathbb{D}_N^\delta} \sigma_i, \end{aligned} \tag{5.52}$$

where the first three sums are over (subsets of) nearest neighbour bonds $(i\,,\,j)$. Following notation introduced in Subsection 2.3, we denote by $\mu_N^{+,\varepsilon,-}$ the corresponding Gibbs measure and by $\Phi_{N,\varepsilon}^{\pm}$ the FK measure.

The counterpart of Theorem 3.2 in this context relies also on the strict convexity assumption of the surface tension **(SC)**.

Theorem 5.1. *Assume* **(SC)** *and let $\delta > 0$, $\varepsilon > 0$ be fixed. For any $\beta \in \mathcal{B}$ and any $\nu > 0$ there exists $c_1 = c_1(\nu) > 0$, such that uniformly in N large enough,*

$$\Phi_{N,\varepsilon}^{\pm}\left(\mathfrak{g} \leftrightarrow \{i \,:\, i_d > \nu\delta N\}\right) \;\leqslant\; \mathrm{e}^{-c_1(\nu)N}. \tag{5.53}$$

The proof is similar to the one of Theorem 3.2.

5.2. **Screening.** Combining the localization of the interface (Theorem 5.1) and the mixing assumption **(MP)**, we obtain a screening property for the Gibbs measure.

Proposition 5.1. *Fix $\beta \in \mathcal{B}_1$. Then there is $c_\beta > 0$ such that for any function g with support included in $\mathcal{S}_{N,\delta} = \left\{i \in \mathbb{D}_N^\delta;\quad i_d = \frac{1}{2}\delta N\right\}$, the following holds uniformly over the boundary conditions in $\partial_b^{\text{int}}\mathbb{D}_N^\delta$*

$$\forall \eta \in \{\pm 1\}^{\partial_b^{\text{int}}\mathbb{D}_N^\delta}, \qquad \left|\mu_N^{+,\varepsilon}(g) - \mu_N^{+,\varepsilon,\eta}(g)\right| \leqslant 2N^{d-1}\,\|g\|_\infty\, \exp(-c_\beta\delta N)\,, \tag{5.54}$$

where $\mu_N^{+,\varepsilon,\eta}$ is the Gibbs measure with η boundary conditions on $\partial_b^{\text{int}}\mathbb{D}_N^\delta$, $+$ boundary conditions on $\partial_t^{\text{ext}}\mathbb{D}_N^\delta$ and $\varepsilon > 0$ boundary conditions elsewhere. $\mu_N^{+,\varepsilon}$ stands for the measure with $\{\eta_i = 1\}_i$.

If $\eta_i = 0$ for all i, the FK counterpart of the Ising measure will be denoted by $\Phi_{N,\varepsilon}^{\text{w/f}}$. In the proof of the Proposition, we are going to show that assumption **(MP)** implies that for any β in $\mathcal{B}_1$, the probabilities that a site in $\mathcal{S}_{N,\delta}$ is connected to the wired boundary conditions under $\Phi^{\mathrm{w}}_{N,\varepsilon}$ or $\Phi^{\mathrm{w/f}}_{N,\varepsilon}$ are almost identical, i.e., there exists $c = c(\beta) > 0$ such that

$$\forall j \in \mathcal{S}_{N,\delta}, \qquad \left| \Phi^{\mathrm{w}}_{N,\varepsilon}(j \leftrightarrow \partial \mathbb{D}^\delta_N) - \Phi^{\mathrm{w/f}}_{N,\varepsilon}(j \leftrightarrow \partial^{\mathrm{ext}} \mathbb{D}^\delta_N) \right| \leqslant \exp(-c\delta N)\,. \tag{5.55}$$

We stress the fact that in general the screening property (5.54) for the Ising measure is stronger than (5.55) since for $\beta \in \mathcal{B}_1$ a phase transition occurs for the Gibbs measure instead the FK measure is unique in the thermodynamic limit. In particular, if the + boundary conditions on $\partial^{\mathrm{ext}}_t \mathbb{D}^\delta_N$ are replaced by the magnetic field ε then (5.54) does not hold for small values of ε instead (5.55) remains valid uniformly in ε (at least for large enough β). The behavior wrt a magnetic field will be investigated in details in Subsection 6.1.

Proof. By definition of the total variation distance

$$\left| \mu^{+,\varepsilon}_N(g) - \mu^{+,\varepsilon,\eta}_N(g) \right| \leqslant \| \tilde{\mu}^{+,\varepsilon}_N - \tilde{\mu}^{+,\varepsilon,\eta}_N \|_{\mathrm{tv}} \, \|g\|_\infty\,,$$

where $\tilde{\mu}$ denotes the projection of the measure on $\mathcal{S}_{N,\delta}$. Furthermore, the total variation can be rewritten as

$$\| \tilde{\mu}^{+,\varepsilon}_N - \tilde{\mu}^{+,\varepsilon,\eta}_N \|_{\mathrm{tv}} = \inf_{\Pi} \left\{ \int d\Pi(\sigma,\sigma')\, 1_{\sigma \neq \sigma'} \right\},$$

where the infimum is taken over the joint probability measure Π on $\left(\{\pm 1\}^{\mathcal{S}_{N,\delta}}\right)^2$ with marginals $\tilde{\mu}^{+,\varepsilon}_N$ and $\tilde{\mu}^{+,\varepsilon,\eta}_N$. As the measures are ordered wrt the boundary conditions, there is a coupling Π which preserves this order.

$$\begin{aligned} \| \tilde{\mu}^{+,\varepsilon}_N - \tilde{\mu}^{+,\varepsilon,\eta}_N \|_{\mathrm{tv}} &\leqslant \inf_{\Pi} \left\{ \int d\Pi(\sigma,\sigma') \left(\sum_{j \in \mathcal{S}_{N,\delta}} |\sigma_j - \sigma'_j| \right) \right\}, \\ &\leqslant \sum_{j \in \mathcal{S}_{N,\delta}} \mu^{+,\varepsilon}_N(\sigma_j) - \mu^{+,\varepsilon,\eta}_N(\sigma_j) \leqslant \sum_{j \in \mathcal{S}_{N,\delta}} \mu^{+,\varepsilon}_N(\sigma_j) - \mu^{+,\varepsilon,-}_N(\sigma_j)\,. \end{aligned} \tag{5.56}$$

In terms of FK representation, this leads to

$$\begin{aligned} \| \tilde{\mu}^{+,\varepsilon}_N - \tilde{\mu}^{+,\varepsilon,\eta}_N \|_{\mathrm{tv}} \leqslant \sum_{j \in \mathcal{S}_{N,\delta}} \Phi^{\mathrm{w}}_{N,\varepsilon}(\quad j \quad &\leftrightarrow \partial^{\mathrm{ext}} \mathbb{D}^\delta_N \cup \partial^{\mathrm{int}}_b \mathbb{D}^\delta_N) \\ &- \quad \Phi^{\pm}_{N,\varepsilon}(j \leftrightarrow \partial^{\mathrm{ext}} \mathbb{D}^\delta_N) + \Phi^{\pm}_{N,\varepsilon}(j \leftrightarrow \mathfrak{g})\,, \end{aligned}$$

where $\partial^{\mathrm{ext}} \mathbb{D}^\delta_N \cup \partial^{\mathrm{int}}_b \mathbb{D}^\delta_N$ is simply the boundary $\partial \mathbb{D}^\delta_N$ of $\mathbb{D}^\delta_N$.

Let us fix $\nu = \frac{1}{4}$.

$$\begin{aligned} \| \tilde{\mu}^{+,\varepsilon}_N - \tilde{\mu}^{+,\varepsilon,\eta}_N \|_{\mathrm{tv}} \leqslant \sum_{j \in \mathcal{S}_{N,\delta}} & \Big\{ \Phi^{\mathrm{w}}_{N,\varepsilon}(j \leftrightarrow \partial \mathbb{D}^\delta_N) \\ & - \Phi^{\pm}_{N,\varepsilon} \left(j \leftrightarrow \partial^{\mathrm{ext}} \mathbb{D}^\delta_N;\ \mathfrak{g} \not\leftrightarrow \{ i\,:\, i_d \geqslant \frac{1}{4} \delta N \} \right) \Big\} \\ & + N^{d-1} \Phi^{\pm}_{N,\varepsilon} \left(\mathfrak{g} \leftrightarrow \{ i\,:\, i_d \geqslant \frac{1}{4} \delta N \} \right)\,. \end{aligned} \tag{5.57}$$

We are going to use now the fact that the interface is localized. Conditioning wrt the bond configuration ξ below $\{i\,:\,i_d=\frac{1}{4}\delta N\}$, we get

$$\Phi^{\pm}_{N,\varepsilon}\left(j\leftrightarrow\partial^{\rm ext}\mathbb{D}^\delta_N;\ \mathfrak{g}\not\leftrightarrow\{i\,:\,i_d\geqslant\frac{1}{4}\delta N\}\right)\ \geqslant\ \inf_{\xi}\left(\Phi^{{\rm w},\xi}_{\tilde{\mathbb{D}}^\delta_N,\varepsilon}\left(j\leftrightarrow\partial^{\rm ext}\tilde{\mathbb{D}}^\delta_N\right)\right)\frac{\Phi^{\rm w}_{N,\varepsilon}\left(\mathfrak{g}\not\leftrightarrow\{i\,:\,i_d\geqslant\frac{1}{4}\delta N\},\ \ \mathfrak{J}_N\right)}{\Phi^{\rm w}_{N,\varepsilon}(\mathfrak{J}_N)}\,,$$

where $\Phi^{{\rm w},\xi}_{\tilde{\mathbb{D}}^\delta_N,\varepsilon}$ denotes the FK measure on $\tilde{\mathbb{D}}^\delta_N=\mathbb{D}^\delta_N\cap\{i\,:\,i_d\geqslant\frac{1}{4}\delta N\}$ with boundary conditions ξ on the lower face of $\tilde{\mathbb{D}}^\delta_N$. As a consequence of Theorem 5.1, we get

$$\Phi^{\pm}_{N,\varepsilon}\left(j\leftrightarrow\partial^{\rm ext}\mathbb{D}^\delta_N;\ \mathfrak{g}\not\leftrightarrow\{i\,:\,i_d\geqslant\frac{1}{4}\delta N\}\right)\ \geqslant\ (1-\exp(-c_1\delta N))\,\Phi^{{\rm w/f}}_{\tilde{\mathbb{D}}^\delta_N,\varepsilon}\left(j\leftrightarrow\partial^{\rm ext}\tilde{\mathbb{D}}^\delta_N\right)\,, \tag{5.58}$$

where $\Phi^{{\rm w/f}}_{\tilde{\mathbb{D}}^\delta_N,\varepsilon}$ denotes the FK measure with free boundary conditions on the bottom face of $\tilde{\mathbb{D}}^\delta_N$ and wired otherwise.

Combining (5.53), (5.57) and (5.58) we finally derive

$$\|\tilde{\mu}^{+,\varepsilon}_N-\tilde{\mu}^{+,\varepsilon,\eta}_N\|_{\rm tv}\leqslant\sum_{j\in\mathcal{S}_{N,\delta}}\Phi^{\rm w}_{N,\varepsilon}(j\leftrightarrow\partial\mathbb{D}^\delta_N)-\Phi^{{\rm w/f}}_{\tilde{\mathbb{D}}^\delta_N,\varepsilon}(j\leftrightarrow\partial^{\rm ext}\tilde{\mathbb{D}}^\delta_N)+N^{d-1}\,\exp(-c_2\delta N)\,.$$

By the FKG property of the random cluster measures,

$$\Phi^{\rm w}_{N,\varepsilon}(j\leftrightarrow\partial\mathbb{D}^\delta_N)\leqslant\Phi^{\rm w}_{\tilde{\mathbb{D}}^\delta_N,\varepsilon}(j\leftrightarrow\partial\tilde{\mathbb{D}}^\delta_N)\,.$$

At this stage, it will be enough to apply the strong mixing inequality (5.55) to conclude.

Finally, it remains to derive (5.55) from the mixing property **(MP)**. First of all, one has to modify the boundary conditions and to replace ε by 0. This rests on the GHS ferromagnetic inequalities which are available only for the Ising measure (see eg. [El]). Using the correspondence between the Ising and the FK measure, we define

$$\forall\varepsilon>0,\qquad\Psi(\varepsilon)=\mu^{+,\varepsilon,+}_{\mathbb{D}^\delta_N}(\sigma_j)-\mu^{+,\varepsilon,0}_{\mathbb{D}^\delta_N}(\sigma_j)=\Phi^{\rm w}_{N,\varepsilon}(j\leftrightarrow\partial\mathbb{D}^\delta_N)-\Phi^{{\rm w/f}}_{N,\varepsilon}(j\leftrightarrow\partial^{\rm ext}\mathbb{D}^\delta_N)\,,$$

where $\mu^{+,\varepsilon,+}_{\mathbb{D}^\delta_N}$ (resp $\mu^{+,\varepsilon,0}_{\mathbb{D}^\delta_N}(\sigma_j)$) denotes the Gibbs measure on the set $\mathbb{D}^\delta_N$ with boundary conditions $+$ on the top face $\partial^{\rm ext}_t\mathbb{D}^\delta_N$, ε on the sides $\partial^{\rm ext}_s\mathbb{D}^\delta_N$ and $+$ on the bottom face $\partial^{\rm int}_b\mathbb{D}^\delta_N$ (resp 0 on the bottom face).

By FKG inequality, the function $\varepsilon\to\Psi(\varepsilon)$ is non negative and we are going to check that it is non increasing. Deriving wrt the parameter ε, we get

$$\Psi'(\varepsilon)=\sum_i\mu^{+,\varepsilon,+}_{\mathbb{D}^\delta_N}(\sigma_j;\sigma_i)-\mu^{+,\varepsilon,0}_{\mathbb{D}^\delta_N}(\sigma_j;\sigma_i)\,,$$

where the sum is restricted to the sites i which interact with the boundary field on the sides $\partial_s^{\text{ext}}\mathbb{D}_N^\delta$ of the box. The GHS inequality ensures that the two point truncated correlation function is a decreasing function of the field (for non negative fields), i.e. $\Psi'(\varepsilon) \leqslant 0$.

Thus, the derivation of (5.55) can be reduce to the case $\varepsilon = 0$ and it is enough to prove that

$$\Psi(0) = \Phi^{\text{w}}_{N,\varepsilon=0}(j \leftrightarrow \partial_t^{\text{ext}}\mathbb{D}_N^\delta \cup \partial_b^{\text{int}}\mathbb{D}_N^\delta) - \Phi^{\text{w/f}}_{N,\varepsilon=0}(j \leftrightarrow \partial_t^{\text{ext}}\mathbb{D}_N^\delta) \leqslant \exp(-c\delta N)\,. \quad (5.59)$$

As $\varepsilon = 0$, the magnetization of σ_j is simply related to the FK connection of j to the top (and possibly to the bottom) face of $\mathbb{D}_N^\delta$.

The mixing property **(MP)** enables us to compare only the probability of events which are locally supported, this is not the case in the previous inequality, thus we need more work to reduce to events with supports independent of N. Let $\mathbb{B}_{\delta N/10}(j)$ be the box centered at j, then Pisztora coarse graining implies that if j is connected to the boundary of $\mathbb{B}_{\delta N/10}(j)$ then with probability at least $1 - \exp(-c\delta N)$ the site j is connected also to the top face of $\mathbb{D}_N^\delta$. Define the set of bond configurations

$$\mathcal{A} = \left\{\xi, \qquad j \leftrightarrow \partial\mathbb{B}_{\delta N/10}(j)\right\}.$$

Then

$$\begin{aligned}\Phi^{\text{w}}_{N,\varepsilon=0}(j \leftrightarrow \partial_t^{\text{ext}}\mathbb{D}_N^\delta \cup \partial_b^{\text{int}}\mathbb{D}_N^\delta) \quad &- \quad \Phi^{\text{w/f}}_{N,\varepsilon=0}(j \leftrightarrow \partial_t^{\text{ext}}\mathbb{D}_N^\delta) \\ &= \quad \Phi^{\text{w}}_{N,\varepsilon=0}(\mathcal{A}) - \Phi^{\text{w/f}}_{N,\varepsilon=0}(\mathcal{A}) + o\big(\exp(-c\delta N)\big)\,.\end{aligned}$$

The previous FK measures are ordered (in the FKG sense). We can consider the joint measure $\nu_N(\xi,\xi')$ such that the first marginal is $\Phi^{\text{w}}_{N,\varepsilon=0}$, the second marginal is $\Phi^{\text{w/f}}_{N,\varepsilon=0}$ and the measure is supported by the configurations $\xi \geqslant \xi'$. By construction

$$\begin{aligned}\Phi^{\text{w}}_{N,\varepsilon=0}(\mathcal{A}) - \Phi^{\text{w/f}}_{N,\varepsilon=0}(\mathcal{A}) \quad &= \quad \nu_N\big(1_{\mathcal{A}}(\xi) - 1_{\mathcal{A}}(\xi')\big) \leqslant \sum_{b \in \mathbb{B}_{\delta N/10}(j)} \nu_N\big(\xi_b \neq \xi'_b\big) \\ &= \quad \sum_{b \in \mathbb{B}_{\delta N/10}(j)} \nu_N\big(\xi_b - \xi'_b\big) = \sum_{b \in \mathbb{B}_{\delta N/10}(j)} \Phi^{\text{w}}_{N,\varepsilon=0}(\xi_b) - \Phi^{\text{w/f}}_{N,\varepsilon=0}(\xi'_b)\,.\end{aligned}$$

For any bound b, the probability on the LHS can be estimated thanks to the mixing property (see Definition 2.2)

$$\Phi^{\text{w}}_{N,\varepsilon=0}(\mathcal{A}) - \Phi^{\text{w/f}}_{N,\varepsilon=0}(\mathcal{A}) \leqslant N^d \exp(-c\delta N)\,,$$

for some $c > 0$. This completes the derivation of (5.55). □

6. Spectral gap estimates

6.1. **Lower bound.** We turn now to the derivation of the lower bound (3.26) on the spectral gap. The proof follows closely the strategy developed by Martinelli [Ma] in the two dimensional case. We will briefly recall the main steps of the proof as they are exposed in the Chapter 6 of [Ma] and focus only on the changes. This comprises a more careful analysis of the boundary effects to take into account the boundary surface tension and a repeated use of Proposition 5.1, whose proof is based on the localization of the interface.

Step 1. The first step is to reduce to a block dynamics in order to estimate the spectrum of the single site Glauber dynamics in $\mathbb{D}_N = \{1, \dots, N\}^d$.

For a given $\delta > 0$, we consider the following covering of $\mathbb{D}_N$ by the overlapping slabs

$$R_i = \left\{ x \in \mathbb{D}_N, \qquad \frac{i}{2}[\delta N] \leqslant x_d < (\frac{i}{2} + 1)[\delta N] \right\} .$$

The total number of sets $\{R_i\}_i$ is independent of N and denoted by $L = L(\delta)$. The sets R_i are simply shifts of the set $\mathbb{D}_N^\delta$ introduced in (5.51). The block dynamics is defined in terms of the generator

$$\forall f \in \mathbb{L}^2(\mu_N^{\mathbf{h}}), \qquad \mathcal{L}_{N,\delta}^{\mathbf{h}} f(\sigma) = \sum_i \left(\mu_{R_i}^{\mathbf{h},\sigma}(f) - f(\sigma) \right) .$$

We recall that the single site dynamics on each R_i has a spectral gap larger than $\exp(-c_\beta \delta N^{d-1})$ (for some $c_\beta > 0$). Therefore, according to Proposition 3.4 of [Ma], the following bound holds for some $C_\beta > 0$

$$\mathrm{SG}(L_N^{\mathbf{h}}) \geqslant \exp(-C_\beta \delta N^{d-1}) \, \mathrm{SG}(\mathcal{L}_{N,\delta}^{\mathbf{h}}) , \tag{6.60}$$

where $\mathrm{SG}(\mathcal{L}_{N,\delta}^{\mathbf{h}})$ denotes the spectral gap of the block dynamics.

Step 2. Thanks to (6.60), it is enough to derive

Lemma 6.1. *Let* $\mathbf{h} = (0, \dots, 0, h_d, 0, \dots, 0)$. *Then, for any* $\delta > 0$,

$$\liminf_{N \to \infty} \frac{1}{N^{d-1}} \log \mathrm{SG}(\mathcal{L}_{N,\delta}^{\mathbf{h}}) \geqslant -\tau(\vec{e}_d) + \Delta_{h_d} .$$

The proof boils down to check that the semi-group associated to $\mathcal{L}_{N,\delta}^{\mathbf{h}}$ is a contraction for some time T, i.e. that there is $r_N > 0$ such that for all N large enough

$$\sup_\eta \left| \mathbb{E}\big(f(\sigma_T^\eta)\big) \right| \leqslant (1 - r_N) \|f\|_\infty , \tag{6.61}$$

for any f such that $\mu_N^{\mathbf{h}}(f) = 0$. In our context r_N will be such that

$$\lim_{N \to \infty} \frac{1}{N^{d-1}} \log r_N = -\tau(\vec{e}_d) + \Delta_{h_d} . \tag{6.62}$$

Iterating (6.61), we get for any f

$$\forall t \geqslant 0, \qquad \left\| \mathbb{E}\big(f(\sigma_t^\eta)\big) - \mu_N^{\mathbf{h}}(f) \right\|_\infty \leqslant \|f\|_\infty \exp\left(-r_N \left[\frac{t}{T} \right] \right) . \tag{6.63}$$

This $\mathbb{L}^\infty$ contraction and (6.62) imply Lemma 6.1.

We turn now to the derivation of (6.61). For technical reasons, it will be convenient to replace the free boundary conditions by a small coupling $\varepsilon > 0$ and to consider the evolution of the Glauber dynamics associated to the generator which takes into account the new boundary conditions. Let us denote by $\mu_N^{\mathbf{h},\varepsilon}$ the corresponding Gibbs measure and by $\mathcal{L}_{N,\delta}^{\mathbf{h},\varepsilon}$ the new generator. The effect of ε is to select the + phase. The two block dynamics are comparable by using the Radon Nykodim derivative; thus as ε vanishes, we recover the result for the original dynamics

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mathrm{SG}(\mathcal{L}^{\mathbf{h}}_{N,\delta}) = \lim_{\varepsilon\to 0} \liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mathrm{SG}(\mathcal{L}^{\mathbf{h},\varepsilon}_{N,\delta}) .$$

Fix a function f such that $\mu_N^{\mathbf{h},\varepsilon}(f) = 0$. Let f_0 be the image of f at time $t = 1$ if only the block R_0 has been updated at the random time t_0

$$f_0(\eta) = \mathbb{E}\big(f(\sigma^\eta_{t=1})\,; 0 < t_0 \leqslant 1 < t_1\big) = p\mu^{\mathbf{h},\varepsilon,\eta}_{R_0}(f)\,,$$

where $p = \frac{1}{L}\mathbb{P}(t_0 < 1 \leqslant t_1)$ is the probability that R_0 is the only update. Furthermore, $\mu^{\mathbf{h},\varepsilon,\eta}_{R_0}$ denotes the Gibbs measure on R_0 with boundary conditions h_d on the bottom face $\{i \in \mathbb{D}_N \,:\, i_d = 0\}$ of R_0, ε on the sides and η at the top face. Notice that p depends on L but not on N.

By construction f_0 satisfies 3 important properties :

(1) f_0 depends only on the spins in $\mathbb{D}_N \setminus R_0$.
(2) $\|f_0\|_\infty \leqslant p\|f\|_\infty$.
(3) $\mu_N^{\mathbf{h},\varepsilon}(f_0) = p\mu_N^{\mathbf{h},\varepsilon}(f) = 0$.

Using the Markov property at time $t = 1$ (see [Ma] page 162), we get

$$\sup_\eta \big|\mathbb{E}\big(f(\sigma^\eta_{t=2})\big)\,\big| \leqslant (1-p)\|f\|_\infty + \sup_\eta \big|\mathbb{E}\big(f_0(\sigma^\eta_{t=1})\big)\,\big|\,.$$

Thus (6.61) will follow if one can derive that for any ψ which does not depend on the spins in R_0 and has zero mean under $\mu_N^{\mathbf{h},\varepsilon}$,

$$\sup_\eta \big|\mathbb{E}\big(\psi(\sigma^\eta_{t=1})\big)\,\big| \leqslant (1 - r_{N,\delta})\|\psi\|_\infty\,, \tag{6.64}$$

where $r_{N,\delta}$ satisfies the asymptotic similar to (6.62)

$$\lim_{\delta\to 0}\lim_{N\to\infty} \frac{1}{N^{d-1}} \log r_{N,\delta} = -\tau(\vec{e}_d) + \Delta_{h_d}\,.$$

Replacing ψ by f_0, we complete Lemma 6.1.

Step 3. We turn now to the derivation of (6.64) for any function ψ which does not depend on the spins in R_0.

We consider a specific evolution up to time $t = 1$ with exactly $L + 1$ updates occurring at the random times $(t_i)_{0\leqslant i\leqslant L}$ (see [Ma] page 159). During the time interval $[0, 1]$, the blocks $R_0, R_1, \dots, R_L$ are successively updated at times $(t_i)_{0\leqslant i\leqslant L}$ such that $0 < t_0 \leqslant t_1 \leqslant \dots \leqslant t_L \leqslant 1 < t_{L+1}$. The k^{th}-update amounts to modify the spin configuration in the slab R_k, thus we introduce the following mappings on the space of configurations

$$\mathcal{T}_k^{\sigma_k}(\sigma) = \begin{cases} \sigma_k(x), & x \in R_k, \\ \sigma(x), & x \notin R_k, \end{cases}$$

To quantify the successive updates, one has to bound

$$\sup_\eta \left| \int \mathrm{d}\mu^\eta_{R_0}(\sigma_0)\ \dots \int \mathrm{d}\mu^{\eta_{j-1}}_{R_j}(\sigma_j)\ \dots \int \mathrm{d}\mu^{\eta_{L-1}}_{R_L}(\sigma_L)\ \psi(\eta_L) \right|,$$

where $\eta_k = \mathcal{T}_k^{\sigma_k} \circ \cdots \circ \mathcal{T}_0^{\sigma_0}(\eta)$. In words, this means that at the j^{th} update the configuration in σ_j is chosen wrt the Gibbs measure on R_j with boundary conditions equal to σ_{j-1} in R_{j-1} and η in R_{j+1}.

We define

$$\forall j \leqslant L, \quad g_j(\eta) = \int \mathrm{d}\mu_{R_j}^{\eta}(\sigma_j) \; \ldots \; \int \mathrm{d}\mu_{R_L}^{\eta_{L-1}}(\sigma_L)\, \psi(\eta_L)\,,$$

where this time $\eta_k = \mathcal{T}_k^{\sigma_k} \circ \cdots \circ \mathcal{T}_j^{\sigma_j}(\eta)$ for $k \geqslant j$. Thus it is enough to estimate $\sup_\eta |\mu_{R_0}^{h,\varepsilon,\eta}(g_1)|$, where the boundary conditions are h_d on $\partial_b^{\text{int}} R_0$, η on $\partial_{\text{t}} R_0$ and ε on the sides. The influence of the boundary condition η will be related to the stability property of the interface and, unlike [Ma], we resort to the FK representation. Let $\mathbb{P}_{R_0}^{\mathbf{h},\varepsilon,\eta}$ be the joint FK measure associated to $\mu_{R_0}^{\mathbf{h},\varepsilon,\eta}$

$$\mathbb{P}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\sigma,\xi) = P_{R_0}^{\xi,\mathbf{h},\eta}(\sigma)\; \Phi_{R_0,\varepsilon}^{\mathbf{h},\mathrm{w}}(\xi \,|\, \mathcal{C}_\eta)\,.$$

The previous formula reads as follows. First a bond configuration is chosen wrt the conditional FK measure; the conditioning $\mathcal{C}_\eta$ imposed by the boundary conditions η is such that ξ can not connect regions of the boundary with different signs. For a given bond configuration ξ, the spin configuration σ is obtained by a random coloring compatible with the bond configuration ξ and the boundary conditions. The random coloring is chosen according to the measure $P_{R_0}^{\xi,\mathbf{h},\eta}$.

As ψ does not depend on the spins in R_0, the support of g_1 is included in $\mathcal{S}_{N,\delta} \cup (R_0 \cup R_1)^c$, where $\mathcal{S}_{N,\delta} = \big\{ i \in R_0; \quad i_d = \frac{1}{2}\delta N \big\}$. We consider the event $\mathcal{A}_\eta$ which decouples the spins in $\mathcal{S}_{N,\delta}$ from the boundary conditions η outside R_0

$$\mathcal{A}_\eta = \big\{ \xi \,\big|\, \{\eta = -1\} \not\leftrightarrow \{ i \,:\, i_d \leqslant \frac{3}{4}\delta N \} \big\}.$$

The domain R_0 is the analog of $\mathbb{D}_N^\delta$ viewed upside down and $\{\eta = -1\} = \{i; \quad \eta_i = -1\}$ replaces $\mathfrak{g}$ (see Section 5). For any η, we write

$$\mu_{R_0}^{\mathbf{h},\varepsilon,\eta}(g_1) = \Phi_{R_0,\varepsilon}^{\mathbf{h},\mathrm{w}}(\mathcal{A}_\eta \,|\, \mathcal{C}_\eta)\, \mathbb{P}_{R_0}^{\mathbf{h},\varepsilon,\eta}\big(g_1 \,|\, \mathcal{A}_\eta\big) + \mathbb{P}_{R_0}^{\mathbf{h},\varepsilon,\eta}\big(g_1\, 1_{\mathcal{A}_\eta^c}\big)\,.$$

This leads to the following decomposition

$$\begin{aligned} \Big|\mu_{R_0}^{\mathbf{h},\varepsilon,\eta}(g_1)\Big| \quad &\leqslant \quad \Phi_{R_0,\varepsilon}^{h,\mathrm{w}}(\mathcal{A}_\eta \,|\, \mathcal{C}_\eta)\, \Big|\mathbb{P}_{R_0}^{\mathbf{h},\varepsilon,\eta}\big(g_1 \,|\, \mathcal{A}_\eta\big) - \mu_{R_0}^{+,\varepsilon,\eta}(g_1)\Big| \\ &\qquad + \Phi_{R_0,\varepsilon}^{h,\mathrm{w}}(\mathcal{A}_\eta \,|\, \mathcal{C}_\eta)\, \big|\mu_{R_0}^{+,\varepsilon,\eta}(g_1)\big| + \Phi_{R_0,\varepsilon}^{h,\mathrm{w}}(\mathcal{A}_\eta^c \,|\, \mathcal{C}_\eta) \|\psi\|_\infty\,, \qquad (6.65) \end{aligned}$$

and $\mu_{R_0}^{+,\varepsilon,\eta}$ denotes the Gibbs measure on R_0 where the boundary magnetic field h_d on $\partial_b^{\text{int}} R_0$ has been replaced by $+1$. In order to complete the evaluation of (6.65), we have to derive the following inequalities :

- A bound involving the surface tension

$$\inf_\eta \Phi_{R_0,\varepsilon}^{\mathbf{h},\mathrm{w}}(\mathcal{A}_\eta \,|\, \mathcal{C}_\eta) \geqslant \exp\big(- N^{d-1}(\tau(\vec{e}_d) - \Delta_{h_d} + O(\varepsilon))\big) = r_{N,\varepsilon}\,. \qquad (6.66)$$

- A characterization of the screening

$$\sup_\eta \Big|\mathbb{P}_{R_0}^{\mathbf{h},\varepsilon,\eta}\big(g_1 \,|\, \mathcal{A}_\eta\big) - \mu_{R_0}^{+,\varepsilon,\eta}(g_1)\Big| \leqslant N^{d-1} \exp(-c\delta N) \|\psi\|_\infty\,. \qquad (6.67)$$

- A proof of the much "faster" relaxation of the dynamics in the + phase. This boils down to check that

$$\sup_\eta \left|\mu_{R_0}^{+,\varepsilon,\eta}(g_1)\right| \leqslant N^{d-1}\exp(-c\delta N)\|\psi\|_\infty \,. \tag{6.68}$$

Combining the 3 previous estimates, there is $c>0$ such that

$$\begin{aligned}\sup_\eta \left|\mu_{R_0}^{\mathbf{h},\varepsilon,\eta}(g_1)\right| &\leqslant \Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\mathcal{A}_\eta \,|\, \mathcal{C}_\eta)\, N^{d-1}\exp(-c\delta N)\|\psi\|_\infty + \Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\mathcal{A}^c_\eta \,|\, \mathcal{C}_\eta)\|\psi\|_\infty \,,\\ &\leqslant \left(1 - r_{N,\varepsilon}(1 - N^{d-1}\exp(-c\delta N))\right)\|\psi\|_\infty \,.\end{aligned}$$

This concludes the proof of (6.64).

6.1.1. *Derivation of inequality* (6.66). The event $\mathcal{A}_\eta \cap \mathcal{C}_\eta$ is supported by the set of bonds $\mathcal{E}_\Delta$ generated by $\Delta = \left\{ i \in R_0, \quad i_d > \frac{3\delta}{4}N \right\}$, i.e. $\mathcal{E}_\Delta = \left\{ (i,j) \in \mathcal{E}, \quad i \in \Delta \right\}$. Since $\mathcal{A}_\eta \cap \mathcal{C}_\eta$ is decreasing, we have

$$\Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\mathcal{A}_\eta \,|\, \mathcal{C}_\eta) = \frac{\Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\mathcal{A}_\eta \cap \mathcal{C}_\eta)}{\Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\mathcal{C}_\eta)} \geqslant \frac{\Phi^{\mathrm{w}}_{\Delta,\varepsilon}(\mathcal{A}_\eta \cap \mathcal{C}_\eta)}{\Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\mathcal{C}_\eta)} \,.$$

In the spin language, it can be rewritten as

$$\frac{\Phi^{\mathrm{w}}_{\Delta,\varepsilon}(\mathcal{A}_\eta \cap \mathcal{C}_\eta)}{\Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\mathcal{C}_\eta)} = \frac{Z_\Delta^{+,\varepsilon,\eta}}{Z_\Delta^{+,\varepsilon,+}}\,\frac{Z_{R_0}^{\mathbf{h},\varepsilon,+}}{Z_{R_0}^{\mathbf{h},\varepsilon,\eta}} \geqslant \frac{Z_\Delta^{+,\varepsilon,-}}{Z_\Delta^{+,\varepsilon,+}}\,\frac{Z_{R_0}^{\mathbf{h},\varepsilon,+}}{Z_{R_0}^{\mathbf{h},\varepsilon,-}} \,,$$

where we used in the last inequality that the the ratio $Z_\Delta^{+,\varepsilon,\eta}\big/Z_{R_0}^{\mathbf{h},\varepsilon,\eta}$ is an increasing function of η.

As in e.g. Lemma 2.1 and Lemma 2.2 in [BIV2] we, taking the thermodynamic limit, recover the surface tension and the surface energy (recall that δN is the height of the box R_0 and that the magnetic filed ε is applied on the lateral sides only):

$$\lim_{\delta\to 0}\,\liminf_{N\to\infty}\,\frac{1}{N^{d-1}}\inf_\eta \log \Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\mathcal{A}_\eta \,|\, \mathcal{C}_\eta) \geqslant -\,\tau(\vec{e}_d) + \Delta_{h_d} \,. \tag{6.69}$$

This conclude (6.66).

6.1.2. *Derivation of inequality* (6.67). Let $\tilde{R}_0$ be the set $R_0 \cap \{i \;:\; i_d \leqslant \frac{3}{4}\delta N\}$. The domain $\tilde{R}_0$ is the counterpart of the domain $\tilde{\mathbb{D}}_N^\delta$ introduced in the proof of Proposition 5.1. An intermediate step is to estimate the total variation distance between the measures $\tilde{\mathbb{P}}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\cdot|\mathcal{A}_\eta)$ and $\tilde{\mu}_{\tilde{R}_0}^{+,\varepsilon}$ which are the projections of the measures $\mathbb{P}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\cdot|\mathcal{A}_\eta)$ and $\mu_{\tilde{R}_0}^{+,\varepsilon}$ on the spin variables in the domain $\mathcal{S}_{N,\delta}$. Before applying (5.56), we need to check that the measure $\tilde{\mathbb{P}}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\cdot|\mathcal{A}_\eta)$ is stochastically dominated by $\tilde{\mu}_{\tilde{R}_0}^{+,\varepsilon}$.

Let ψ be an increasing function supported by $\{\pm 1\}^{\mathcal{S}_{N,\delta}}$

$$\tilde{\mathbb{P}}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\psi|\mathcal{A}_\eta) = \frac{1}{\Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\mathcal{A}_\eta \,|\, \mathcal{C}_\eta)}\sum_{\sigma,\xi} P_{R_0}^{\xi,\mathbf{h},\eta}(\sigma)\;\Phi^{\mathbf{h},\mathrm{w}}_{R_0,\varepsilon}(\xi \,|\, \mathcal{C}_\eta) 1_{\mathcal{A}_\eta}(\xi)\psi(\sigma) \,.$$

Let us decompose ξ into (ξ', ξ''), where ξ' is the restriction of ξ to $\tilde{R}_0$. Conditioning wrt ξ'', we get

$$\tilde{\mathbb{P}}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\psi|\mathcal{A}_\eta) = \frac{1}{\Phi_{R_0,\varepsilon}^{\mathbf{h},\mathrm{w}}(\mathcal{A}_\eta\,|\,\mathcal{C}_\eta)} \Phi_{R_0,\varepsilon}^{\mathbf{h},\mathrm{w}}\left(1_{\mathcal{A}_\eta} \sum_{\sigma,\xi'} P_{R_0}^{\xi,\mathbf{h},\eta}(\sigma)\ \Phi_{R_0,\varepsilon}^{\mathbf{h},\mathrm{w}}(\xi'\,|\,\xi'')\psi(\sigma) \,\Big|\, \mathcal{C}_\eta \right).$$

As ξ'' belongs to $\mathcal{A}_\eta$, the coloring measure $P_{R_0}^{\xi,\mathbf{h},\eta}$ does not take into account the constraint imposed by η. Thus one can write

$$\sum_{\sigma,\xi'} P_{R_0}^{\xi,\mathbf{h},\eta}(\sigma)\ \Phi_{R_0,\varepsilon}^{\mathbf{h},\mathrm{w}}(\xi'\,|\,\xi'')\psi(\sigma) = \int m_{\xi''}(\mathrm{d}\,\omega)\mu_{\tilde{R}_0}^{\mathbf{h},\varepsilon,\omega}(\psi)\,.$$

where $m_{\xi''}$ is a measure on the boundary conditions ω outside $\tilde{R}_0$. As the RHS of the previous inequality is always smaller than $\tilde{\mu}_{\tilde{R}_0}^{+,\varepsilon}$ the stochastic domination holds.

Using the property that the measures are ordered, we have according to (5.56)

$$\begin{aligned}\|\tilde{\mu}_{\tilde{R}_0}^{+,\varepsilon} - \tilde{\mathbb{P}}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\cdot|\mathcal{A}_\eta)\|_{\mathrm{tv}} &\leqslant \sum_{j\in\mathcal{S}_{N,\delta}} \mu_{\tilde{R}_0}^{+,\varepsilon}(\sigma_j) - \mathbb{P}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\sigma_j|\mathcal{A}_\eta)\\ &\leqslant \sum_{j\in\mathcal{S}_{N,\delta}} \Phi_{\tilde{R}_0,\varepsilon}^{\mathrm{w}}\big(j \leftrightarrow \partial\tilde{R}_0 \cup \partial_b^{\mathrm{int}}\tilde{R}_0\big) - \Phi_{R_0,\varepsilon}^{\mathbf{h},\mathrm{w}}(j \leftrightarrow \partial^{\mathrm{ext}}R_0\,|\,\mathcal{A}_\eta\cap\mathcal{C}_\eta)\,.\end{aligned}$$

By FKG inequality, this leads to

$$\|\tilde{\mu}_{\tilde{R}_0}^{+,\varepsilon} - \tilde{\mathbb{P}}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\cdot|\mathcal{A}_\eta)\|_{\mathrm{tv}} \leqslant \sum_{j\in\mathcal{S}_{N,\delta}} \Phi_{\tilde{R}_0,\varepsilon}^{\mathrm{w}}\big(j \leftrightarrow \partial^{\mathrm{ext}}\tilde{R}_0 \cup \partial_b^{\mathrm{int}}\tilde{R}_0\big) - \Phi_{\tilde{R}_0,\varepsilon}^{h,\mathrm{w/f}}(j \leftrightarrow \partial^{\mathrm{ext}}R_0)\,.$$

As β is in $\mathcal{B}_1$, the strong mixing property implies that for some $c > 0$

$$\|\tilde{\mu}_{\tilde{R}_0}^{+,\varepsilon} - \tilde{\mathbb{P}}_{R_0}^{\mathbf{h},\varepsilon,\eta}(\cdot|\mathcal{A}_\eta)\|_{\mathrm{tv}} \leqslant N^{d-1}\exp(-c\delta N)\,. \tag{6.70}$$

By Proposition 5.1, the total variation distance between the measures $\tilde{\mu}_{\tilde{R}_0}^{+,\varepsilon}$ and $\tilde{\mu}_{R_0}^{+,\varepsilon,\eta}$ is exponentially small, thus (6.67) is proven.

6.1.3. *Derivation of inequality* (6.68). The proof is based on a repeated use of the screening property obtained in Proposition 5.1.

$$\mu_{R_0}^{+,\varepsilon,\eta}(g_1) = \mu_{R_0}^{+,\varepsilon,\eta}(g_1) - \mu_{R_0\cup R_1}^{+,\varepsilon,\eta}(g_1) + \mu_{R_0\cup R_1}^{+,\varepsilon,\eta}(g_1)\,.$$

As g_1 is supported by $\mathcal{S}_{N,\delta} \cup (R_0 \cup R_1)^c$, we can apply Proposition 5.1 to get

$$\left|\mu_{R_0}^{+,\varepsilon,\eta}(g_1) - \mu_{R_0\cup R_1}^{+,\varepsilon,\eta}(g_1)\right| \leqslant \exp(-c\delta N)\|f\|_\infty\,. \tag{6.71}$$

Since $\mu_{R_0\cup R_1}^{+,\varepsilon,\eta}(g_1) = \mu_{R_0\cup R_1}^{+,\varepsilon,\eta}(g_2)$, the previous argument can be iterated

$$\mu_{R_0}^{+,\varepsilon,\eta}(g_1) = \sum_{i=0}^{L-1}\Big(\mu_{R_0\cup\cdots\cup R_i}^{+,\varepsilon,\eta}(g_{i+1}) - \mu_{R_0\cup\cdots\cup R_{i+1}}^{+,\varepsilon,\eta}(g_{i+1})\Big) + \mu_N^{+,\varepsilon}(f)\,,$$

where we used that $f = g_{L+1}$ and $\mathbb{D}_N = R_0 \cup \cdots \cup R_L$. Using the fact that for $i \geqslant 1$, the restriction of g_{i+1} to $R_0\cup\cdots\cup R_{i+1}$ is measurable wrt $R_0\cup\cdots\cup R_{i-1}$ an estimate similar to (6.71) holds.

For $\varepsilon > 0$, an argument similar to the one used in Proposition 5.1 implies

$$\left|\mu_N^{+,\varepsilon}(f) - \mu_N^{h,\varepsilon}(f)\right| \leqslant \exp(-c\delta N)\|f\|_\infty .$$

By construction $\mu_N^{\mathbf{h},\varepsilon}(f) = 0$. Summarizing the previous estimates, there exists $c > 0$ such that

$$\sup_\eta \left|\mu_{R_0}^{+,\varepsilon,\eta}(g_1)\right| \leqslant LN^{d-1}\exp(-c\delta N)\|f\|_\infty .$$

Thus (6.68) holds.

6.2. **Upper Bound on the spectral gap.** We turn now to the derivation of Theorem 3.5. For any $m \in]-m^*(\beta), m^*(\beta)[$, we set

$$A_m = \left\{\sigma \in \{-1,1\}^{\mathbb{D}_N} \mid \quad \mathbb{M}_N \leqslant m\right\},$$

where $\mathbb{M}_N$ denotes the averaged magnetization $\mathbb{M}_N = 1/N^d \sum_{i\in\mathbb{D}_N}\sigma_i$. Applying formula (3.25) to the test function $f(\sigma) = 1_{\{\sigma\in A_m\}}$, we get the following upper bound on the spectral gap

$$\mathrm{SG}(N,\mathbf{h}) \leqslant (2N)^d \frac{\mu_N^{\mathbf{h}}\big(\partial A_m\big)}{\mu_N^{\mathbf{h}}(A_m)(1-\mu_N^{\mathbf{h}}(A_m))}, \tag{6.72}$$

where ∂A_m is the boundary of the set A_m

$$\partial A_m = \left\{\sigma \in \{-1,1\}^{\mathbb{D}_N} \mid \exists x \in \mathbb{D}_N, \quad \sigma \in A_m,\ \sigma^x \notin A_m, \quad \text{or} \quad \sigma \notin A_m,\ \sigma^x \in A_m\right\}.$$

Optimizing this inequality over m will enables us to bound the spectral gap in terms of equilibrium quantities.

For any β in $\mathcal{B}$, the measure $\mu_N^{\mathbf{h}}$ converges to the pure phase μ^+ in the thermodynamic limit as soon as one of the coordinates of h is positive. As a consequence $\mu_N^{\mathbf{h}}(A_m)$ vanishes as N goes to infinity for m in $[-m^*(\beta), m^*(\beta)[$.

The set A_m contains the configurations in the $-$ phase which can be associated, on the macroscopic level, to the function u uniformly equal to -1. In this case there is no interface in the bulk and the interfacial energy is concentrated along the boundary. A straightforward adaptation of proposition 4.1 of [BIV2] implies

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_N^{\mathbf{h}}(A_m) \geqslant -\mathcal{F}^{\mathbf{h}}(-m^*(\beta)) = -\sum_{i=1}^{2d} \Delta_{h_i}. \tag{6.73}$$

Proposition 4.2 of [BIV2] implies the following upper bound

$$\limsup_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_N^{\mathbf{h}}\big(\{\mathbb{M}_N \in [m-\delta, m+\delta]\}\big) \leqslant -\mathcal{F}^{\mathbf{h}}(m).$$

The previous inequality rests upon the lower semi-continuity of the functional $\mathcal{W}_{\mathbf{h}}$ which, for the sake of completeness, is proven in the Appendix.

For any $\delta > 0$, ∂A_m is included in the set $\{\mathbb{M}_N \in [m-\delta, m+\delta]\}$, thus

$$\forall m \in]-m^*(\beta), m^*(\beta)[, \qquad \limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_N^{\mathbf{h}}(\partial A_m) \leqslant -\mathcal{F}^{\mathbf{h}}(m). \tag{6.74}$$

Combining estimates (6.73) and (6.74), we conclude Theorem 3.5.

7. Appendix

7.1. **Lower semi-continuity.** By considering appropriate boundary conditions, we are going to reduce $\mathcal{W}^{\mathbf{h}}$ to a functional which does not explicitly take into account the boundary field. We set

$$g^{\mathbf{h}}(x) \;=\; \begin{cases} \dfrac{|\Delta_{h_i}|}{2\tau(\vec{e}_i)}, & \text{if } x_i - 1/2 \geqslant \max_j\{|x_j - 1/2|\}\,, \\ \dfrac{|\Delta_{h_{2i}}|}{2\tau(\vec{e}_i)}, & \text{if } x_i - 1/2 \leqslant \min_j\{-|x_j - 1/2|\}\,. \end{cases}$$

To any function u of bounded variation, we associate Du the vector measure of its first partial derivatives and $|Du|$ the positive measure obtained by taking the total variation of Du. Finally, we denote by $\vec{n}$ the vector function obtained as the Radon Nykodim derivative of $\frac{dDu}{d|Du|}$. For any function u in $\mathrm{BV}(\mathrm{int}\widehat{\mathbb{D}}, \{\pm 1\})$, one sets

$$\mathfrak{W}^{\mathbf{h}}(u) = \frac{1}{2}\int_{\mathcal{O}} \tau_\beta(\vec{n}_x)\, d|Du \vee g^{\mathbf{h}}|(x). \tag{7.75}$$

This functional is lower semi-continuous w.r.t. $\mathbb{L}^1$-convergence. Let $s_i = \frac{|\Delta_{h_i}|}{2\tau_\beta(\vec{e}_i)}$ and recall that $\mathcal{P} = \cup_{i=1}^{2d}\mathcal{P}_i$ denotes the faces of the cube $\widehat{\mathbb{D}}$. This functional can be rewritten as follows

$$\begin{aligned} \mathfrak{W}^{\mathbf{h}}(u) \;=\; & \int_{\partial^*_{g^{\mathbf{h}}} u \setminus \mathcal{P}} \tau_\beta(\vec{n}_x) d\mathcal{H}^{(d-1)}_x \\ & + \sum_i (|\mathcal{P}_i| - S_i)(1 - s_i)\tau_\beta(\vec{e}_i) + S_i(1 + s_i)\tau_\beta(\vec{e}_i) + C(\mathcal{O}), \end{aligned}$$

where S_i stands for the Hausdorff measure of $\partial^*\{\{u = -1\} \vee g^{\mathbf{h}}\} \cap \mathcal{P}_i$ and $C(\mathcal{O})$ is the variation of g in $\mathcal{O} \setminus \widehat{\mathbb{D}}$. We recover $\mathcal{W}^{\mathbf{h}}(u)$ up to a constant

$$\mathcal{W}^{\mathbf{h}}(u) = \mathfrak{W}^{\mathbf{h}}(u) - \sum_i |\mathcal{P}_i|\,(\tau_\beta(\vec{e}_i) - \Delta_{h_i}/2) - C(\mathcal{O}). \tag{7.76}$$

This implies that the functional $\mathcal{W}^{\mathbf{h}}(u)$ is lower semi-continuous.

7.2. **Proof of Proposition 2.1.** .

We split the proof into several steps:

Step 1. If τ_β is strictly convex at $\vec{e}_d$, then also τ_β^ϵ is strictly convex at $\vec{e}_d$. Indeed, define $\mathbf{x} = (0, \ldots, 0, \tau_\beta(\vec{e}_d))$. Thus, $\mathbf{x}$ belongs to $\partial\mathcal{K}$, it is just a point where the $\vec{e}_d$-orthogonal hyperplane touches $\partial\mathcal{K}$. Of course, $\tau_\beta^\epsilon(\vec{e}_d) = (\mathbf{x}, \vec{e}_d)$ for every $\epsilon > 0$. The inequality (2.14) can be equivalently reformulated as follows: at least for one of the vectors $\vec{v}_k$, $\tau_\beta(\vec{v}_k) > (\mathbf{x}, \vec{v}_k)$, or, in other words,

$$\{x \in \mathbb{R}^d \,:\, (x - \mathbf{x}, \vec{v}_k) = 0\} \cap \mathrm{int}(\mathcal{K}) \neq \emptyset.$$

Since the Wulff shape $\mathcal{K}$ is convex and has a non-empty interiour, the latter is equivalent to

$$\{x \in \mathbb{R}^d \,:\, (x - \mathbf{x}, \vec{v}_k) = 0\} \cap \mathrm{int}(\mathcal{K}^\epsilon) \neq \emptyset$$

for every $\epsilon > 0$. Hence $\tau_\beta^\epsilon(\vec{v}_k) > (\mathbf{x}, \vec{v}_k)$ as well.

Step 2. Below we use a simplified notation $\tau \equiv \tau_\beta^\epsilon$. Let $E \subset \{x : x_d \geqslant 0\}$ be a bounded set of finite perimeter and positive volume (d-dimensional Hausdorff measure). Let $\partial^* E$ be the reduced boundary [EG] of E. Let us split it as $\partial^* E = A \cup \Sigma$, where

$$\Sigma = \partial^* E \setminus \{x \in \mathbb{R}^d : x_d = 0\} = \partial^* E \setminus A.$$

We claim that

$$\int_\Sigma \tau(\vec{n}_x) d\mathcal{H}_x^{d-1} > \mathcal{H}^{d-1}(A)\tau(\vec{e}_d). \tag{7.77}$$

By the Gauss-Green formula [EG],

$$\mathcal{H}^{d-1}(A)\vec{e}_d = \int_\Sigma \vec{n}_x d\mathcal{H}_x^{d-1}. \tag{7.78}$$

In view of (2.14) it is enough to show that one can find a decomposition of Σ into a disjoint union $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_d$, such that the vectors

$$\vec{v}_k \triangleq \int_{\Sigma_k} \vec{n}_x d\mathcal{H}_x^{d-1} \quad k = 1, \ldots, d, \tag{7.79}$$

are in the general position. At this stage the positivity of the volume of E enters the picture. By the continuity one can pick positive numbers $0 = a_0 < a_1 < \cdots < a_{d-1} < a_d = \infty$ such that

(i) $\min_{k \leqslant d} \mathcal{H}^d \left(E \cap \{x : a_{k-1} < x_d < a_k\}\right) > 0$.

(ii) $\min_{k \leqslant d-1} \mathcal{H}^{d-1} \left(E \cap \{x : x_d = a_k\}\right) > 0$.

Of course, $E \cap \{x : a_{k-1} < x_d < a_k\}$ is just the part of E which is chopped out by $\vec{e}_d$-orthogonal hyperplanes through the points $\mathbf{x}_{k-1} \triangleq (0, \ldots, 0, a_{k-1})$ and $\mathbf{x}_k \triangleq (0, \ldots, 0, a_k)$ respectively. Since E is a set of finite perimeter we may in addition assume that small perturbations of these hyperplanes retain both properties above. Specifically, there exist positive numbers $\delta_1, \ldots, \delta_{d-1} > 0$, such that the sets ($k = 2, \ldots, d-1$)

$$S_k \triangleq E \cap \{x : (x - \mathbf{x}_k, \vec{e}_d + \delta_k \vec{e}_k) < 0 < (x - \mathbf{x}_{k-1}, \vec{e}_d + \delta_{k-1}\vec{e}_{k-1})\},$$

$S_1 \triangleq \{x : (x - \mathbf{x}_1, \vec{e}_d + \delta_k \vec{e}_1) < 0 < (x, \vec{e}_d)\}$ and $S_d = E \setminus \cup_{k=1}^{d-1} S_k$, are disjoint and, furthermore, each and everyone of the corresponding portions of their boundaries, which we denote as $\Sigma_k = \Sigma \cap S_k$; $k = 1, \ldots, d$ and

$$A_k = E \cap \{x : (x - \mathbf{x}_k, \vec{e}_d + \delta_k \vec{e}_k) = 0\};\ k = 1, \ldots, d-1,$$

has a positive $(d-1)$-dimensional Hausdorff measure.

Subsequent application of the Gauss-Green formula on each of the sets $S_1, \ldots, S_d$ leads now to the following chain of equalities for the vectors $\vec{v}_1, \ldots, \vec{v}_d$ defined in

(7.79):

$$
\begin{aligned}
0 \neq \vec{v}_d \quad &= (\vec{e}_d + \delta_{d-1}\vec{e}_{d-1})\,\mathcal{H}^{d-1}(A_{d-1}) \\
0 \neq \vec{v}_{d-1} &= -\,(\vec{e}_d + \delta_{d-1}\vec{e}_{d-1})\,\mathcal{H}^{d-1}(A_{d-1}) + (\vec{e}_d + \delta_{d-2}\vec{e}_{d-2})\,\mathcal{H}^{d-1}(A_{d-2}) \\
&\dots \\
0 \neq \vec{v}_1 \quad &= -\,(\vec{e}_d + \delta_1\vec{e}_1)\,\mathcal{H}^{d-1}(A_1) + \vec{e}_d\mathcal{H}^{d-1}(A).
\end{aligned}
$$

Recall that by (7.78) $\sum_1^d \vec{v}_k = \vec{e}_d\mathcal{H}^{d-1}(A)$. Consequently, $\vec{v}_1, \dots, \vec{v}_d$ span $\mathbb{R}^d$ and (7.77) follows.

Step 3. Finally we turn to the proof of Proposition 2.1 proper. Let $u \in \mathrm{BV}\left(\mathrm{int}\,\widehat{\mathbb{D}}, \{\pm 1\}\right)$. Set $E = \{x : u(x) = -1\}$. As in **Step 2**, split $\partial^* E = \Sigma \cup A$. The functional $\widehat{\mathcal{W}}_{\beta,\epsilon}(u|g)$ can be then written (in the notation $\tau \equiv \tau_\beta^\epsilon$) as

$$
\widehat{\mathcal{W}}_{\beta,\epsilon}\left(u \,\middle|\, g\right) \;=\; \int_\Sigma \tau(\vec{n}_x) d\mathcal{H}_x^{d-1} \;+\; \left(1 - \mathcal{H}^{d-1}(A)\right)\tau(\vec{e}_d).
$$

By (7.77),

$$
\widehat{\mathcal{W}}_{\beta,\epsilon}\left(u \,\middle|\, g\right) \;>\; \widehat{\mathcal{W}}_{\beta,\epsilon}\left(\mathbb{I}(\cdot) \,\middle|\, g\right),
$$

as soon as $\mathcal{H}^d(E) > 0$. But this is precisely the claim of the Proposition.

□

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