THE WULFF CONSTRUCTION IN THREE AND MORE DIMENSIONS.

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ABSTRACT. In this paper we prove the Wulff construction in three and more dimensions for an Ising model with nearest neighbor interaction.

1. Introduction

The problem of phase separation for two dimensional Ising model and the study of the equilibrium shape of crystals (Wulff shape) has been initiated by Dobrushin, Kotecky and Shlosman [DKS]. Among other things, they proved that if at very low temperatures we decrease the averaged magnetization in the + pure phase, we observe the creation of a macroscopic droplet of the $-$ phase which has a deterministic shape on the macroscopic scale.

The proof has been first simplified by Pfister [Pf] and then extended to the whole of the phase transition region by Ioffe [I1], [I2] (see also [SS] and [PV]). Recently, Ioffe and Schonmann [IS] have completed the DKS theory up to the critical temperature and greatly simplified the original proofs. Moderate deviations in the exact canonical ensemble are also studied in [IS].

In two dimensions, the proofs have been based on duality arguments and on a coarse graining procedure (skeleton). These arguments do not seem to apply in higher dimensions.

For more than two dimensions, an alternative procedure has been proposed by Alberti, Bellettini, Cassandro and Presutti [ABCP], [BCP] for Ising systems with Kac potentials. They rephrase the whole problem in terms of L^1 theory and prove large deviations for the appearance of a droplet of the minority phase in a scaling limit when the size of the domain diverges not much faster than the range of the Kac potentials. This amounts to a weak large deviation principle which is obtained by proving Γ-convergence of a functional associated to the spins system [ABCP]. A large deviation principle has then been proved via a tightness property [BCP].

Their approach has been generalized by Benois, Bodineau, Butta and Presutti [BBBP], [BBP] by taking first the thermodynamic limit and then letting the range of interaction go to infinity. The first paper [BBBP] was devoted to the proof of a weak large deviation principle for the macroscopic magnetization which is

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equivalent to the computation of surface tension. The main idea has been to introduce a coarse graining in order to use the L^1 setting. Namely, events in L^1 were related to mesoscopic quantities by an argument which we will refer to later as minimal section argument. An exact expression of surface tension was difficult to recover from coarse grained estimates and surface tension was only derived in the Kac limit, i.e. when the range of interactions tends to infinity. The second step [BBP] consisted of proving a tightness property by using the compactness in L^1 of the set of functions of bounded variation with finite perimeter.

Wulff construction for three dimensional independent percolation has been proven by Cerf [Ce] using a procedure similar to the one of [BBBP] and a novel definition of surface tension. In this case, the dependence on boundary conditions is weaker and, the minimal section argument enables to prove directly a weak large deviation principle by using this appropriate definition of surface tension. As percolation occurs in an infinite volume, there is an extra difficulty and different compactness arguments have been required.

In this paper we proceed as in [BBBP]. The main difficulty is to recover surface tension from a constraint on the averaged magnetization. The surface tension is defined as $\log\left(\frac{Z^+}{Z^{+,-}}\right)$ where the partition functions are computed with $+$ boundary conditions and with mixed boundary conditions ($+$ at the top and $-$ at the bottom), see for instance the paper of Messager, Miracle-Solé and Ruiz [MMR]. To use directly this definition, one would have to find in the bulk surfaces of $+$ spins or of $-$ spins which in fact may not exist. A way to circumvent this problem is to prove that surface tension can be produced by averaging the boundary conditions, choosing the spins with respect to the $+$ pure phase and to the $-$ pure phase.

For Ising model with nearest neighbor interaction, the coarse graining developed by Pisztora [Pi1] will play an analogous role to the one used for Kac model. Pisztora's coarse graining is one of the most profound and powerful technique for the study of the Ising (Potts) model, it provides an accurate description of the Ising model in a non perturbative regime up to a temperature $\hat{T}_c$ which is conjectured to agree with the critical temperature. In the following, we will mention which of our results hold up to $\hat{T}_c$. As Pisztora's coarse graining is defined via the FK representation, several quantities need to be rewritten in terms of the FK representation. In particular, our approach to the surface tension (Section 4) is built upon the FK representation and, is motivated by the corresponding construction in [Ce]. This is a key to obtain precise surface order estimates on the logarithmic scale. This is also the only point at which we refer to [Ce], the core philosophy of our proof is based on the renormalization ideas of [BBBP] and [BBP], including the appropriate setup of the geometric measure theory. The coarse graining schemes of the latter works, however, depend on specific properties of Kac potentials and, one of our main technical tasks here is to develop a relevant modification of these renormalization procedures in the nearest neighbor context. A step further in the understanding of the surface tension will be to prove a phase separation theorem for Kac model with finite range interactions by using a coarse graining defined only in terms of the Gibbs measure [Bo].

After introducing the main notation, we state in Section 2 the results and an overview of the paper (see subsection 2.3).

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2. Notation and results

For simplicity, notation and results are stated in three dimensions, but they are valid for any dimension larger or equal to three.

2.1. Notation.

We introduce the following norms on $\mathbb{Z}^3$

$$\forall x \in \mathbb{Z}^3, \qquad \|x\|_1 = \sum_{i=1}^{3} |x_i| \qquad \text{and} \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{3} |x_i|^2}.$$

Two vertices x, y in $\mathbb{Z}^3$ are nearest neighbors if $\|x-y\|_1 \leq 1$ and we denote it by $x \sim y$. For any finite subset Λ of $\mathbb{Z}^3$, we define its boundary by

$$\partial\Lambda = \{x \in \Lambda^c \mid y \in \Lambda, \quad x \sim y\},$$

and denote its cardinality by $|\Lambda|$.

We consider the Ising model on $\mathbb{Z}^3$ with nearest neighbor interaction. Each spin σ_i, attached at the lattice site i in $\mathbb{Z}^3$, can take values ± 1. For any integer N, we set $\Delta_N = \{1, N\}^3$ and denote the space of configurations in Δ_N by $\Sigma_{\Delta_N} = \{\pm 1\}^{\Delta_N}$. Let σ_{Δ_N} be the spin configuration restricted to Δ_N. We introduce the Hamiltonian associated to σ_{Δ_N} with boundary conditions $\sigma_{\partial\Delta_N}$

$$H(\sigma_{\Delta_N} \mid \sigma_{\partial\Delta_N}) = -\frac{1}{2} \sum_{\substack{i \sim j \\ i,j \in \Delta_N}} \sigma_i \sigma_j - \sum_{\substack{i \sim j \\ i \in \Delta_N, j \in \partial\Delta_N}} \sigma_i \sigma_j .$$

The Gibbs measure on Σ_{Δ_N} at inverse temperature $\beta > 0$ is

$$\mu_{\beta,\Delta_N}(\sigma_{\Delta_N} \mid \sigma_{\partial\Delta_N}) = \frac{1}{Z_\beta(\sigma_{\partial\Delta_N})} \exp\Big(-\beta H(\sigma_{\Delta_N} \mid \sigma_{\partial\Delta_N})\Big),$$

where the partition function $Z_\beta(\sigma_{\partial\Delta_N})$ is the normalizing factor. When the boundary conditions $\sigma_{\partial\Delta_N}$ are identically equal to 1, we simply write μ^+_{β,Δ_N}. There is a critical value $\beta_c > 0$ and for all β larger than β_c, there exists $m_\beta > 0$ such that

$$\lim_{N \to \infty} \mu^+_{\beta,\Delta_N}(\sigma_0) = m_\beta > 0.$$

The Gibbs measure μ^+_β on $\Sigma_{\mathbb{Z}^3}$ obtained by taking the thermodynamic limit of μ^+_{β,Δ_N} is called the $+$ pure phase and m_β is the equilibrium value of the magnetization.

2.2. **Surface tension.** Let us recall the definition of surface tension and related results which can be found in [MMR].

The set of unit vectors in $\mathbb{R}^3$ is denoted by $\mathbb{S}^2$. We fix $\vec{n}$ a vector in $\mathbb{S}^2$ and $\vec{e_1}, \vec{e_2}$ two vectors orthogonal to $\vec{n}$. Let h be a positive constant and $N \to f(N)$ a positive function which diverges as N goes to infinity. For any integer N, we denote by $\bar{\Lambda}(hN, hN, f(N))$ the parallelepiped of $\mathbb{R}^3$ centered at 0 with faces parallel to the axis $(\vec{e_1}, \vec{e_2}, \vec{n})$ such that the lengths of the sides parallel to $(\vec{e_1}, \vec{e_2})$ are hN, hN and the ones parallel to $\vec{n}$ is $f(N)$. We introduce Λ_N the set of vertices $\bar{\Lambda}(hN, hN, f(N)) \cap \mathbb{Z}^3$. The boundary $\partial \Lambda_N$ is split into 2 sets

$$\begin{aligned} \partial^+ \Lambda_N &= \{ i \in \partial \Lambda_N \mid \vec{i}.\vec{n} \geq 0 \}, \\ \partial^- \Lambda_N &= \{ i \in \partial \Lambda_N \mid \vec{i}.\vec{n} < 0 \}. \end{aligned}$$

We call $\partial^+ \Lambda_N$ the upper and $\partial^- \Lambda_N$ the lower part of $\partial \Lambda_N$. We fix the boundary conditions outside Λ_N to be equal to 1 on $\partial^+ \Lambda_N$ and to -1 on $\partial^- \Lambda_N$. The corresponding partition function on Λ_N is denoted by $Z^{+,-}_{\Lambda_N}$.

Definition 2.1. *The surface tension in the direction $\vec{n} \in \mathbb{S}^2$ is defined by*

$$\tau(\vec{n}) = \lim_{N \to \infty} \ - \frac{1}{h^2 N^2} \log \frac{Z^{+,-}_{\Lambda_N}}{Z^{+}_{\Lambda_N}}.$$

The surface tension defined above coincides with the one defined in [MMR] (see Appendix 8.1). It depends neither on h nor on f as proven in [MMR] (Theorem 2). Let us extend τ by homogeneity

$$\forall \vec{v} \in \mathbb{R}^3 - \{0\}, \quad \tilde{\tau}(\vec{v}) = \|\vec{v}\|_2 \, \tau \left(\frac{\vec{v}}{\|\vec{v}\|_2} \right) \qquad \text{and} \qquad \tilde{\tau}(0) = 0. \tag{2.1}$$

The pyramidal inequality proven in Theorem 3 of [MMR] ensures that $\tilde{\tau}$ is convex. As $\tilde{\tau}$ is locally bounded and convex, it is continuous. It was proven by Lebowitz and Pfister [LP] that for all β larger than β_c, the surface tension $\tau(\vec{n}_0)$ in the direction $\vec{n}_0 = (1,0,0)$ is positive. From the symmetries and the convexity of τ, we check that τ is uniformly positive on $\mathbb{S}^2$.

The spin configuration σ should be seen as a microscopic representation of the system. The macroscopic state of the system is instead determined by the value of an order parameter (the averaged magnetization) which specifies the phase of the system. As β is fixed, it is convenient to replace the order parameter by a parameter u with values ± 1. We suppose that the macroscopic region of $\mathbb{R}^3$ where our system is confined is $\mathcal{T} = [0,1]^3$. We denote by $\mathrm{BV}(\mathcal{T}, \{+1,-1\})$ the set of functions of bounded variation in $\mathcal{T}$ with values ± 1 (see [EG] for a review). The fact that $u_r = 1$ for some r in $\mathcal{T}$ means that locally at r the system is in equilibrium in the phase $+m_\beta$. The precise correspondence between σ and functions on $\mathcal{T}$ is described in Section 3, where we approximate σ by a coarse graining procedure, introducing a mesoscopic scale.

For all u in $\mathrm{BV}(\mathcal{T}, \{+1,-1\})$, we denote by ∂u the boundary of the set $\{u = -1\}$. If the set ∂u has finite perimeter, there exists a set $\partial^* u$, called the reduced boundary, such that one can define in each point x of $\partial^* u$ the outer normal denoted by $\vec{n_x}$. Let us introduce the functional $\mathcal{F}$ on $L^1(\mathcal{T}, [-\frac{1}{m_\beta}, \frac{1}{m_\beta}])$

$$\forall u \in \mathrm{BV}(\mathcal{T}, \{+1,-1\}), \qquad \mathcal{F}(u) = \int_{\partial^* u} \tau(\vec{n_x}) d\mathcal{H}_x, \tag{2.2}$$

where $d\mathcal{H}$ is the 2 dimensional Hausdorff measure in $\mathbb{R}^3$. If u is in $L^1(\mathcal{T}, [-\frac{1}{m_\beta}, \frac{1}{m_\beta}])$ but not in $\mathrm{BV}(\mathcal{T}, \{+1,-1\})$ then we set $\mathcal{F}(u) = \infty$. To any subset A of $\mathcal{T}$, we associate the function $\mathbb{I}_A = 1_{A^c} - 1_A$ and simply write $\mathcal{F}(A) = \mathcal{F}(\mathbb{I}_A)$.

An important property is the lower semi-continuity of $\mathcal{F}$ with respect to L^1 convergence. As $\tilde{\tau}$ is convex (see (2.1)), the lower semi-continuity is a consequence of a result by Ambrosio and Braides (see [AmBr] Theorem 2.1 and example 2.8).

The equilibrium crystal shape $\mathcal{W}_m$, called Wulff shape, is a solution of the following isoperimetric problem

$$\min \Big\{ \mathcal{F}(u) \;\Big|\; u \in \mathrm{BV}(\mathcal{T}, \{+1,-1\}), \qquad m_\beta \int_{\mathcal{T}} u_r \, dr \leq m \Big\}, \tag{2.3}$$

where m belongs to $]m^*, m_\beta[$. We will restrict the parameter m so that, for m in $]m^*, m_\beta[$ the minimizers of the variational problems in $\mathcal{T}$ and $\mathbb{R}^3$ are the same. This enables us to avoid boundary problems. The shape $\mathcal{W}_m$ can be explicitly constructed (the Wulff construction) by dilating the set

$$\mathcal{W} = \bigcap_{\vec{n} \in \mathbb{S}^2} \{ x \in \mathbb{R}^3; \qquad \vec{x}.\vec{n} \leq \tau(\vec{n}) \}$$

in order to satisfy the volume constraint $m_\beta \int_{\mathcal{T}} \mathbb{I}_{\mathcal{W}_m}(r)\, dr = m$. As $\mathcal{W}_m = \lambda_m \mathcal{W}$, one has $\mathcal{F}(\mathcal{W}_m) = \lambda_m \mathcal{F}(\mathcal{W})$. Thus $\mathcal{F}(\mathcal{W}_m)$ is continuous with respect to m.

Taylor [Ta] proved that $\mathcal{W}_m$ is a closed convex surface and that all other minimizers of (2.3) are deduced from $\mathcal{W}_m$ by shifts. In the following, we suppose that $\mathcal{W}_m$ is centered in $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

2.3. **Heuristics and Results.** The total magnetization $\frac{1}{N^3} \sum_{i \in \Delta_N} \sigma_i$ will be denoted by $\mathbf{M}_{\Delta_N}$. A shift of the magnetization from its equilibrium value leads to large deviations controlled by a surface order

Theorem 2.1. *There is β_0 positive such that for any β larger than β_0 and m in $]m^*, m_\beta[$*

$$\lim_{N \to \infty} \frac{1}{N^2} \log \mu^+_{\beta, \Delta_N} \Big(\mathbf{M}_{\Delta_N} \leq m \Big) = -\mathcal{F}(\mathcal{W}_m),$$

where m^* and $\mathcal{W}_m$ were defined in (2.3).

More precisely, a phase separation occurs on the macroscopic level. In order to describe it, we introduce an intermediate scale called mesoscopic : the magnetization is locally averaged on boxes of size N^α with $\alpha \in]0,1[$ $(1 \ll N^\alpha \ll N)$. For any integer L, we define the sub-lattice

$$\mathcal{L}_L = \Big\{ \Big(L x_1 + \frac{L}{2}, L x_2 + \frac{L}{2}, L x_3 + \frac{L}{2} \Big) \;\Big|\; x = (x_1, x_2, x_3) \in \mathbb{Z}^3 \Big\}. \tag{2.4}$$

We introduce also $B(x,L)$ the box of length L centered in x in $\mathcal{L}_L$

$$B(x,L) = \Big\{ y \in \mathbb{Z}^3 \ | \quad \forall i \in \{1,2,3\}, \quad -\frac{L}{2} < y_i - x_i \leq \frac{L}{2} \Big\}. \tag{2.5}$$

To simplify the notation, we fix α in $\mathbb{Q} \cap]0,1[$ and suppose from now that N is of the form $2^{\alpha^{-1}k}$, where k and $\frac{k}{\alpha}$ are integers. The set Δ_N is partitioned into boxes $B(x,N^\alpha)$ of side length N^α centered in x in $\mathcal{L}_{N^\alpha}$. For general values of N, one would need to partition Δ_N with boxes which may have different sizes. This is a standard technique and we refer the reader to Pisztora [Pi1].

The local magnetization $\mathcal{M}$ is a piecewise constant function on $\mathcal{T}$

$$\forall r \in \mathcal{T}, \qquad \mathcal{M}_r = \frac{1}{N^{3\alpha}} \sum_{j \in B(x,N^\alpha)} \sigma_j \qquad \text{if} \ \ \forall \mathrm{i}, \quad -\frac{\mathrm{L}}{2} < \mathrm{N r_i} - \mathrm{x_i} \leq \frac{\mathrm{L}}{2}.$$

As explained in the introduction, it is convenient to formulate the problem of phase separation in terms of L^1 theory. For any function u in $L^1(\mathcal{T},[-\frac{1}{m_\beta},\frac{1}{m_\beta}])$, we denote by $\mathcal{V}(u,\delta)$ the δ-neighborhood of u

$$\mathcal{V}(u,\delta) = \Big\{ v \in L^1\Big(\mathcal{T},[-\frac{1}{m_\beta},\frac{1}{m_\beta}]\Big) \ \Big| \qquad \int |u_r - v_r| \, dr \leq \delta \Big\}.$$

We can now state a theorem on phase separation which says that for β large enough with $\mu^+_{\beta,\Delta_N}\big(\cdot \ \big| \mathbf{M}_{\Delta_N} \leq m \big)$-probability converging to 1, the function $\mathcal{M}$ is close to some translate of the Wulff shape $m_\beta 1\!\!1_{\mathcal{W}_m}$.

Theorem 2.2. *There is β_0 positive such that for any β larger than β_0 and m in $]m^*, m_\beta[$*

$$\forall \delta > 0, \qquad \lim_{N\to\infty} \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \bigcup_{r \in \mathcal{T}'} \mathcal{V}(1\!\!1_{\mathcal{W}_m + r},\delta) \ \Big| \mathbf{M}_{\Delta_N} \leq m \Big) = 1,$$

where m^*, $\mathcal{W}_m$ were defined in (2.3) and $\mathcal{T}' = \{ r \in \mathcal{T} \ | \ \mathcal{W}_m + r \subset \mathcal{T} \}$.

This result is far less sharp than those obtained in the 2 dimensional case (see [IS]).

We will follow the scheme of [BBBP] and deduce Theorems 2.1 and 2.2 from the following statements.

Proposition 2.1. *Let β be large enough. Then for all u in $BV(\mathcal{T},\{+1,-1\})$ such that $\mathcal{F}(u)$ is finite*

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \ \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(u,\delta) \Big) \leq -\mathcal{F}(u).$$

Proposition 2.2. *For β large enough and m in $]m^*, m_\beta[$*

$$\lim_{\delta\to 0} \liminf_{N\to\infty} \ \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(1\!\!1_{\mathcal{W}_m},\delta) \Big) \geq -\mathcal{F}(\mathcal{W}_m),$$

where m^* and $\mathcal{W}_m$ were defined in (2.3).

Since Proposition 2.1 is only a weak large deviation principle, we need to strengthen it by proving an exponential tightness property which is similar to the one in [BBP]. For any a positive, the set

$$K_a = \Big\{u \in \mathrm{BV}(\mathcal{T}, \{+1,-1\}) \;\Big|\; \mathcal{F}(u) \leq a\Big\}$$

is compact with respect to convergence in measure : As $\mathcal{F}$ is lower semi-continuous, K_a is closed and as the surface tension τ is larger than a positive constant τ_0, the set K_a is included in the compact set of functions of bounded variation in $\mathcal{T}$ with perimeter smaller than $\frac{a}{\tau_0}$ (see [EG] Section 5.2.3).

Proposition 2.3. *We fix β large enough. Then there exists a constant C_β such that for all a and δ positive*

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(K_a,\delta)^c\Big) \leq -C_\beta\, a,$$

where $\mathcal{V}(K_a,\delta)$ is the δ-neighborhood of K_a in $L^1(\mathcal{T},[-\frac{1}{m_\beta},\frac{1}{m_\beta}])$.

The proofs of Theorems 2.1 and 2.2 are based on well known large deviations arguments (see [DS]). For completeness we prove Theorem 2.1, the proof of Theorem 2.2 is similar

Proof of Theorem 2.1.

In order to prove the upper bound, we fix δ positive and split the closed set

$$F = \Big\{u \in L^1\Big(\mathcal{T},[-\frac{1}{m_\beta},\frac{1}{m_\beta}]\Big) \;\Big|\; m_\beta \int_{\mathcal{T}} u_r\, dr \leq m\Big\}$$

into 2 sets

$$\mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in F\Big) \leq \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in F \cap \mathcal{V}(K_a,\delta)\Big) + \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(K_a,\delta)^c\Big).$$

We choose a such that $C_\beta a$ is much larger than $\mathcal{F}(\mathcal{W}_m)$, then Proposition 2.3 enables us to bound the last term in the RHS. Let us fix ε positive. Since K_a is compact, we cover it with a finite number ℓ of neighborhoods $\mathcal{V}(u_i,\varepsilon_i)$, where each ε_i belongs to $]0,\varepsilon]$ and is chosen such that Proposition 2.1 implies

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(u_i,\varepsilon_i)\Big) \leq -\mathcal{F}(u_i) + \varepsilon.$$

For δ small enough, we cover $F \cap \mathcal{V}(K_a,\delta)$ with the above neighborhoods which intersect F. Since u_i belongs to $\mathcal{V}(F,\varepsilon)$, we get from Lemma 2.1.2 of [DS]

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in F \cap \mathcal{V}(K_a,\delta)\Big) \leq -\lim_{\varepsilon\to 0} \inf_{u\in\mathcal{V}(F,\varepsilon)} \mathcal{F}(u) \leq -\inf_{u\in F} \mathcal{F}(u).$$

As $\mathcal{W}_m$ minimizes the variational problem (2.3), the upper bound holds.

To prove the lower bound, we fix ε positive and check that $\{\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(\mathbb{1}_{\mathcal{W}_{m-\varepsilon}},\delta)\}$ is included in $\{\mathbf{M}_{\Delta_N} \leq m\}$ for δ small enough. Proposition 2.2 implies

$$\lim_{\delta\to 0}\liminf_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(\mathbb{1}_{\mathcal{W}_{m-\varepsilon}},\delta)\Big) \geq -\mathcal{F}(\mathcal{W}_{m-\varepsilon}).$$

Letting ε go to 0, we complete Theorem 2.1. $\square$

Let us comment on Propositions 2.2, 2.1 and 2.3. A shift of the averaged magnetization can be realized by 2 competing effects. The first one, which consists of producing a large droplet of $-$ inside the bulk, is controlled by surface tension (Propositions 2.2 and 2.1). The second one consists of increasing homogeneously the number of small $-$ contours. This requires a lot of energy, but may be favored by entropy. This effect is ruled out by Proposition 2.3 which is a combination of an estimate in the phase of small contours with a Peierls type estimate for large contours. In fact, the underlying phenomena are more subtle and it was shown by [IS] in the case of dimension 2, that on the level of moderate deviations the second effect may be the most important.

We start by defining a coarse graining on the mesoscopic scale which keeps more details of the microscopic structure than $\mathcal{M}$. This is done, in Section 3, via the FK representation by using Pisztora's results [Pi1]. This coarse graining procedure imposes that β is larger than a critical value $\tilde{\beta}_c$ related to the slab percolation threshold (see [Pi1]) and to condition (3.3). It is conjectured that $\tilde{\beta}_c$ equals the critical value β_c. In Section 4, motivated by [Ce], we use an alternative definition of surface tension in terms of the FK representation. We prove the equivalence of several expressions for surface tension which will enable us to compare different boundary conditions.

In Section 5, Proposition 2.1 is proven along the lines of the argument developed in [BBBP]. It states that the most likely configurations in $\left\{\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(u,\delta)\right\}$ are those for which the $+$ and $-$ phases coexist along the boundary of ∂u, this coexistence induces deviations proportional to a surface order. The L^1 constraint $\left\{\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(u,\delta)\right\}$ imposed on the magnetization is not strong enough to localize the interface close to ∂u : there might be mesoscopic fingers of one phase percolating into the other. Following [BBBP], we prove by the minimal section argument that one can chop off these fingers without changing too much the probability of the event. The renormalization is an essential feature in the previous procedure. Once the interface is localized on the mesoscopic level, the main problem is to identify surface tension. Note that in the case of percolation [Ce], the minimal section argument enables to cut the microscopic fingers which connect the domains separated by ∂u. Therefore, one can identify the surface tension factor, because for independent percolation it is defined as the probability that no cluster connects one domain to the other. In the case of spin systems, one would need to find a microscopic surface of $+$ spins on one side of ∂u and another one of $-$ spins on the other side in order to use directly the definition of surface tension. This would seem difficult to achieve because mesoscopic contours enable only to control the averaged magnetization and do not ensure the existence of such microscopic surfaces. We proceed differently and use an alternative definition of surface tension in terms of the FK measure. This requires β to be large.

Proposition 2.2 is proven in Section 6 under the condition that β is larger than $\tilde{\beta}_c$. The coarse graining is only useful to get the lower bound up to $\tilde{\beta}_c$ and it could be avoided if one considers only β large, in which case, a direct proof without using the FK representation is possible.

Section 7 is devoted to the proof of Proposition 2.3. Besides its probabilistic interpretation, i.e. the proof of an exponential tightness property, Proposition 2.3 deals with a physical phenomenon of a different nature than the surface tension : it states that the occurrence of many small contours is unlikely. The production of surface tension supposes a balance between energy and entropy. It is a general feature of DKS theory that energy is the dominant factor which rules out the occurrence of small contours. The techniques developed in [SS] and [IS] to control the phase of small contours, for the two dimensional Ising model, are robust enough to be extended to higher dimensions provided Peierls estimate holds. This observation was used in [BBP]. For β large enough, one could have proceed as in [BBP] and worked only with the Gibbs measure. We use an alternative approach borrowed to [I2] and deduce directly estimates on the phase of small contours from Pisztora's results [Pi1]. Thus Proposition 2.3 holds as soon as β is larger than $\tilde{\beta}_c$.

As noticed in the papers on Ising model with Kac potentials, the strategy described above can be applied in any dimension larger or equal to three. As a final remark, we stress the fact that the above results could be easily extended to prove a large deviation principle for the measures μ^+_{β,Δ_N} with action functional $\mathcal{F}$. This setting was developed in [BBP] and also used for percolation [Ce]. This requires a modification of Proposition 2.2 which is described in remark 8.3 at the end of subsection 8.3.

3. Coarse graining and mesoscopic scale

3.1. **The FK representation.** We describe now the FK representation of Ising model. For a review of FK measures, we refer the reader to [Pi1], [Gri] and [ACCN].

The set of edges is $\mathbb{E} = \Big\{ \{x,y\} \mid x \sim y \Big\}$. For bond percolation, the configurations ω belong to $\Omega = \{0,1\}^{\mathbb{E}}$. An edge b in $\mathbb{E}$ is open if $\omega_b = 1$ and closed otherwise. To any subset Λ of $\mathbb{Z}^3$ and π included in $\partial\Lambda$, we associate a set of edges

$$[\Lambda]^\pi_e = \Big\{ \{x,y\} \mid x \sim y,\ x \in \Lambda,\ y \in \Lambda \cup \pi \Big\},$$

and the space of configurations in Λ is $\Omega^\pi_\Lambda = \{0,1\}^{[\Lambda]^\pi_e}$.

Let ω be a configuration in Ω, an open path $(x_1,\dots,x_n)$ is a finite sequence of distinct nearest neighbors $x_1,\dots,x_n$ such that on each edge $\omega_{\{x_i,x_{i+1}\}} = 1$. We write $\{A \leftrightarrow B\}$ for the event such that there exists an open path joining a site of A to one of B. A $*$-connected path $(x_1,\dots,x_n)$ is a finite sequence of distinct vertices such that $\|x_k - x_{k+1}\|_2$ is smaller than $\sqrt{3}$ for all k.

The connected components of the set of open edges of ω are called ω-clusters. The ω-cluster associated to the site i is denoted by $C_i(\omega)$.

Let us now describe the FK representation of the Ising model (see Edwards and Sokal [ES]). Let Λ be a finite subset in $\mathbb{Z}^3$ and π a subset of $\partial\Lambda$. The first step is to introduce a measure on Ω^π_Λ. A vertex x of Λ is called π-wired if it is connected by an open path to π. We call π-clusters the clusters defined with respect to the boundary condition π : a π-cluster is a connected set of open edges in Ω^π_Λ and we identify to be the same cluster all the clusters which are π-wired, i.e. connected to π. For a given p in $[0,1]$, we define the FK measure on Ω_Λ^π with boundary conditions π by

$$\Phi_\Lambda^{\pi,p}(\omega) = \frac{1}{Z_\Lambda^{\pi,p}} \left(\prod_{b \in [\Lambda]_e^\pi} (1-p)^{1-\omega_b} p^{\omega_b} \right) 2^{c^\pi(\omega)},$$

where $Z_\Lambda^{\pi,p}$ is a normalization factor and $c^\pi(\omega)$ is the number of clusters which are not π-wired.

If $\pi = \partial\Lambda$ then the boundary conditions are said to be wired and the corresponding FK measure on $\Omega_\Lambda^{\mathrm{w}}$ is denoted by $\Phi_\Lambda^{\mathrm{w,p}}$. If $\pi = \emptyset$, we write $\Phi_\Lambda^{\mathrm{f,p}}$ for the measure on $\Omega_\Lambda^{\mathrm{f}}$. For any subset Δ of Λ, we denote by $\mathcal{F}_\Lambda^\Delta$ the σ-field generated by finite dimensional cylinders associated with configurations in $\Omega_\Lambda^{\mathrm{w}}/\Omega_\Delta^{\mathrm{f}}$, then strong FKG property (see [Pi1]) implies that for every increasing function g supported by $\Omega_\Delta^{\mathrm{f}}$

$$\Phi_\Lambda^{\mathrm{w,p}} - \text{a.s}, \qquad \Phi_\Delta^{\mathrm{f,p}}(g) \le \Phi_\Lambda^{\mathrm{w,p}}(g \,|\, \mathcal{F}_\Lambda^\Delta) \le \Phi_\Delta^{\mathrm{w,p}}(g). \tag{3.1}$$

In particular, one has

$$\Phi_\Delta^{\mathrm{f,p}}(g) \le \Phi_\Lambda^{\mathrm{f,p}}(g) \le \Phi_\Lambda^{\mathrm{w,p}}(g) \le \Phi_\Delta^{\mathrm{w,p}}(g). \tag{3.2}$$

In order to recover the Gibbs measure $\mu_{\beta,\Lambda}^+$, we fix the percolation parameter $p_\beta = 1 - \exp(-\beta)$ and generate the edges configuration ω in $\Omega_\Lambda^{\mathrm{w}}$ according to the measure $\Phi_\Lambda^{\mathrm{w},\mathrm{p}_\beta}$.

Given ω, we associate to the wired cluster the sign $+1$ and equip randomly each ω-cluster with a color ± 1 with probability $\frac{1}{2}$ independently from the others. This amounts to introduce the measure P_Λ^ω on $\{-1,1\}^\Lambda$ such that the spin $\sigma_i = 1$ if $C_i(\omega)$ is π-wired and to be the chosen color of $C_i(\omega)$ otherwise. The Gibbs measure $\mu_{\beta,\Lambda}^+$ can be viewed as the first marginal of the coupled measure $P_\Lambda^\omega(\sigma)\Phi_\Lambda^{\mathrm{w},\mathrm{p}_\beta}(\omega)$

$$\forall \sigma_\Lambda \in \Sigma_\Lambda, \qquad \mu_{\beta,\Lambda}^+(\sigma_\Lambda) = \int_{\Omega_\Lambda^{\mathrm{w}}} P_\Lambda^\omega(\sigma) \Phi_\Lambda^{\mathrm{w},\mathrm{p}_\beta}(d\omega).$$

By abuse of notation, the joint measure will be also denoted by $\mu_{\beta,\Lambda}^+$.

As a consequence of this representation one has

$$m_\beta = \lim_{N\to\infty} \mu_{\beta,\Delta_N}^+(\sigma_0) = \lim_{N\to\infty} \Phi_{\Delta_N}^{\mathrm{w},\mathrm{p}_\beta}(\{0 \leftrightarrow \partial\Delta_N\}) = \Theta.$$

In the following, we use m_β or Θ depending on the context.

In Theorems 2.1 and 2.2, we consider only the case β large. The first reason to do so is to satisfy the hypothesis of Theorem 5.3 of [Gri] which implies that for β large enough

$$\lim_{N\to\infty} \Phi_{\Delta_N}^{\mathrm{f},\mathrm{p}_\beta}(\{0 \leftrightarrow \partial\Delta_N\}) = \lim_{N\to\infty} \Phi_{\Delta_N}^{\mathrm{w},\mathrm{p}_\beta}(\{0 \leftrightarrow \partial\Delta_N\}) = \Theta. \tag{3.3}$$

Throughout the paper we suppose that (3.3) holds. The assumption β large will also be useful for technical reasons in the proof of Lemma 4.3.

3.2. **Coarse graining.** We recall the renormalization procedure introduced by Pisztora [Pi1], [DP] for the FK measure. For our purposes, it is preferable to use an alternative construction of the coarse graining [Pi2]. The results of this section hold for β larger than $\hat\beta_c$, where $\hat\beta_c$ was defined in [Pi1] in terms of slab percolation threshold. Let $\tilde\beta_c$ be the smallest value such that (3.3) is satisfied and $\tilde\beta_c \geq \hat\beta_c$. It is conjectured that $\tilde\beta_c$ coincides with the critical value β_c.

Let $\gamma = \frac{1}{9}$ and $\alpha = \frac{1}{3} + \frac{\gamma}{9}$. In fact γ could be any positive parameter small enough. As in subsection 2.3, we partition the domain $\Delta_N = \{1, N\}^3$ into disjoint boxes $B(x, N^\alpha)$ of length N^α centered in x in $\mathcal{L}_{N^\alpha}$ (see (2.4) and (2.5)). For each x in $\mathcal{L}_{N^\alpha}$, we consider also the bigger box $B(x, \frac{5}{4}N^\alpha)$ containing $B(x, N^\alpha)$. Note that if x and y are $*$-neighbors in $\mathcal{L}_{N^\alpha}$ the boxes $B(x, \frac{5}{4}N^\alpha)$ and $B(y, \frac{5}{4}N^\alpha)$ overlap. Following [Pi1], we introduce events which occur on the box $B(x, \frac{5}{4}N^\alpha)$ for each x in $\mathcal{L}_{N^\alpha}$

$$U_x = \left\{\omega \in \Omega^{\mathrm{w}}_{\Delta_N} \;\middle|\; \text{there is a unique crossing cluster } C^* \text{ in } B(x, \frac{5}{4}N^\alpha)\right\}.$$

A crossing cluster is a cluster which intersects all the faces of the box.

$$\begin{aligned} R_x \quad &= \quad U_x \bigcap \Big\{\omega \in \Omega^{\mathrm{w}}_{\Delta_N} \;\Big|\; \text{every open path in } B(x, \frac{5}{4}N^\alpha) \text{ with diameter} \\ &\qquad\qquad \text{larger than } N^\gamma \text{ is contained in } C^*\Big\}, \end{aligned}$$

where the diameter of a subset A of $\mathbb{Z}^3$ is $\sup_{x,y\in A} \|x-y\|_1$. We also define

$$\begin{aligned} O_x \quad &= \quad R_x \bigcap \Big\{\omega \in \Omega^{\mathrm{w}}_{\Delta_N} \;\Big|\; C^* \text{ crosses every sub-box of side length } N^\gamma \\ &\qquad\qquad \text{contained in } B(x, \frac{5}{4}N^\alpha)\Big\}. \end{aligned}$$

Finally, we consider an event which imposes that the density of the crossing cluster is close to Θ (see (3.3)) in $B(x, N^\alpha)$ with accuracy $\zeta > 0$

$$V_x^\zeta = U_x \bigcap \Big\{\omega \in \Omega^{\mathrm{w}}_{\Delta_N} \;\Big|\; |C^*| \in [\Theta - \zeta, \Theta + \zeta]\, |B(x, N^\alpha)|\Big\}.$$

In the following, parameters α, γ will be fixed, therefore we omit the dependence on these parameters in notation. We will only consider different coarse graining for different values of ζ.

Each box $B(x, N^\alpha)$ is labelled by the random variable $Y_x^\zeta(\omega)$ depending only on the configuration ω in $\Omega^{\mathrm{w}}_{\Delta_N}$

$$\begin{aligned} Y_x^\zeta(\omega) &= 1 \qquad \text{if} \qquad \omega \in O_x \cap V_x^\zeta, \\ Y_x^\zeta(\omega) &= 0 \qquad \text{otherwise.} \end{aligned}$$

Let $\{x_1, \dots, x_\ell\}$ be vertices in $\mathcal{L}_{N^\alpha}$ not $*$-neighbors of x, then [Pi1] implies that there is an integer N_ζ such that

$$\forall N \geq N_\zeta, \qquad \Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Delta_N}(Y_x^\zeta = 0 \mid Y_{x_1}^\zeta, \dots, Y_{x_\ell}^\zeta) \leq \exp(-cN^\gamma) + \exp(-c'_\zeta N^\alpha),$$

where c'_ζ depends only on ζ and c is a constant. From [LSS] (Theorem 1.3), we deduce that for N large enough, the random variables $\{Y_x^\zeta\}$ are dominated by a Bernoulli product measure π_{ρ_N}

$$\pi_{\rho_N}(X=0) = \rho_N \leq \exp(-c_\zeta N^\gamma), \tag{3.4}$$

where c_ζ is a positive constant. A similar result was already stated in [Pi1].

The random variables Y^ζ are only related to ω, therefore the next step is to define a family of random variables which depend on (σ,ω). We denote by $\mathbf{M}_x$ the averaged magnetization in the box $B(x,N^\alpha)$

$$\mathbf{M}_x = \mathcal{M}_{\frac{x}{N}} = \frac{1}{N^{3\alpha}} \sum_{i\in B(x,N^\alpha)} \sigma_i. \tag{3.5}$$

Pisztora's results [Pi1] give a control of the deviation of the averaged magnetization from its equilibrium values $\pm m_\beta$ in the boxes $B(x,N^\alpha)$. If $Y_x^\zeta = 1$, this deviation comes from the random coloring of the small clusters (those of diameter less than N^γ) included in $B(x,N^\alpha)$: this random coloring is independent of the boxes around $B(x,N^\alpha)$. Let ζ be positive and define the new random variables $\{Z_x^\zeta\}$ which depend on the joint law of (σ,ω)

$$\begin{aligned} Z_x^\zeta(\sigma,\omega) &= \operatorname{sign}(C^*) \qquad \text{if} \qquad Y_x^\zeta(\omega)=1 \;\text{ and }\; |\mathbf{M}_x - \operatorname{sign}(C^*)\, m_\beta| < 2\zeta, \\ Z_x^\zeta(\sigma,\omega) &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

Combining results of [Pi1] and [LSS], we check that there is N_ζ such that for all N larger than N_ζ the random variables $\{|Z_x^\zeta|\}$, taking values in $\{0,1\}$, are dominated by a Bernoulli product measure $\pi_{\rho'_N}$

$$\pi_{\rho'_N}(X=0) = \rho'_N \leq \exp(-c_\zeta N^\gamma), \tag{3.6}$$

where c_ζ is a positive constant depending only on ζ. Since the setting is different from [Pi1], we sketch the proof in Appendix 8.2.

3.3. **Mesoscopic scale.** In subsection 2.3, we already used a homogenization procedure on the mesoscopic scale N^α. We introduce now a different mesoscopic representation which takes into account more details of the microscopic structure.

For a given ζ positive, we associate to any configuration (σ,ω) in $\Sigma_{\Delta_N} \times \Omega^{\mathrm{w}}_{\Delta_N}$ the piecewise constant function T^ζ on $\mathcal{T}$

$$\forall r \in \mathcal{T}, \qquad T_r^\zeta(\sigma,\omega) = Z_x^\zeta(\sigma,\omega) \qquad \text{if } \;\forall i, \quad -\frac{L}{2} < N r_i - x_i \leq \frac{L}{2}. \tag{3.7}$$

If (σ,ω) is close to an equilibrium phase on a mesoscopic scale then T^ζ has the sign of this phase. The 2 pure phases are represented by functions T^ζ constantly equal to 1 or -1. From (3.6), one knows that for β larger than $\tilde{\beta}_c$

$$\lim_{N\to\infty} \mu^+_{\beta,\Delta_N}(\{T_r^\zeta = 1, \qquad \forall r \in \mathcal{T}\}) = 1.$$

The next lemma proves that a knowledge of the asymptotic of T^ζ is sufficient to control the local magnetization $\mathcal{M}$. Therefore to prove Propositions 2.1, 2.2 and

2.3, it will be enough to replace $\mathcal{M}$ by T^ζ. The accuracy of the approximation depends on the parameter ζ which controls the coarse graining.

Lemma 3.1. *For any δ positive, we set $\zeta = \frac{1}{4}\delta$, then*

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N} \left(\int_{\mathcal{T}} \left| m_\beta T^\zeta_r - \mathcal{M}_r \right| dr \geq \delta \right) = -\infty.$$

Proof. One has

$$\int_{\mathcal{T}} \left| m_\beta T^\zeta_r - \mathcal{M}_r \right| dr \leq \left(\frac{N^\alpha}{N} \right)^3 \sum_{B(x,N^\alpha)} \left| m_\beta Z^\zeta_x - \mathbf{M}_x \right|,$$

this implies

$$\int_{\mathcal{T}} \left| m_\beta T^\zeta_r - \mathcal{M}_r \right| dr \leq \left(\frac{N^\alpha}{N} \right)^3 \sum_{B(x,N^\alpha)} 1_{Z^\zeta_x = 0} + 2\zeta.$$

Since ζ is small enough

$$\mu^+_{\beta,\Delta_N} \left(\int_{\mathcal{T}} \left| m_\beta T^\zeta_r - \mathcal{M}_r \right| dr \geq \delta \right) \leq \mu^+_{\beta,\Delta_N} \left(\#\{Z^\zeta_x = 0\} \geq \frac{\delta}{2} N^{3(1-\alpha)} \right),$$

where $\#\{Z^\zeta_x = 0\}$ is the number of boxes with label 0. Therefore the lemma above will be a consequence of

Lemma 3.2. *For any δ and ζ positive*

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N} \left(\#\{Z^\zeta_x = 0\} \geq \delta N^{3(1-\alpha)} \right) = -\infty.$$

Proof. One has

$$\mu^+_{\beta,\Delta_N} \left(\#\{Z^\zeta_x = 0\} \geq \delta N^{3(1-\alpha)} \right) \leq \sum_{k=\delta N^{3(1-\alpha)}}^{N^{3(1-\alpha)}} \mu^+_{\beta,\Delta_N} \left(\#\{Z^\zeta_x = 0\} = k \right).$$

The random variables $|Z^\zeta_x|$ are dominated by independent variables (3.6), thus for N large enough

$$\mu^+_{\beta,\Delta_N} \left(\#\{Z^\zeta_x = 0\} \geq \delta N^{3(1-\alpha)} \right) \leq 2^{N^{3(1-\alpha)}} \exp(-c_\zeta \delta N^{3(1-\alpha)+\gamma}).$$

This implies

$$\mu^+_{\beta,\Delta_N} \left(\#\{Z^\zeta_x = 0\} \geq \delta N^{3(1-\alpha)} \right) \leq \exp\left(\ln 2\, N^{3(1-\alpha)} - c_\zeta \delta N^{2+\frac{2}{3}\gamma} \right).$$

As $3(1-\alpha) < 2$, the entropic factor is negligible and the Lemma follows. □

□

4. Surface tension

As explained in the introduction, the main problem to derive Wulff construction is to recover surface tension from general boundary conditions. In this section we rewrite the surface tension in terms of the FK measure and prove that this new expression depends weakly on boundary conditions. This expression is reminiscent to the one introduced by Cerf [Ce] in the context of percolation.

We keep notation of subsection 2.2. Throughout this section, we fix the direction $\vec{n}$ and without loss of generality, we set $h = 1$. We also suppose that $\frac{f(N)}{\log(N)}$ diverges to infinity as N goes to infinity.

4.1. First step.

The next lemma will be useful to prove Proposition 2.2

Lemma 4.1. *Let $\{\partial^+\Lambda_N \not\leftrightarrow \partial^-\Lambda_N\}$ be the event such that there is no open path inside $[\Lambda_N]_e^{\mathrm{w}}$ joining $\partial^+\Lambda_N$ to $\partial^-\Lambda_N$. Then*

$$\tau(\vec{n}) = \lim_{N\to\infty} -\frac{1}{N^2} \log \Phi_{\Lambda_N}^{\mathrm{w},p_\beta}\Big(\{\partial^+\Lambda_N \not\leftrightarrow \partial^-\Lambda_N\}\Big). \tag{4.1}$$

Note that the event $\{\partial^+\Lambda_N \not\leftrightarrow \partial^-\Lambda_N\}$ takes only into account the paths inside Λ_N and not the identification produced by wired boundary conditions.

Proof. We rewrite the quantities in terms of the FK representation. A well known argument implies that for $p_\beta = 1 - \exp(-\beta)$

$$Z_{\Lambda_N}^{+} = \sum_{\sigma\in\Sigma_{\Lambda_N}} \prod_{<x,y>\in[\Lambda_N]_e^{\mathrm{w}}} \exp\Big(\beta(\delta_{\sigma_x,\sigma_y} - 1)\Big) = \sum_{\omega\in\Omega_{\Lambda_N}^{\mathrm{w}}} \prod_{b\in[\Lambda_N]_e^{\mathrm{w}}} (1-p_\beta)^{1-\omega_b} p_\beta^{\omega_b} 2^{c^{\mathrm{w}}(\omega)},$$

where $c^{\mathrm{w}}(\omega)$ is the number of clusters which are not wired.

We prove now an equivalent formula for

$$Z_{\Lambda_N}^{+,-} = \sum_{\sigma\in\Sigma_{\Lambda_N}} \prod_{<x,y>\in[\Lambda_N]_e^{\mathrm{w}}} \exp\Big(\beta(\delta_{\sigma_x,\sigma_y} - 1)\Big),$$

where boundary conditions are equal to 1 on $\partial\Lambda_N^+$ and to -1 on $\partial\Lambda_N^-$. We get

$$Z_{\Lambda_N}^{+,-} = \sum_{\sigma\in\Sigma_{\Lambda_N}} \prod_{<x,y>\in[\Lambda_N]_e^{\mathrm{w}}} \Big(1 - p_\beta + p_\beta\,\delta_{\sigma_x,\sigma_y}\Big),$$

this gives

$$Z_{\Lambda_N}^{+,-} = \sum_{\sigma\in\Sigma_{\Lambda_N}} \sum_{\omega\in\Omega_{\Lambda_N}^{\mathrm{w}}} \prod_b (1-p_\beta)^{1-\omega_b} p_\beta^{\omega_b} \prod_{\substack{b=<x,y>\\ \omega_b=1}} \delta_{\sigma_x,\sigma_y}.$$

Therefore

$$Z_{\Lambda_N}^{+,-} = \sum_{\omega\in\Omega_{\Lambda_N}^{\mathrm{w}}} \prod_b (1-p_\beta)^{1-\omega_b} p_\beta^{\omega_b} \sum_{\sigma\in\Sigma_{\Lambda_N}} \prod_{\substack{b=<x,y>\\ \omega_b=1}} \delta_{\sigma_x,\sigma_y}.$$

The boundary conditions imply that configurations ω containing a path joining $\partial^+\Lambda_N$ to $\partial^-\Lambda_N$ are not taken into account. We keep the definition of wired boundary conditions identifying all the clusters which touch the boundary $\partial\Lambda_N$

$$Z_{\Lambda_N}^{+,-} = \sum_{\omega\in\Omega_{\Lambda_N}^{\mathrm{w}}} 1_{\{\partial^+\Lambda_N \not\leftrightarrow \partial^-\Lambda_N\}}(\omega) \prod_b (1-p_\beta)^{1-\omega_b} p_\beta^{\omega_b} 2^{c^{\mathrm{w}}(\omega)}.$$

Taking the ratio $\frac{Z^{+,-}_{\Lambda_N}}{Z^{+}_{\Lambda_N}}$, we recover $\Phi^{\mathrm{w,p}_\beta}_{\Lambda_N}\Big(\{\partial^+\Lambda_N \not\leftrightarrow \partial^-\Lambda_N\}\Big)$. □

4.2. **Second step.** In the following, we denote by $\Lambda'_N = \bar\Lambda(N,N,\frac{1}{2}f(N))\cap\mathbb{Z}^3$ the parallelepiped included in Λ_N.

Lemma 4.2. *One has*

$$\tau(\vec n) = \lim_{N\to\infty} -\frac{1}{N^2}\log\Phi^{\mathrm{w,p}_\beta}_{\Lambda_N}\Big(\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}\Big).$$

Proof. By definition $\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}$ is included in $\{\partial^+\Lambda_N \not\leftrightarrow \partial^-\Lambda_N\}$. Therefore, Lemma 4.1 implies

$$\tau(\vec n) \le \liminf_{N\to\infty} -\frac{1}{N^2}\log\Phi^{\mathrm{w,p}_\beta}_{\Lambda_N}\Big(\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}\Big).$$

Let us prove the reverse inequality. The event $\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}$ is decreasing and supported by $[\Lambda'_N]^{\mathrm{w}}_e$. Thus (3.2) gives

$$\Phi^{\mathrm{w,p}_\beta}_{\Lambda_N}\Big(\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}\Big) \ge \Phi^{\mathrm{w,p}_\beta}_{\Lambda'_N}\Big(\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}\Big). \tag{4.2}$$

Since surface tension does not depend on the function f, Lemma 4.1 implies

$$\tau(\vec n) = \lim_{N\to\infty} -\frac{1}{N^2}\log\Phi^{\mathrm{w,p}_\beta}_{\Lambda'_N}\Big(\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}\Big).$$

Thus using (4.2), the lemma is proven. □

4.3. **Third step.** Let $\Phi^{\mathrm{f,w}}_{\Lambda_N}$ be the FK measure with wired boundary conditions on the sides of Λ_N parallel to $\vec n$ and free on the sides orthogonal to $\vec n$.

Lemma 4.3. *There is a constant β_0 independent of $\vec n$ and f such that for any β larger than β_0*

$$\tau(\vec n) = \lim_{N\to\infty} -\frac{1}{N^2}\log\Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}\Big).$$

Proof. The event $\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}$ is denoted by S_N. Applying (3.1), one observes that $\Phi^{\mathrm{f,w}}_{\Lambda_N}(S_N) \ge \Phi^{\mathrm{w,p}_\beta}_{\Lambda_N}(S_N)$ so that Lemma 4.2 implies

$$\tau(\vec n) \ge \limsup_{N\to\infty} -\frac{1}{N^2}\log\Phi^{\mathrm{f,w}}_{\Lambda_N}(S_N). \tag{4.3}$$

To prove the reverse inequality, we introduce the slabs $\mathrm{Sl_N}^+$ and $\mathrm{Sl_N}^-$ in Λ_N

$$\begin{aligned}
\mathrm{Sl_N}^+ &= \Big(\bar\Lambda\Big(N,N,\frac{1}{10}f(N)\Big) + \frac{3}{8}f(N)\vec n\Big)\cap\mathbb{Z}^3,\\
\mathrm{Sl_N}^- &= \Big(\bar\Lambda\Big(N,N,\frac{1}{10}f(N)\Big) - \frac{3}{8}f(N)\vec n\Big)\cap\mathbb{Z}^3.
\end{aligned}$$

For any ω in $\Omega^{\mathrm{w}}_{\Lambda_N}$, we call a vertex x white if $\omega_b = 1$ for all edge b incident with x and black otherwise. Let A^+_N (resp A^-_N) be the event such that there is a surface of white vertices which crosses the slab $\mathrm{Sl_N}^+$ (resp $\mathrm{Sl_N}^-$) and separates the two sides of the slab orthogonal to $\vec n$. Equivalently, one can define ${A^+_N}^c$ as the set of configurations ω_{Λ_N} which contain a $*$-connected path of black vertices intersecting the 2 sides of $\mathrm{Sl_N}^+$ orthogonal to $\vec n$.

One has

$$\Phi^{\mathrm{f,w}}_{\Lambda_N}(S_N) = \Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(S_N \cap A^+_N \cap A^-_N\Big) + \Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(S_N \cap (A^+_N \cap A^-_N)^c\Big). \tag{4.4}$$

First we estimate the last term in the RHS. It is enough to prove an upper bound for $\Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(S_N \cap {A^+_N}^c\Big)$. The events S_N and ${A^+_N}^c$ have distinct supports, so that we can take the conditional expectation with respect to $\partial\omega$, the configuration outside ${\mathrm{Sl_N}}^+$

$$\Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(S_N \cap {A^+_N}^c\Big) = \Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(S_N\, \Phi^{\partial\omega}_{{\mathrm{Sl_N}}^+}({A^+_N}^c)\Big).$$

Since ${A^+_N}^c$ is decreasing, (3.1) implies

$$\Phi^{\partial\omega}_{{\mathrm{Sl_N}}^+}({A^+_N}^c) \le \Phi^{\mathrm{f,p}_\beta}_{{\mathrm{Sl_N}}^+}({A^+_N}^c),$$

where the free boundary conditions are outside the domain $[{\mathrm{Sl_N}}^+]^{\mathrm{w}}_e$. In order to control this term, we use a Peierls argument (see [Gri] p. 1486). By the comparison result of Aizenman, Chayes, Chayes and Newman [ACCN], the above probability is bounded by the percolation (product) measure $\bar\Phi^{p'_\beta}_{{\mathrm{Sl_N}}^+}$ with $p'_\beta = \frac{p_\beta}{p_\beta+(1-p_\beta)^2}$

$$\Phi^{\mathrm{f,p}_\beta}_{{\mathrm{Sl_N}}^+}({A^+_N}^c) \le \bar\Phi^{\mathrm{p}'_\beta}_{{\mathrm{Sl_N}}^+}({A^+_N}^c).$$

We choose β large enough so that p'_β is close to 1. Then Peierls estimate holds and there is a constant $c > 0$ such that the probability that a $*$-connected black path joins 2 vertices x and y on both sides of ${\mathrm{Sl_N}}^+$ is less than $\exp(-\frac{c}{10} f(N))$. This comes from the fact that the length of such a path is at least $\frac{1}{10} f(N)$

$$\bar\Phi^{\mathrm{p}'_\beta}_{{\mathrm{Sl_N}}^+}({A^+_N}^c) \le N^2 \exp\Big(-\frac{c}{10} f(N)\Big).$$

One finally obtains

$$\Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(S_N \cap {A^+_N}^c\Big) \le N^2 \exp\Big(-\frac{c}{10} f(N)\Big)\Phi^{\mathrm{f,w}}_{\Lambda_N}(S_N). \tag{4.5}$$

We turn now to the estimate of $\Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(S_N \cap A^+_N \cap A^-_N\Big)$. For a given ω in A^+_N, we are going to define the surface $\mathcal{S}^+(\omega)$ of white vertices which is the closest to the "upper" side of ${\mathrm{Sl_N}}^+$.

First we construct the black set $\mathcal{B}^+(\omega)$ as follows : $\mathcal{B}^+(\omega)$ contains the vertices in the "upper" side of $\partial{\mathrm{Sl_N}}^+$, i.e. the vertices at distance less than 2 of the hyperplan parallel to $(\vec{e_1}, \vec{e_2})$ and centered in $\frac{17}{40} f(N)\vec{n}$. Furthermore $\mathcal{B}^+(\omega)$ contains all the black vertices linked by a $*$-connected path of black vertices to the boundary of the "upper" side of ${\mathrm{Sl_N}}^+$. A vertex x is in $\mathcal{S}^+(\omega)$ if it belongs to the boundary of $\mathcal{B}^+(\omega)$ and if there is a path of vertices joining x to 0 without crossing $\mathcal{B}^+(\omega)$. By construction the vertices in $\mathcal{S}^+(\omega)$ are white. In the same way we define $\mathcal{S}^-(\omega)$ as the surface of white vertices which is the closest to the "lower" side of ${\mathrm{Sl_N}}^-$, i.e. to the set of the vertices at distance less than 2 of the hyperplan parallel to $(\vec{e_1}, \vec{e_2})$ and centered in $-\frac{17}{40} f(N)\vec{n}$.

The region between the surfaces $\mathcal{S}^+$ and $\mathcal{S}^-$ is denoted by $\mathcal{S}_N$ and by construction Λ'_N is included in $\mathcal{S}_N$. Therefore we can consider the conditional expectation of S_N with respect to the configurations outside $\mathcal{S}_N$ (the measurability is discussed in [Gri] p. 1487). Since $\mathcal{S}^+$ and $\mathcal{S}^-$ contain only white vertices, one gets

$$\Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(S_N \cap A_N^+ \cap A_N^-\Big) = \Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(A_N^+ \cap A_N^- \; \Phi^{\mathrm{w,p}_\beta}_{\mathcal{S}_N}(S_N)\Big).$$

The event S_N is decreasing, thus strong FKG property (3.2) implies

$$\Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(S_N \cap A_N^+ \cap A_N^-\Big) \le \Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(A_N^+ \cap A_N^-\Big)\,\Phi^{\mathrm{w,p}_\beta}_{\Lambda_N}(S_N) \le \Phi^{\mathrm{w,p}_\beta}_{\Lambda_N}(S_N). \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), we obtain

$$\Phi^{\mathrm{f,w}}_{\Lambda_N}(S_N) \le \Phi^{\mathrm{w,p}_\beta}_{\Lambda_N}(S_N) + N^2 \exp\Big(-\frac{c}{10} f(N)\Big)\Phi^{\mathrm{f,w}}_{\Lambda_N}(S_N).$$

Applying Lemma 4.2, we get

$$\liminf_{N\to\infty} -\frac{1}{N^2}\log \Phi^{\mathrm{f,w}}_{\Lambda_N}(S_N) \ge \tau(\vec{n}).$$

The Lemma is completed. □

4.4. **Fourth step.** Now, we will modify the boundary conditions and prove that the surface tension remains unchanged. The following lemma will be important in the proof of Proposition 2.1. It requires the assumption β large.

We denote by $\partial^{\mathrm{top}}\Lambda'_N$ (resp $\partial^{\mathrm{bot}}\Lambda'_N$) the face of $\partial^+\Lambda'_N$ (resp $\partial^-\Lambda'_N$) orthogonal to $\vec{n}$. Let $\{\partial^{\mathrm{top}}\Lambda'_N \not\leftrightarrow \partial^{\mathrm{bot}}\Lambda'_N\}$ be the event such that there is no open path inside $[\Lambda'_N]^{\mathrm{w}}_e$ connecting $\partial^{\mathrm{top}}\Lambda'_N$ to $\partial^{\mathrm{bot}}\Lambda'_N$. Finally, we set

$$\delta = \limsup_{N\to\infty} \frac{f(N)}{N},$$

and suppose that δ is finite.

Lemma 4.4. *There is a constant β_0 independent of $\vec{n}$ and f such that for any β larger than β_0*

$$\limsup_{N\to\infty} \frac{1}{N^2}\log\left(\sup_\pi \; \Phi^{\pi,\mathrm{p}_\beta}_{\Lambda_N}\Big(\{\partial^{\mathrm{top}}\Lambda'_N \not\leftrightarrow \partial^{\mathrm{bot}}\Lambda'_N\}\Big)\right) \le -\tau(\vec{n}) + c_\beta\delta,$$

where the constant c_β depends only on β. The above inequality holds uniformly over the boundary conditions π outside $[\Lambda_N]^{\mathrm{w}}_e$.

Proof. As $\{\partial^{\mathrm{top}}\Lambda'_N \not\leftrightarrow \partial^{\mathrm{bot}}\Lambda'_N\}$ is decreasing, strong FKG property (3.1) implies

$$\sup_\pi \Phi^{\pi,\mathrm{p}_\beta}_{\Lambda_N}\Big(\{\partial^{\mathrm{top}}\Lambda'_N \not\leftrightarrow \partial^{\mathrm{bot}}\Lambda'_N\}\Big) \le \Phi^{\mathrm{f,p}_\beta}_{\Lambda_N}\Big(\{\partial^{\mathrm{top}}\Lambda'_N \not\leftrightarrow \partial^{\mathrm{bot}}\Lambda'_N\}\Big),$$

where the free boundary conditions are outside $[\Lambda_N]^{\mathrm{w}}_e$. Note also that

$$\Phi^{f,\mathrm{p}_\beta}_{\Lambda_N}\Big(\{\partial^{\mathrm{top}}\Lambda'_N \not\leftrightarrow \partial^{\mathrm{bot}}\Lambda'_N\}\Big) \le 2^{4f(N)N}\;\Phi^{\mathrm{f,w}}_{\Lambda_N}\Big(\{\partial^{\mathrm{top}}\Lambda'_N \not\leftrightarrow \partial^{\mathrm{bot}}\Lambda'_N\}\Big). \tag{4.7}$$

We fix a configuration ω in $\{\partial^{\mathrm{top}}\Lambda'_N \not\leftrightarrow \partial^{\mathrm{bot}}\Lambda'_N\}$. The inner boundary of Λ_N is defined by

$$\partial^*\Lambda_N = \{x \in \Lambda_N \mid \exists y \notin \Lambda_N,\; y \sim x\}.$$

For any vertex x on the sides of $\partial^*\Lambda_N$ parallel to $\vec{n}$, we modify the edges of ω incident with x into closed edges and denote by $\bar{\omega}$ the new configuration. By construction $\bar{\omega}$ belongs to $\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}$. Noticing that the number of edges which have been modified is smaller than $50f(N)N$ and using (4.7), one has

$$\Phi^{\mathrm{f},\mathrm{p}_\beta}_{\Lambda_N}\Big(\{\partial^{\mathrm{top}}\Lambda'_N \not\leftrightarrow \partial^{\mathrm{bot}}\Lambda'_N\}\Big) \leq \exp(c_\beta f(N)N)\ \Phi^{\mathrm{f},\mathrm{w}}_{\Lambda_N}\Big(\{\partial^+\Lambda'_N \not\leftrightarrow \partial^-\Lambda'_N\}\Big),$$

where c_β depends only on β. Using Lemma 4.3, we complete the proof. □

5. Upper bound : Proposition 2.1

Throughout this Section, we fix u in $\mathrm{BV}(\mathcal{T}, \{+1,-1\})$ such that $\mathcal{F}(u)$ is finite. We split the proof into 3 steps.

5.1. **Approximation.** First we suppose that ∂u is included in the interior of $\mathcal{T}$. The general case will be treated in subsection 5.3. We approximate the boundary of u with a finite number of parallelepipeds. Similar Theorems were already stated in [ABCP] and [Ce]. The following result is proven in Appendix 8.3.

Theorem 5.1. *For any δ positive, there exists h positive such that there are ℓ disjoint parallelepipeds $R^1,\dots,R^\ell$ included in $\mathcal{T}$ with cubic basis $B^1,\dots,B^\ell$ of size h and height δh. The basis B^i divides R^i in 2 parallelepipeds $R^{i,+}$ and $R^{i,-}$ and we denote by $\vec{n}_i$ the normal to B^i. Furthermore, the parallelepipeds satisfy the following properties*

$$\int_{R^i} |\chi_{R^i}(r) - u(r)|\, dr \leq \delta\, \mathrm{vol}(R^i) \quad \text{and} \quad \Big|\sum_{i=1}^{\ell} \int_{B^i} \tau(\vec{n}_i)\, d\mathcal{H}_x - \mathcal{F}(u)\Big| \leq \delta,$$

where $\chi_{R^i} = 1_{R^{i,+}} - 1_{R^{i,-}}$ and the volume of R^i is $\mathrm{vol}(R^i) = \delta h^3$. The area $\int_{B^i} d\mathcal{H}_x$ of B^i is h^2.

We fix δ positive. The approximation procedure implies

$$\lim_{\delta'\to 0} \limsup_{N\to\infty} \frac{1}{N^2} \quad \log \quad \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(u,\delta')\Big) \leq$$
$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \bigcap_{i=1}^{\ell} \mathcal{V}(R^i, 2\delta \mathrm{vol}(R^i))\Big),$$

where the ε-neighborhood of R^i is

$$\mathcal{V}(R^i,\varepsilon) = \Big\{ v \in L^1\Big(\mathcal{T}, [-\frac{1}{m_\beta}, \frac{1}{m_\beta}]\Big) \ \Big|\ \int_{R^i} |v(r) - \chi_{R^i}(r)|\, dr \leq \varepsilon \Big\}.$$

According to Lemma 3.1, there is ζ small enough, depending on δ and h, such that

$$\limsup_{N\to\infty} \frac{1}{N^2} \quad \log \quad \mu^+_{\beta,\Delta_N}\Big(\frac{\mathcal{M}}{m_\beta} \in \bigcap_{i=1}^{\ell} \mathcal{V}(R^i, 2\delta \mathrm{vol}(R^i))\Big) \leq$$
$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(T^\zeta \in \bigcap_{i=1}^{\ell} \mathcal{V}(R^i, 3\delta \mathrm{vol}(R^i))\Big).$$

Therefore to prove Proposition 2.1, it is enough to show that

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(T^\zeta \in \bigcap_{i=1}^{\ell} \mathcal{V}(R^i, 3\delta \mathrm{vol}(R^i))\Big) \leq -\mathcal{F}(u) + C_{\beta,u}\delta,$$

where the constant $C_{\beta,u}$ depends only on β and u. Each box can be labelled by 3 values $0, \pm 1$, thus the number of configurations T^ζ is less than $3^{N^{3(1-\alpha)}}$. As $3(1-\alpha) < 2$, this term has no entropic effect. Thus it remains to check

$$\limsup_{N\to\infty} \frac{1}{N^2} \log\left(\sup_{T^\zeta \in \mathcal{U}^\delta} \mu^+_{\beta,\Delta_N}\left(\{T^\zeta\}\right)\right) \leq -\mathcal{F}(u) + C_{\beta,u}\delta, \tag{5.1}$$

where $\mathcal{U}^\delta$ denotes $\bigcap_{i=1}^{\ell} \mathcal{V}(R^i, 3\delta \mathrm{vol}(R^i))$ and $\{T^\zeta\}$ is the set of configurations (σ,ω) which realize T^ζ.

5.2. **Minimal section argument.** The microscopic domain associated to R^i is $R^i_N = NR^i \cap \Delta_N$. We also set $R^{i,+}_N = NR^{i,+} \cap \Delta_N$ and $R^{i,-}_N = R^i_N / R^{i,+}_N$. Let $\mathbb{L}^i_N$ be the subset of boxes $B(x, N^\alpha)$ intersecting R^i_N. The number of boxes intersecting $R^{i,+}_N$ is

$$\mathcal{N}^{i,+}_N = N^{3(1-\alpha)} \mathrm{vol}(R^i) \left(\frac{1}{2} + o(N^{2(1-\alpha)})\right), \tag{5.2}$$

where the error term $o(N^{2(1-\alpha)})$ goes to 0 as N increases. This error is due to the fact that the partition may not be exact on the sides of $R^{i,+}_N$. A similar estimate holds for $\mathcal{N}^{i,-}_N$, the number of boxes intersecting $R^{i,-}_N$.

Let us fix T^ζ in $\bigcap_{i=1}^{\ell} \mathcal{V}(R^i, 3\delta\mathrm{vol}(R^i))$. To any configuration (σ,ω) in $\{T^\zeta\}$, we associate the set of *bad* boxes which are the boxes in $\mathbb{L}^i_N$ labelled by $Z^\zeta_x = 0$ and the ones intersecting $R^{i,+}_N$ (resp $R^{i,-}_N$) labelled by $Z^\zeta_x = -1$ (resp $Z^\zeta_x = 1$).

We will now use the L^1-constraint to derive bounds on the number of *bad* boxes. The number of boxes in $\mathbb{L}^i_N$ not included in R^i_N is smaller than $50h^2N^{2(1-\alpha)}$, therefore

$$\left| N^{3(1-\alpha)} \int_{R^{i,+}} |T^\zeta_r - 1|\, dr - \sum_{B(x,N^\alpha)\cap R^{i,+}_N \neq \emptyset} |Z^\zeta_x - 1| \right| \leq 100h^2N^{2(1-\alpha)}.$$

Since T^ζ belongs to $\mathcal{V}(R^i, 3\delta\mathrm{vol}(R^i))$, one gets from (5.2) that for N large enough the number of *bad* boxes in $R^{i,+}_N$ is smaller than $10\delta\mathcal{N}^{i,+}_N$. In the same way, we check that the number of *bad* boxes in $R^{i,-}_N$ is smaller than $10\delta\mathcal{N}^{i,-}_N$.

Let $R^{i\,\prime}$ be the parallelepiped included in R^i with basis B^i and height $\frac{\delta}{2}h$. Its microscopic counterpart is $R^{i\,\prime}_N$. We will apply the minimal section argument introduced in [BBBP] and relate the expectation of $\{T^\zeta\}$ to the one of $\bigcap_{i=1}^{\ell}\{\partial^{\mathrm{top}} R^{i\,\prime}_N \not\leftrightarrow \partial^{\mathrm{bot}} R^{i\,\prime}_N\}$.

For any integer k, we set $B^{i,k} = B^i + 10\frac{k}{N^{1-\alpha}}\vec{n}_i$. Let $B^{i,k}_N$ be the microscopic subset of $R^{i\,\prime}_N$ associated to $B^{i,k}$

$$B^{i,k}_N = \left\{ j \in R^{i\,\prime}_N \mid \exists r \in B^{i,k}, \qquad \|j - Nr\|_1 \leq 10 \right\}.$$

We define $\mathcal{B}^k_i$ as the smallest connected set of mesoscopic boxes containing

$$\left\{ B(y, N^\alpha) \in \mathbb{L}^i_N \mid \qquad B(y,N^\alpha) \cap B^{i,k}_N \neq \emptyset \right\}.$$

By construction the $\mathcal{B}_i^k$ are disjoint surfaces of boxes. For k positive, let $n_i^+(k)$ be the number of *bad* boxes in $\mathcal{B}_i^k$ and define

$$n_i^+ = \min\Big\{n_i^+(k); \qquad 0 < k < \frac{\delta h}{30} N^{1-\alpha}\Big\}.$$

Call k^+ the smallest location where the minimum is achieved and define the minimal section as $\mathcal{B}_i^{k^+}$. For k non positive, we denote by $\mathcal{B}_i^{k^-}$ the minimal section in $R_N^{i,-}$ and n_i^- the number of *bad* boxes in $\mathcal{B}_i^{k^-}$.

For any configuration (σ, ω) in $\{T^\zeta\}$, we will check that the total number of *bad* boxes is bounded by

$$\sum_{i=1}^{\ell} n_i^+ + n_i^- \leq C_u \delta N^{2(1-\alpha)}, \tag{5.3}$$

where C_u is a constant depending only on u. By definition, one has

$$\frac{\delta h}{30} N^{1-\alpha}\, n_i^+ \leq \sum_{B(x,N^\alpha)\cap R_N^{i,+}\neq\emptyset} 1_{Z_x^\zeta \neq 1} \leq 10\delta \mathcal{N}_N^{i,+}.$$

For N large enough, (5.2) implies that $n_i^+ \leq 10^3 \delta h^2 N^{2(1-\alpha)}$. Note that h^2 is in fact the area of B^i, therefore the approximation procedure implies that ℓh^2 is bounded by a constant depending on the perimeter of ∂u. Thus (5.3) holds.

We are now going to use all the previous estimates. We define

$$\mathcal{A} = \Big\{\omega \in \Omega^{\mathrm{w}}_{\Delta_N} \;\Big|\; \exists \sigma \text{ such that } (\sigma,\omega) \in \{T^\zeta\}\Big\}.$$

Any configuration ω in $\mathcal{A}$ will be mapped into $\bar\omega$ by the following procedure. For any *bad* box $B(x, N^\alpha)$ in the minimal sections, we change the open edges of ω located on the sides of the box $B(x, \frac{5}{4}N^\alpha)$ into closed edges. The new configuration $\bar\omega$ belongs to $\{\partial^{\mathrm{top}} {R_N^i}' \not\leftrightarrow \partial^{\mathrm{bot}} {R_N^i}'\}$, because any open path of ω which joins $\partial^{\mathrm{bot}} {R_N^i}'$ to $\partial^{\mathrm{top}} {R_N^i}'$ intersects at least one of the minimal section on a *bad* block and therefore is cut by the above procedure. Let $\mathbf{C}$ be an open path of ω joining $\partial^{\mathrm{top}} {R_N^i}'$ to $\partial^{\mathrm{bot}} {R_N^i}'$ and suppose that $\mathbf{C}$ crosses the minimal sections without intersecting a *bad* box. Then $\mathbf{C}$ intersects the boxes $B(x^+, N^\alpha)$ and $B(x^-, N^\alpha)$ in $\mathcal{B}_i^{k^+}$ and $\mathcal{B}_i^{k^-}$ with labels $Y^\zeta_{x^+} = Y^\zeta_{x^-} = 1$. This would imply that the crossing clusters of $B(x^+, N^\alpha)$ and $B(x^-, N^\alpha)$ are connected to $\mathbf{C}$, so that $Z^\zeta_{x^+} = Z^\zeta_{x^-}$. Therefore one of these boxes has to be a *bad* box.

Around the *bad* boxes, we change at most $20(n_i^+ + n_i^-)N^{2\alpha}$ edges. From (5.3) the total number of edges involved in the previous procedure is bounded by $100 C_u \delta N^2$. Therefore we get

$$\mu^+_{\beta,\Delta_N}(\{T^\zeta\}) \leq \Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Delta_N}\big(\mathcal{A}\big) \leq \exp\big(C_{\beta,u}\delta N^2\big)\Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Delta_N}\Big(\bigcap_{i=1}^{\ell}\{\partial^{\mathrm{top}} {R_N^i}' \not\leftrightarrow \partial^{\mathrm{bot}} {R_N^i}'\}\Big),$$

where the constant $C_{\beta,u}$ depends only on β and u.

Conditioning outside each domain R_N^i and using Lemma 4.4, we derive

$$\limsup_{N\to\infty} \frac{1}{N^2} \log\left(\sup_{T^\zeta \in \mathcal{U}^\delta} \Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Delta_N}\Big(\{T^\zeta\}\Big)\right) \leq -\sum_{i=1}^{\ell} \int_{B_i} \tau(\vec{n}_i)\, d\mathcal{H}_x + C_{\beta,u}\,\delta + c_\beta \ell h^2\,\delta,$$

where c_β was defined in Lemma 4.4. Noticing that ℓh^2 is bounded in terms of the perimeter of u and using Theorem 5.1, we derive (5.1).

5.3. **Boundary conditions.** Let U be the intersection of the reduced boundary $\partial^* u$ and of $\partial\mathcal{T}$. Suppose that U has a positive 2 dimensional Hausdorff measure. In this case we cannot approximate the surface U as in Theorem 5.1 with parallelepipeds included in $\mathcal{T}$. We state a variant of Theorem 5.1 proven in Appendix 8.3

Theorem 5.2. *For any δ positive, there exist h positive and ℓ disjoint squares $B^1,\dots,B^\ell$ in $\partial\mathcal{T}$ of size h and normal $\vec{n}_i$ such that*

$$\left| \sum_{i=1}^{\ell} \int_{B^i} \tau(\vec{n}_i)\, d\mathcal{H}_x - \int_U \tau(\vec{n}_x)\, d\mathcal{H}_x \right| \leq \delta.$$

Furthermore, there are ℓ disjoint parallelepipeds $R^1,\dots,R^\ell$ included in $\mathcal{T}$ such that one of the face of R^i is B^i and the height of R^i is δh. The parallelepipeds also satisfy

$$\forall i \leq \ell, \qquad \int_{R^i} |1+u_r|\, dr \leq \delta\, \mathrm{vol}(R^i).$$

The proof of the upper bound is based on local estimates in each parallelepiped, thus we will simply explain how to adapt the previous proof to obtain

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mu^+_{\beta,\Delta_N}\Big(T^\zeta \in \bigcap_{i=1}^{\ell} \mathcal{V}(R^i, \delta \mathrm{vol}(R^i))\Big) \leq -\int_U \tau(\vec{n}_x)\, d\mathcal{H}_x + C_{\beta,u}\delta, \tag{5.4}$$

where $C_{\beta,u}$ is a constant and $\mathcal{V}(R^i, \delta\mathrm{vol}(R^i))$ is

$$\mathcal{V}(R^i, \delta\mathrm{vol}(R^i)) = \left\{ v \in L^1\Big(\mathcal{T}, [-\frac{1}{m_\beta}, \frac{1}{m_\beta}]\Big) \;\Big|\; \int_{R^i} |v_r + 1|\, dr \leq \delta\mathrm{vol}(R^i) \right\}.$$

Combining estimates (5.1) and (5.4), one derives Proposition 2.1 for any function of bounded variation u in $\mathcal{T}$.

Let R_N^i be the microscopic set associated to R^i. The set of *bad* boxes is the set of boxes $B(x, N^\alpha)$ intersecting R_N^i and labelled by 0 or 1. Using the L^1 constraint, we see that the number of *bad* boxes is smaller than $10\delta^2 h^3 N^{3(1-\alpha)}$. Let $R^{i\,\prime}$ be the parallelepiped included in R^i with height $\frac{\delta h}{2}$ and such that one of its faces is B^i. The previous argument implies that the minimal section contains less than $100\delta h^2 N^{2(1-\alpha)}$ bad boxes. Therefore we can cut the wired open paths which cross the minimal section and obtain for N large enough

$$\mu^+_{\beta,\Delta_N}\Big(T^\zeta \in \bigcap_{i=1}^{\ell} \mathcal{V}(R^i, \delta\mathrm{vol}(R^i))\Big) \leq \exp\Big(C_{\beta,u}\delta N^2\Big) \Phi^{\mathrm{w}}_{\Delta_N}\Big(\bigcap_{i=1}^{\ell} \{\partial^{\mathrm{top}} {R_N^i}' \not\leftrightarrow \partial^{\mathrm{bot}} {R_N^i}'\}\Big).$$

Let $\bar{R}^i$ be the union of R^i and of the parallelepiped in $\mathcal{T}^c$ with height $\frac{\delta h}{2}$ and one of its faces is equal to B^i. Denote by $\bar{R}^i_N$ the corresponding microscopic domain, then by inequality (3.1) one has for any boundary condition π outside $\bar{R}^i_N$

$$\bar{\Phi}^{\pi,\mathrm{w}}_{R^i_N}\Big(\{\partial^{\mathrm{top}} {R^i_N}' \not\leftrightarrow \partial^{\mathrm{bot}} {R^i_N}'\}\Big) \leq \Phi^{\mathrm{f},\mathrm{p}_\beta}_{\bar{R}^i_N}\Big(\{\partial^{\mathrm{top}} {R^i_N}' \not\leftrightarrow \partial^{\mathrm{bot}} {R^i_N}'\}\Big),$$

where $\bar{\Phi}^{\pi,\mathrm{w}}_{R^i_N}$ is the gibbs measure with wired boundary conditions on the face of R^i_N which coincides with $\partial \Delta_N$ and π otherwise. This enables us to apply Lemma 4.4 and to recover (5.4).

6. Lower Bound : Proposition 2.2

The proof rests only on Lemma 4.1, therefore Proposition 2.2 holds as soon as β is larger than $\tilde{\beta}_c$ (see subsection 3.2). The proof is divided into 2 steps.

6.1. **Approximation procedure.**

We first state an approximation theorem which will be proven in Appendix 8.3. We call polyhedral set, a set which has a boundary included in the union of a finite number of hyperplans.

Theorem 6.1. *For any δ positive, there exists a polyhedral set W such that*

$$1\!\!\mathrm{I}_W \in \mathcal{V}(1\!\!\mathrm{I}_{\mathcal{W}_m}, \frac{\delta}{3}) \qquad \text{and} \qquad \Big| \mathcal{F}(W) - \mathcal{F}(\mathcal{W}_m) \Big| \leq \frac{\delta}{2}.$$

For any h small enough there are ℓ disjoint cubes $R^1, \dots, R^\ell$ of size h with basis $B^1, \dots, B^\ell$ included in ∂W. Furthermore, the squares $B^1, \dots, B^\ell$ cover ∂W up to a set of measure less than δ denoted by $U^\delta = \partial W / \bigcup_{i=1}^\ell B^i$ and they satisfy

$$\Big| \sum_{i=1}^{\ell} \int_{B^i} \tau(\vec{n}_i)\, d\mathcal{H}_x - \mathcal{F}(\mathcal{W}_m) \Big| \leq \delta,$$

where the normal to B^i is denoted by $\vec{n}_i$.

We fix δ positive and choose a set W approximating $\mathcal{W}_m$, then

$$\Big\{ \frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(T^\zeta, \frac{\delta}{3}) \Big\} \bigcap \Big\{ T^\zeta \in \mathcal{V}(1\!\!\mathrm{I}_W, \frac{\delta}{3}) \Big\} \subset \Big\{ \frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(1\!\!\mathrm{I}_{\mathcal{W}_m}, \delta) \Big\}.$$

Lemma 3.1 implies that there exists ζ such that the event $\Big\{ \frac{\mathcal{M}}{m_\beta} \notin \mathcal{V}(T^\zeta, \frac{\delta}{3}) \Big\}$ has a probability which vanishes exponentially fast, therefore

$$\liminf_{N \to \infty} \frac{1}{N^2} \log \mu^+_{\beta, \Delta_N} \Big(\frac{\mathcal{M}}{m_\beta} \in \mathcal{V}(1\!\!\mathrm{I}_{\mathcal{W}_m}, \delta) \Big) \geq \liminf_{N \to \infty} \frac{1}{N^2} \log \mu^+_{\beta, \Delta_N} \Big(T^\zeta \in \mathcal{V}(1\!\!\mathrm{I}_W, \frac{\delta}{3}) \Big).$$

It remains to find a lower bound for the term in the RHS. For any ε positive, we construct a shell around ∂W which splits $\mathcal{T}$ into 2 domains

$$\mathcal{S}_\varepsilon = \{ r \in \mathcal{T} \mid \mathrm{dist}(r, \partial W) \leq \varepsilon \}.$$

We set

$$W^+_\varepsilon = \{ r \in W^c \mid \mathrm{dist}(r, \partial W) \geq \varepsilon \} \quad \text{and} \quad W^-_\varepsilon = \{ r \in W \mid \mathrm{dist}(r, \partial W) \geq \varepsilon \}.$$

Let $W^\pm_{\varepsilon,N}$ be the set of mesoscopic boxes included in $N W^\pm_\varepsilon \cap \Delta_N$. We fix ε such that the volume of $\mathcal{S}_\varepsilon$ is smaller than $\frac{\delta}{10}$ and choose h smaller than $\frac{\varepsilon}{2}$.

6.2. **Exponential bound.** The microscopic domain associated to the cube R^i is denoted by $R^i_N = NR^i \cap \Delta_N$. We set $\mathcal{A}_N = \bigcap_{i=1}^{\ell}\{\partial^+ R^i_N \not\leftrightarrow \partial^- R^i_N\}$. The microscopic domain

$$U^\delta_N = \Big\{x \in \Delta_N \;\Big|\; \exists r \in U^\delta,\ \|x - Nr\|_1 \leq 10\Big\}$$

is an enlargement of the surface U^δ defined in the approximation procedure. We introduce $\mathcal{B}_N$, the set of configurations with closed edges in U^δ_N. Hypotheses on U^δ imply that $\mathcal{B}_N$ is supported by at most $10\delta N^2$ edges.

We decompose $\mathcal{A}_N \cap \mathcal{B}_N$ into 2 disjoint sets

$$\begin{aligned}\mu^+_{\beta,\Delta_N}\Big(\mathcal{A}_N \cap \mathcal{B}_N\Big) \;&=\; \mu^+_{\beta,\Delta_N}\Big(\mathcal{A}_N \cap \mathcal{B}_N \cap \{\forall B(x,N^\alpha) \subset W^+_{\varepsilon,N} \cup W^-_{\varepsilon,N}; \quad |Z^\zeta_x| = 1\}\Big)\\ &+\; \mu^+_{\beta,\Delta_N}\Big(\mathcal{A}_N \cap \mathcal{B}_N \cap \{\exists B(x,N^\alpha) \subset W^+_{\varepsilon,N} \cup W^-_{\varepsilon,N}; \quad Z^\zeta_x = 0\}\Big).\end{aligned}$$

We first estimate the last term in the RHS. By definition the events $\mathcal{A}_N \cap \mathcal{B}_N$ and $\{(\sigma,\omega) \mid \exists B(x,N^\alpha) \subset W^+_{\varepsilon,N} \cup W^-_{\varepsilon,N};\ Z^\zeta_x = 0\}$ have disjoint supports. Taking the conditional expectation with respect to $\mathcal{A}_N \cap \mathcal{B}_N$ and using the stochastic domination (3.6) (see also remark 8.1), we get

$$\mu^+_{\beta,\Delta_N}\Big(\{\exists B(x,N^\alpha) \subset W^+_{\varepsilon,N} \cup W^-_{\varepsilon,N}; \quad Z^\zeta_x = 0\} \;\Big|\; \mathcal{A}_N \cap \mathcal{B}_N\Big) \leq N^{3(1-\alpha)} \exp(-c_\zeta N^\gamma).$$

Therefore, for N large enough

$$\mu^+_{\beta,\Delta_N}\Big(\mathcal{A}_N \cap \mathcal{B}_N\Big) \leq 2\mu^+_{\beta,\Delta_N}\Big(\mathcal{A}_N \cap \mathcal{B}_N \cap \{\forall B(x,N^\alpha) \subset W^+_{\varepsilon,N} \cup W^-_{\varepsilon,N}; \quad |Z^\zeta_x| = 1\}\Big).$$

By construction, no configuration ω of $\mathcal{A}_N \cap \mathcal{B}_N$ contains an open path joining the 2 connected components of $\Delta_N / \Big(\bigcup_{i=1}^{\ell} R^i_N \bigcup U^\delta_N\Big)$. Therefore any configuration in $\mathcal{A}_N \cap \mathcal{B}_N \cap \{\forall B(x,N^\alpha) \subset W^+_{\varepsilon,N} \cup W^-_{\varepsilon,N}; \quad |Z^\zeta_x| = 1\}$ contains 2 disconnected microscopic crossing clusters. The cluster connected to $\partial\Delta_N$ is denoted by C^+ and the other one by C^-. The wired constraint imposes the sign 1 to C^+. With probability $\frac{1}{2}$ we choose the sign of C^- to be -1. We define the event

$$\begin{aligned}\mathcal{C}_N(\sigma,\omega) = \mathcal{A}_N \bigcap \mathcal{B}_N \bigcap \{\forall B(x,N^\alpha) \subset W^+_{\varepsilon,N}; \quad Z^\zeta_x = 1\}\\ \bigcap \{\forall B(x,N^\alpha) \subset W^-_{\varepsilon,N}; \quad Z^\zeta_x = -1\}.\end{aligned}$$

then $\mu^+_{\beta,\Delta_N}(\mathcal{A}_N \cap \mathcal{B}_N) \leq 4\mu^+_{\beta,\Delta_N}(\mathcal{C}_N)$. Thus for any configuration (σ,ω) in $\mathcal{C}_N$, the function $T^\zeta(\sigma,\omega)$ is equal to 1 on W^+_ε and to -1 on W^-_ε (see (3.7)). Since the volume of $\mathcal{S}_\varepsilon$ is less than $\frac{\delta}{10}$, we have

$$\mathcal{C}_N \subset \Big\{(\sigma,\omega) \;\Big|\; T^\zeta(\sigma,\omega) \in \mathcal{V}(\mathbb{1}_W, \frac{\delta}{3})\Big\},$$

this leads to

$$\mu^+_{\beta,\Delta_N}\Big(T^\zeta(\sigma,\omega) \in \mathcal{V}(\mathbb{1}_W, \frac{\delta}{3})\Big) \geq \frac{1}{4}\mu^+_{\beta,\Delta_N}\Big(\mathcal{A}_N \cap \mathcal{B}_N\Big).$$

As $\mathcal{A}_N \cap \mathcal{B}_N$ depends only on the variable ω, we replace the coupled measure by the FK measure $\Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Delta_N}$. The support of $\mathcal{B}_N$ contains less than $10\delta N^2$ edges, so that

$$\mu^+_{\beta,\Delta_N}\Big(T^\zeta \in \mathcal{V}(\mathbb{1}_W, \frac{\delta}{3})\Big) \geq \frac{1}{4}\exp(-c_\beta \delta N^2)\Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Delta_N}\Big(\bigcap_{i=1}^{\ell}\{\partial^+ R^i_N \not\leftrightarrow \partial^- R^i_N\}\Big),$$

where c_β is a constant depending on β. The events $\{\partial^+ R_N^i \not\leftrightarrow \partial^- R_N^i\}$ occur on disjoint supports, taking the conditional expectation with respect to the configuration $\partial\omega_i$ outside R_N^i, we have

$$\mu_{\beta,\Delta_N}^+\Big(T^\zeta \in \mathcal{V}(\mathbb{1}_W, \frac{\delta}{3})\Big) \geq \frac{\exp(-c_\beta \delta N^2)}{4} \Phi_{\Delta_N}^{\mathrm{w},\mathrm{p}_\beta}\Big(\prod_{i=1}^{\ell} \Phi_{R_N^i}^{\partial\omega_i}(\{\partial^+ R_N^i \not\leftrightarrow \partial^- R_N^i\})\Big).$$

Since the events $\{\partial^+ R_N^i \not\leftrightarrow \partial^- R_N^i\}$ are non increasing, (3.1) implies

$$\mu_{\beta,\Delta_N}^+\Big(T^\zeta \in \mathcal{V}(\mathbb{1}_W, \frac{\delta}{3})\Big) \geq \frac{\exp(-c_\beta \delta N^2)}{4} \prod_{i=1}^{\ell} \Phi_{R_N^i}^{\mathrm{w},\mathrm{p}_\beta}\Big(\{\partial^+ R_N^i \not\leftrightarrow \partial^- R_N^i\}\Big).$$

Taking the limit as N goes to infinity and using Lemma 4.1, we obtain

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mu_{\beta,\Delta_N}^+\Big(T^\zeta \in \mathcal{V}(\mathbb{1}_W, \frac{\delta}{3})\Big) \geq -\sum_{i=1}^{\ell} \int_{B^i} \tau(\vec{n}_i) d\mathcal{H}_x - c_\beta \delta.$$

The proof is completed by letting δ go to 0. □

7. Exponential tightness : Proposition 2.3

In view of Lemma 3.1, Proposition 2.3 will be a consequence of

Lemma 7.1. *Let β be larger than $\tilde{\beta}_c$ (see subsection 3.2), then there is C_β such that for any a positive*

$$\forall \zeta, \delta > 0, \qquad \limsup_{N\to\infty} \frac{1}{N^2} \log \mu_{\beta,\Delta_N}^+\Big(T^\zeta \in \mathcal{V}(K_a, \delta)^c\Big) \leq -C_\beta\, a.$$

A high density of small contours can be interpreted on the mesoscopic scale as a high density of random variables $Z^\zeta = 0$. Such events are ruled out by Lemma 3.2. If T^ζ belongs to $\mathcal{V}(K_a, \delta)^c$ and does not contain many boxes labelled by $Z^\zeta = 0$, then there are mainly mesoscopic sets of constant sign. By definition of the variables Z^ζ, two sets of different signs are disconnected on the microscopic level and therefore are separated by a layer of boxes with label $Z^\zeta = 0$. The problem is that the probability of the event $\{Z_x^\zeta = 0\}$ may only be of the order of $\exp(-N^\gamma)$ which is much higher than the expected surface order $\exp(-N^{2\alpha})$. Thus we introduce another coarse graining in order to recover the surface order. The scheme of the following proof was suggested by D. Ioffe.

Proof. As noticed in subsection 5.2, the number of configurations T^ζ is less than $3^{N^{3(1-\alpha)}}$. Thus it is enough to fix T^ζ in $\mathcal{V}(K_a, \delta)^c$ and to estimate $\mu_{\beta,\Delta_N}^+(\{T^\zeta\})$. Lemma 3.2 enables us to consider only configurations T^ζ with a number of boxes labelled by 0 smaller than $\delta N^{3(1-\alpha)}$. This amounts to say that

$$\int_{\mathcal{T}} 1_{T_r^\zeta = 0}\, dr \leq \delta. \tag{7.1}$$

Each realization of T^ζ splits $\mathcal{T}$ into $\mathcal{T} = \mathcal{T}_+ \cup \mathcal{T}_- \cup \mathcal{T}_0$, where T^ζ is constantly equal to ± 1 on $\mathcal{T}_{\pm 1}$ and to 0 on $\mathcal{T}_0$. From (7.1), the measure of $\mathcal{T}_0$ is smaller than δ. The microscopic counterparts of $\mathcal{T}_+$ and $\mathcal{T}_-$ will be denoted by $\Delta_{N,+}$ and

$\Delta_{N,-}$. Moreover as T^ζ belongs to $\mathcal{V}(K_a,\delta)^c$, for any regular set A of $\mathcal{T}$ such that $\mathcal{T}_- \subset A \subset \mathcal{T} \setminus \mathcal{T}_+$

$$\int_{\partial A} d\mathcal{H}_x \geq a, \tag{7.2}$$

where $\int_{\partial A} d\mathcal{H}_x$ is the perimeter of ∂A. Note that for each configuration in $\mathcal{V}(K_a,\delta)^c$ the set $\mathcal{T}_-$ is not empty.

Let L be an integer large enough which divides N^α and is independent of N. We partition Δ_N into boxes $B(i,L)$ where i is in $\mathcal{L}_L$. We also introduce the boxes $B(i,\frac{5}{4}L)$ and following subsection 3.2, define a coarse graining on the scale L. Let $\{y_i\}$ be the family of random variables equal to 1 if the event O_i (for the box $B(i,\frac{5}{4}L)$) is satisfied and 0 otherwise. We define the microscopic set $\tilde{A}_N$ as the union of $\Delta_{N,-}$ and of the boxes $B(i,L)$ labelled by 1 such that there is a $*$-connected path of boxes $B(j,L)$ labelled by 1 joining $B(i,L)$ to $\Delta_{N,-}$. For any configuration (σ,ω) in $\{T^\zeta\}$ there is no microscopic path connecting $\Delta_{N,+}\cup\partial\Delta_N$ to $\Delta_{N,-}$. Therefore there is no $*$-connected path of boxes $B(j,L)$ labelled by 1 connecting $\Delta_{N,+}\cup\partial\Delta_N$ to $\Delta_{N,-}$. This implies that

$$\Delta_{N,-} \subset \tilde{A}_N \subset \Delta_N \setminus \Delta_{N,+}.$$

We define also $G_N = \Delta_N \setminus \tilde{A}_N$. Let $G_N = \bigcup_{i=1}^{\ell} G_{N,i}$ be the decomposition of G_N into maximal connected components composed of boxes $B(j,L)$. The components $G_{N,i}$ such that $|G_{N,i}| < N^{3\alpha}$ cannot intersect $\Delta_{N,+}$. We set

$$A_N = \tilde{A}_N \bigcup_{|G_{N,i}|<N^{3\alpha}} G_{N,i}$$

which satisfies $\Delta_{N,-} \subset A_N \subset \Delta_N \setminus \Delta_{N,+}$. To any $G_{N,i}$ with cardinality larger than $N^{3\alpha}$, we associate the contour

$$\partial^L G_{N,i} = \left\{ B(j,L) \;\middle|\; B(j,L) \subset G_{N,i} \quad \text{and} \quad B(j,2L) \cap A_N \neq \emptyset \right\}.$$

By construction $\partial^L G_{N,i}$ is a connected set of boxes $B(j,L)$ such that $y_j = 0$. Furthermore, whenever $|G_{N,i}| \geq N^{3\alpha}$

$$\left|\partial^L G_{N,i}\right| \geq c\left(\frac{N^\alpha}{L}\right)^2, \tag{7.3}$$

where c is a universal constant. Finally, from inequality (7.2), we see that the cardinality of the boundary ∂A_N is larger than caN^2, thus

$$\sum_{|G_{N,i}|\geq N^{3\alpha}} \left|\partial^L G_{N,i}\right| \geq ca\left(\frac{N}{L}\right)^2. \tag{7.4}$$

Stochastic domination (3.4) implies that for L large enough, a Peierls estimate holds : there is a positive constant $C_{L,\beta}$ such that the probability that a connected surface $\mathcal{S}$ of boxes $B(i,L)$ labelled by 0 contains more than n cubes is bounded by

$$\Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Delta_N}(\#\mathcal{S} \geq n) \leq N^3 \exp(-C_{L,\beta}\, n),$$

where $\#\mathcal{S}$ is the number of cubes in $\mathcal{S}$. By definition the contours $\partial^L G_{N,i}$ are connected surfaces of boxes $B(j,L)$ labelled by 0, so that using (7.3) and (7.4), the proof is concluded by a standard Peierls argument. $\square$

8. Appendix

8.1. **Surface tension.** In [MMR], the surface tension τ' was defined by tilting the boundary conditions outside the domain $\Delta_M = \{-M, M\}^3$, where M is an integer. For a given $\vec{n}$ in $\mathbb{S}^2$, we denote by D_M the intersection of $[-M, M]^3$ with the hyperplan centered in 0 and orthogonal to $\vec{n}$. The surface tension τ' is

$$\tau'(\vec{n}) = \lim_{M\to\infty} -\frac{1}{S_M} \log \frac{Z^{+,-}_{\Delta_M}}{Z^{+}_{\Delta_M}},$$

where S_M is the area of D_M and the mixed boundary conditions in $\partial\Delta_M$ are $\sigma_i = 1$ if $\vec{i}.\vec{n} > 0$ and $\sigma_i = -1$ otherwise. The boundary conditions above D_M are equal to 1 and to -1 below.

This definition coincides with definition 2.1. This is straightforward from the argument developed by [MMR] (Theorem 2). We recall this argument for completeness. First, we show that

$$\tau'(\vec{n}) \leq \liminf_{N\to\infty} -\log \frac{Z^{+,-}_{\Lambda_N}}{Z^{+}_{\Lambda_N}}. \tag{8.1}$$

Let M, N be 2 integers such that $N \ll M$. We shall write

$$F'_M = -\log \frac{Z^{+,-}_{\Delta_M}}{Z^{+}_{\Delta_M}} \qquad \text{and} \qquad F(\Lambda_N) = -\log \frac{Z^{+,-}_{\Lambda_N}}{Z^{+}_{\Lambda_N}},$$

where the parallelepiped Λ_N was introduced in definition 2.1.

We tile D_M with k_M squares B^i of side length hN and at distance 10 from each others. These squares are chosen such that the area of D_M not covered by $\bigcup_{i=1}^{k_M} B^i$ is smaller than $C(Mf(N) + k_M hN)$, where C is some positive constant independent of M and N. If the center of B^i does not coincide with a site on the lattice, we translate B^i at a distance smaller than 1 such that the center of its translate $B^{i'}$ belongs to $\mathbb{Z}^3$. Let Λ_N^i be the parallelepiped of basis $B^{i'}$ deduced from Λ_N by translation. There is a choice of the squares B^i such that all the parallelepipeds Λ_N^i are included in Δ_M. Note that $F(\Lambda_N^i) = F(\Lambda_N)$.

Inequality (C.2) of [MMR] implies

$$F'_M \leq \sum_{i=1}^{k_M} F(\Lambda_N^i) + KC\Big(Mf(N) + k_M hN\Big),$$

where K is some positive constant. Since $|S_M - k_M h^2 N^2| \leq C(Mf(N) + k_M hN)$, one gets as M goes to infinity

$$\tau'(\vec{n}) \leq \frac{1}{h^2N^2} F(\Lambda_N) + \frac{KC}{hN}.$$

Letting N go to infinity, we obtain (8.1). The reverse inequality

$$\tau'(\vec{n}) \geq \limsup_{N\to\infty} -\log \frac{Z^{+,-}_{\Lambda_N}}{Z^{+}_{\Lambda_N}} \tag{8.2}$$

is derived in the same way by choosing $M \ll N$ and by partitioning the basis of Λ_N with translates of D_M. Combining (8.1) and (8.2), we see that $\tau(\vec{n}) = \tau'(\vec{n})$.

8.2. **Stochastic domination.** The following result is a consequence of [I2] and of the proof of Theorem 1.1 of [Pi1], we sketch the proof for completeness

Theorem 8.1. *For any ζ positive, there is N_ζ such that for all N larger than N_ζ, the family of random variables $|Z_x^\zeta|$ is dominated by a product Bernoulli measure $\pi_{\rho'_N}$ (see (3.6)).*

Proof. According to [LSS], it is enough to check that

$$\mu^+_{\beta,\Delta_N}\Big(|Z_x^\zeta| = 0 \;\Big|\; |Z_{x_1}^\zeta| = \varepsilon_1, \dots, |Z_{x_\ell}^\zeta| = \varepsilon_\ell\Big) \le \exp(-c_\zeta N^\gamma),$$

where the vertices $\{x_1, \dots, x_\ell\}$ are not $*$-neighbors of x in $\mathcal{L}_{N^\alpha}$ and each ε_i belongs to $\{0,1\}$. Using notation from subsection 3.1, one has

$$\begin{aligned}
&\mu^+_{\beta,\Delta_N}\Big(|Z_x^\zeta| = 0, |Z_{x_1}^\zeta| = \varepsilon_1, \dots, |Z_{x_\ell}^\zeta| = \varepsilon_\ell\Big) \le \\
&\quad \Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Delta_N}\Big(Y_x^\zeta = 1;\; P^\omega_{\Delta_N}\Big(|\mathbf{M}_x - \mathrm{sign}(C^*) m_\beta| > 2\zeta, |Z_{x_1}^\zeta| = \varepsilon_1, \dots, |Z_{x_\ell}^\zeta| = \varepsilon_\ell\Big)\Big) \\
&\qquad + \Phi^{\mathrm{w},\mathrm{p}_\beta}_{\Delta_N}\Big(Y_x^\zeta = 0, Y_{x_1}^\zeta = \varepsilon_1, \dots, Y_{x_\ell}^\zeta = \varepsilon_\ell;\; P^\omega_{\Delta_N}\Big(|Z_{x_1}^\zeta| = \varepsilon_1, \dots, |Z_{x_\ell}^\zeta| = \varepsilon_\ell\Big)\Big).
\end{aligned}$$

Because of stochastic domination (3.4), the last term in the RHS is bounded by

$$\exp(-c_\zeta N^\gamma)\, \mu^+_{\beta,\Delta_N}\Big(|Z_{x_1}^\zeta| = \varepsilon_1, \dots, |Z_{x_\ell}^\zeta| = \varepsilon_\ell\Big).$$

In order to control the other term, we have to take into account the deviations occurring from the random coloring of the small clusters, i.e. those of diameter less than N^γ. We enumerate the small clusters $C_1, \dots, C_{k_0}$ included in $B(x, N^\alpha)$. Their cardinals are denoted $c_1, \dots, c_{k_0}$ and their signs $s_1, \dots, s_{k_0}$. The random variables $s_1, \dots, s_{k_0}$ are iid Bernoulli. For N large enough, one has

$$\Big| N^{3\alpha}\mathbf{M}_x - \mathrm{sign}(C^*)|C^*| - \sum_{i=1}^{k_0} s_i c_i \Big| < \frac{\zeta}{2} N^{3\alpha}.$$

This comes from the fact that all the clusters intersecting the boundary of $B(x, N^\alpha)$ and distinct from C^* have length smaller than N^γ when $Y_x^\zeta = 1$. Thus the total magnetization produced by these clusters is less than $6N^{2\alpha+\gamma}$ and does not contribute. By definition of the event V_x^ζ, the unique crossing cluster C^* in $B(x, N^\alpha)$ satisfies $|m_\beta N^{3\alpha} - |C^*|\,| \le \zeta N^{3\alpha}$. Therefore, we just need to prove large deviations for $P^\omega_{\Delta_N}\Big(|\sum_{i=1}^{k_0} s_i c_i| > \frac{\zeta}{2} N^{3\alpha}\Big)$, for configurations ω which satisfy $Y_x^\zeta(\omega) = 1$. By symmetry, it is enough to bound $P^\omega_{\Delta_N}\Big(\sum_{i=1}^{k_0} s_i c_i > \frac{\zeta}{2} k_o\Big)$, with $k_0 \ge \frac{\zeta}{2} N^{3(\alpha-\gamma)}$ (note that k_0 is always smaller than $N^{3\alpha}$). We follow the argument of [I2] (p. 325). For all t positive, Chebyshev's inequality implies

$$P^\omega_{\Delta_N}\Big(\sum_{i=1}^{k_0} s_i c_i > \frac{\zeta}{2} k_o\Big) \le \exp\Big(-t\frac{\zeta k_0}{2}\Big) \prod_{i=1}^{k_0} P^\omega_{\Delta_N}\Big(\exp(t c_i s_i)\Big).$$

As each c_i is smaller than $N^{3\gamma}$, one has

$$\frac{1}{k_0} \log P^\omega_{\Delta_N}\Big(\sum_{i=1}^{k_0} s_i c_i > \frac{\zeta}{2} k_o\Big) \le -t\frac{\zeta}{2} + \frac{1}{k_0}\sum_{i=1}^{k_0} \log\cosh(t c_i) \le -t\frac{\zeta}{2} + \log\cosh(t N^{3\gamma}),$$

Let Λ^* be the Legendre-Laplace transform of the Bernoulli measure $\frac{1}{2}(\delta_1 + \delta_{-1})$. There exists c_ζ positive such that for N large enough

$$\frac{1}{k_0} \log P^{\omega}_{\Delta_N}\Big(\sum_{i=1}^{k_0} s_i c_i > \frac{\zeta}{2} k_o \Big) \leq -\Lambda^*\Big(\frac{\zeta}{2N^{3\gamma}} \Big) \leq -\frac{c_\zeta}{N^{6\gamma}}.$$

Since $k_0 \geq \frac{\zeta}{2} N^{3(\alpha-\gamma)}$, this leads to

$$P^{\omega}_{\Delta_N}\Big(\sum_{i=1}^{k_0} s_i c_i > \frac{\zeta}{2} k_o \Big) \leq \exp\Big(- c_\zeta \frac{\zeta}{2} N^{3\alpha - 9\gamma} \Big).$$

As $\gamma = \frac{1}{9}$, we obtain the expected upper bound. $\square$

Remark 8.1. Let $\mathcal{A}$ be a subset of $\Omega^{\mathrm{w}}_{\Delta_N}$ with support disjoint from the box $B(x, \frac{5}{4}N^\alpha)$. Then the following holds for N large enough

$$\mu^+_{\beta,\Delta_N}\Big(|Z^\zeta_x| = 0 \;\Big|\; \mathcal{A} \Big) \leq \exp(-c_\zeta N^\gamma).$$

This is straightforward from the previous arguments. From [Pi1], we know that $\Phi^{\omega,p_\beta}_{\Delta_N}(Y^\zeta_x = 0)$ vanishes exponentially fast for arbitrary boundary conditions ω outside the box $B(x, \frac{5}{4}N^\alpha)$. Furthermore, if the magnetization differs from its equilibrium values $\pm m_\beta$, the deviation occurs from the random coloring of small clusters independent of $\mathcal{A}$.

8.3. **Approximation.** Before proving Theorems 5.1 and 5.2, let us recall some basic notions of geometric measure Theory. Throughout this section, we fix u in $\mathrm{BV}(\mathcal{T}, \{+1,-1\})$ such that $\mathcal{F}(u) < \infty$ and δ in $]0,1]$. As τ is bounded, the perimeter of ∂u, which is $\int_{\partial^* u} d\mathcal{H}_x$, is also finite. The ball of radius r centered in y will be denoted by $B(y,r)$. For y in $\partial^* u$, we introduce the half-spaces

$$\begin{aligned} H^+(y) &= \Big\{ z \in \mathbb{R}^3 \mid \quad \vec{n}_y.(z-y) \geq 0 \Big\}, \\ H^-(y) &= \Big\{ z \in \mathbb{R}^3 \mid \quad \vec{n}_y.(z-y) \leq 0 \Big\}, \end{aligned}$$

where $\vec{n}_y$ is the normal to $\partial^* u$ in y. Let $H(y)$ be the hyperplan $H^+(y) \cap H^-(y)$.

We fix ζ positive. According to Theorem 2 (p. 205) of [EG], the reduced boundary $\partial^* u$ equals $\bigcup_{i=1}^n K_i \cup N$ where the 2 dimensional Hausdorff measure of N is less than ζ and each K_i is a compact subset of a C^1-hypersurface S_i. For all x in K_i, the normal $\vec{n}_x$ is also normal to S_i and there is $r_0 > 0$ such that uniformly on K_i

$$\begin{aligned} \forall i,\ \forall r \leq r_0,\ \forall y \in K_i, \qquad & \mathrm{vol}\Big(B(y,r) \cap \{u = -1\} \cap H^+(y) \Big) < \zeta r^3, \\ & \mathrm{vol}\Big(B(y,r) \cap \{u = +1\} \cap H^-(y) \Big) < \zeta r^3. \end{aligned} \tag{8.3}$$

In the decomposition of $\partial^* u$, one can choose each set K_i such that it is either included in $\partial \mathcal{T}$ or at a positive distance from $\partial \mathcal{T}$.

Proof of Theorem 5.1

We first approximate the compact sets K_i which do not touch the boundary. The following construction is the same for each S_i so it is enough to present it for one S_i, that we shall denote by S (with corresponding $K \subset S$).

As S is C^1, we can find M pairwise disjoint open subsets $\Sigma_1,\dots,\Sigma_M$ of S which cover S up to a set of measure less than ζ and such that each Σ_i is congruent to the graph of a real function $f_i : U_i \to \mathbb{R}$ of class C^1, where U_i is a bounded open set of $\mathbb{R}^2$ and f_i satisfies the bound $|\nabla f_i| \le \zeta$. To any x in U_i, we associate the point $g_i(x) = (x, f_i(x))$ of S. Let K_i be the compact subset of U_i such that $g_i(K_i) = K \cap \Sigma_i$. We choose h in $]0, \frac{r_0}{10}[$ (see (8.3)) arbitrarily small and cover U_i with pairwise disjoint cubes $C^j \subset U_i$ of side h up to a set of measure less than $\frac{\zeta}{M}$.

For each cube C^j centered in x_j and intersecting K_i, we denote by B^j the translate of C^j centered in $g_i(x_j)$. The parallelepiped R^j is defined as $R^{j,+} \cup R^{j,-}$, where both parallelepipeds $R^{j,+}$ and $R^{j,-}$ have a common face B^j and height $\frac{\delta}{2}h$ (one above and the other below B^j). Let y be in $C^j \cap K_i$. The parallelepiped R^j is included in the ball $B(g_i(y), 10h)$. As $|\nabla f_i| \le \zeta$, the intersection of the hyperplan $H(g_i(y))$ and R^j is contained in $\{z \in \mathbb{R}^3 \mid \ \mathrm{dist}(z, B^j) \le 2\zeta h\}$. Therefore (8.3) implies that

$$\int_{R^j} |\chi_{R^j}(r) - u(r)|\, dr \le 2\zeta h^3 + 10^3 \zeta h^3 \le 10^4 \frac{\zeta}{\delta} \mathrm{vol}(R^i). \tag{8.4}$$

The upper bound of τ is denoted by $\|\tau\|_\infty$. It remains to check that

$$\left| \sum_{i=1}^{k} \int_{B^i} \tau(\vec{n}_i)\, d\mathcal{H}_x - \int_K \tau(\vec{n}_x)\, d\mathcal{H}_x \right| \le \|\tau\|_\infty\, C_K\, \zeta, \tag{8.5}$$

where $\vec{n}_i$ is the normal to B^i and C_K depends on the Hausdorff measure of K.

Let $\mathcal{C}^i$ be the union of cubes C^j which intersect K_i, then for h small enough the measure of $\mathcal{C}^i \Delta K_i$ (the symmetric difference) is smaller than $\frac{\zeta}{M}$ and one has

$$\left| \int_{g_i(\mathcal{C}^i)} \tau(\vec{n'}_x)\, d\mathcal{H}_x - \int_{g_i(K_i)} \tau(\vec{n}_x)\, d\mathcal{H}_x \right| \le \frac{\zeta \|\tau\|_\infty}{M},$$

where $\vec{n'}$ is the normal vector to the surface S which coincides with $\vec{n}$ on K. The normal $\vec{n'}$ is uniformly continuous on any compact. Therefore for h small enough, the following holds for any cube C^j in U_i

$$\forall x, y \in C^j, \qquad \left| \tau(\vec{n'}_{g_i(x)}) - \tau(\vec{n'}_{g_i(y)}) \right| \le \zeta.$$

Using the fact that $|\nabla f_i| \le \zeta$ on each U_i, we derive (8.5).

Let us go back to the previous notation and denote by $B^1, \dots B^\ell$ the collection of sets which approximate the union of sets K_i which are not in $\partial^* u$. We set also $U = \partial^* u \bigcap \partial \mathcal{T}$. As τ is bounded

$$\int_N \tau(\vec{n}_x)\, d\mathcal{H}_x \le \|\tau\|_\infty \zeta.$$

We deduce from (8.5) that

$$\left| \sum_{i=1}^{\ell} \int_{B^i} \tau(\vec{n}_i)\, d\mathcal{H}_x - \int_{\partial^* u / U} \tau(\vec{n}_x)\, d\mathcal{H}_x \right| \le C_u\, \zeta, \tag{8.6}$$

where C_u depends only on the perimeter of u. From (8.4) and (8.6), we derive Theorem 5.1 for ζ small enough. ◻

Proof of Theorem 5.2

We are going now to approximate the compact sets K_i included in $U = \partial\mathcal{T} \cap \partial^* u$. One can also suppose that each K_i is included in one face of $\partial\mathcal{T}$. Note that $\mathcal{H}$-almost surely for y in K_i, the normal $\vec{n}_y$ is orthogonal to $\partial\mathcal{T}$.

For a given ζ positive and h in $]0, \frac{r_0}{10}[$ small enough, there is a covering of K_i with pairwise disjoint cubes $B^j \subset \mathcal{T}$ of size h up to a set of measure less than ζ. We denote by R^j the parallelepiped in $\mathcal{T}$ with one face equal to B^j and height δh. Let y be in $B^j \cap K_i$. The parallelepiped R^j is included in the ball $B(g_i(y), 10h)$ and (8.3) implies

$$\int_{R^j} |1 + u(r)| \, dr \leq 10^3 \zeta h^3 \leq 10^3 \frac{\zeta}{\delta} \mathrm{vol}(R^j). \tag{8.7}$$

Furthermore, for h small enough

$$\left| \sum_{j=1}^{k} \int_{B^j} \tau(\vec{n}_j) \, d\mathcal{H}_x - \int_U \tau(\vec{n}_x) \, d\mathcal{H}_x \right| \leq \|\tau\|_\infty \, \zeta, \tag{8.8}$$

where $\vec{n}_i$ is the normal to B^i. Combining (8.7) and (8.8), we conclude the proof. $\square$

One could have also modified the proof of Lemma 6.4 [Ce] and replaced the approximation in terms of balls, by cubes.

Proof of Theorem 6.1

Theorem 6.1 can be viewed as a consequence of a general approximation procedure developed by Alberti and Bellettini [AlBe]. We briefly recall their proof and refer the reader to [AlBe] for details. Let u be a function of bounded variation, then general results of measure Theory imply the existence of a sequence $\{u_n\}$ of polyhedral functions converging to u in $L^1(\mathcal{T})$ and such that the vectors measure of the partial first derivatives Du_n converge weakly to Du and also that the perimeters of ∂u_n converge to the one of ∂u. Since τ is continuous, a Theorem of Reshetnyak (see [LM]) implies that $\mathcal{F}(u_n)$ converge to $\mathcal{F}(u)$.

Therefore for any δ positive, there exists a polyhedral set W such that

$$1\!\!\mathrm{I}_W \in \mathcal{V}(1\!\!\mathrm{I}_{\mathcal{W}_m}, \frac{\delta}{3}) \qquad \text{and} \qquad \left| \mathcal{F}(W) - \mathcal{F}(\mathcal{W}_m) \right| \leq \frac{\delta}{2}.$$

For any h small enough, we approximate the polyhedral set W with disjoint cubes $R^1, \ldots, R^\ell$ of size h and basis $B^1, \ldots B^\ell$. The set $\partial W / \bigcup_{i=1}^{\ell} B^i$ has arbitrarily small area and is denoted by U^δ. As τ is bounded, one has

$$\left| \sum_{i=1}^{\ell} \int_{B^i} \tau(\vec{n}_i) \, d\mathcal{H}_x - \mathcal{F}(W) \right| \leq \|\tau\|_\infty \int_{U^\delta} d\mathcal{H}_x \leq \|\tau\|_\infty \delta.$$

This concludes the Theorem. $\square$

Remark 8.2. Since we are only interested in approximating the Wulff shape, one could have also used Aleksandrov's Theorem (see [EG]) which ensures that the boundary of a convex function has almost surely a second derivative.

Remark 8.3. As explained previously, Theorem 6.1 holds for any function u of bounded variation. Thus following the arguments developed in Section 6, one can prove the lower bound (Proposition 2.2) for any function u of bounded variation. This implies that a large deviation principle for the measures μ^+_{β,Δ_N} holds.

References

[AlBe] G. Alberti, G. Bellettini : Asymptotic behavior of a non local anisotropic model for phase transition, *J. Math. Ann.*, **310**, No. 3, 527-560, (1998).

[ABCP] G. Alberti, G. Bellettini, M. Cassandro, E. Presutti : Surface tension in Ising system with Kac potentials, *J. Stat. Phys.*, **82**, No. 3-4, 743-796, (1996).

[AmBr] L. Ambrosio, A. Braides : Functionals defined on partitions in sets of finite perimeter II : Semicontinuity, relaxation and homogenization, *J. Math. pures et appl.*, **69**, 307-333, (1990).

[ACCN] M. Aizenman, J. Chayes, L. Chayes, C. Newman : Discontinuity of the magnetization in the one-dimensional $\frac{1}{|x-y|^2}$ Ising and Potts model, *J. Stat. Phys.*, **50**, 1-40, (1988).

[BCP] G. Bellettini, M. Cassandro, E. Presutti : Constrained minima of non local free energy functionals, *J. Stat. Phys.*, **84**, 1337-1349, (1996).

[BBBP] O. Benois, T. Bodineau, P. Butta, E. Presutti : On the validity of van der Waals theory of surface tension, *Mark. Proc. and Rel. Fields* , **3**, No. 2, 175-198, (1997).

[BBP] O. Benois, T. Bodineau, E. Presutti : Large deviations in the van der Waals limit, *Stoch. Proc. and Appl.*, **75**, 89-104, (1998).

[Bo] T. Bodineau : Wulff construction for Ising model with finite range Kac potentials, *in preparation.*

[Ce] R. Cerf : Large deviations for three dimensional supercritical percolation, *Preprint*, October (1998).

[DKS] R.L. Dobrushin, R. Kotecky, S. Shlosman : *Wulff construction: a global shape for non local interaction*, Providence, Amer. Math. Soc., (1992).

[DP] J. D. Deuschel, A. Pisztora : Surface order large deviations for high-density percolation, *Prob. Theo. Rel. Fields*, **104**, 467-482, (1996).

[DS] J. D. Deuschel, D. Stroock : *Large deviations*, Academic Press, (1989).

[EG] L. Evans, R. Gariepy : *Measure Theory and Fine Properties of Functions*, CRC Press, London (1992).

[ES] R. Edwards, A. Sokal : Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm, *Phys. Rev. D*, **Vol 38**, No. 6, 2009-2012, (1988).

[Gri] G. Grimmett : The stochastic random cluster process and the uniqueness of random cluster measures, *Ann. Prob.*, **23**, No. 4, 1461-1510, (1995).

[I1] D. Ioffe : Large deviations for the 2D Ising model : a lower bound without cluster expansions, *J. Stat. Phys.*, **74**, 411-432, (1994).

[I2] D. Ioffe : Exact deviation bounds up to T_c for the Ising model in two dimensions, *Prob. Theo. Rel. Fields*, **102**, 313-330, (1995).

[IS] D. Ioffe, R. Schonmann : Dobrushin-Kotecky-Shlosman Theorem up to the critical temperature, *Comm. Math. Phys.*, **199**, 117-149 , (1998).

[LM] S. Luckaus, L. Modica : The Gibbs-Thompson relation within the gradient theory of phase transitions, *Arch. Rat. Mech. Anal.*, **107**, 71-83, (1989).

[LP] J. Lebowitz, C.E. Pfister : Surface tension and phase coexistence, *Phys Rev. Let.*, **46**, No. 15, 1031-1033, (1981).

[LSS] T. Liggett, R. Schonmann, A. Stacey : Domination by product measures, *Ann. Prob.*, **25**, No. 1, 71-95, (1997).

[MMR] A. Messager, S. Miracle-Solé, J. Ruiz : Convexity property of the surface tension and equilibrium crystals, *J. Stat. Phys.*, **67**, No. 3/4, 449-470, (1992).

[Pi1] A. Pisztora : Surface order large deviations of Ising, Potts and percolation models, *Prob. Theo. Rel. Fields*, **104**, 427-466, (1996).

[Pi2] A. Pisztora : Lectures in IHP, *Unpublished*, Paris, June (1998).

[Pf] C.E. Pfister : Large deviations and phase separation in the two dimensional Ising model, *Helvetica Physica Acta*, **64**, 953-1054, (1991).

[PV] C.E. Pfister, Y. Velenik : Large deviations and continuum limit in the $2D$ Ising model, *Prob. Theo. Rel. Fields*, **109**, No. 4, 435-506, (1997).

[Ta] J. Taylor : Crystalline variational problems, *Bull. Am. Math. Soc.*, **84**, No. 4, 568-588, (1978).

[SS] R. Schonmann, S. Shlosman : Complete analyticity for the 2d Ising model completed, *Comm. Math. Phys.*, **170**, 453-482, (1996).