2. Classical RMT results

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CAPITAL FUND MANAGEMENT

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Spectral Transforms

- Stieltjes transform, Green and Blue functions
  - $\rho(\lambda) = N^{-1} \sum_i \delta(\lambda - \lambda_i)$
  - Stieltjes transform:

$$\mathcal{S}(z) = \int \mathrm{d}\lambda \frac{\rho(\lambda)}{\lambda - z} = \frac{1}{N} \mathrm{Tr} \left[(\mathbf{H} - z\mathbf{I})^{-1} \right]$$

  - Green function:

$$G(z) \equiv -\mathcal{S}(z); \qquad \rho(\lambda) = \lim_{\epsilon \to 0} \frac{1}{\pi} \Im \left(G(\lambda - \mathrm{i}\epsilon) \right)$$

  - Blue function: $B[G(z)] = z$

Spectral Transforms

- R-transforms and S-transforms
  - R-transform: $R(z) = B(z) - z^{-1}$
  - Properties:

$$R_{aH}(z) = aR_H(az)$$

$$R(z) = \sum_{k=1}^{\infty} c_k z^{k-1} \qquad c_k : \qquad \text{Generalized cumulants}$$

  - S-transform:

$$\eta(y) \equiv -\frac{1}{y} G\left(-\frac{1}{y}\right); \qquad S(z) = -\frac{1+z}{z}\eta^{-1}(1+z)$$

Spectral Transforms

- Example 1: Wigner semi-circle

$$G(z) = \frac{z \pm \sqrt{z^2 - 4}}{2} \qquad R(z) = z$$

- Example 2: Marcenko-Pastur $Q = T/N$, $q = 1/Q$

$$\rho(\lambda) = (1-Q)^+\delta(\lambda) + \frac{\sqrt{4\lambda q - (\lambda + q - 1)^2}}{2\pi\lambda q} \qquad \lambda \in [(1-\sqrt{q})^2, (1+\sqrt{q})^2]$$

$$G(z) = \frac{(z + q - 1) - \sqrt{(z + q - 1)^2 - 4zq}}{2zq}, \; R(z) = \frac{1}{1 - qz}, \; S(z) = \frac{1}{1 + qz}$$

Four-ways to the semi-circle

- 1. From the full multivariate density. The GOE measure:

$$P(\mathbf{H}) \propto \exp[-\frac{1}{2}\mathrm{Tr}\mathbf{H}^2] \to \rho(\lambda_1, \lambda_2, ...\lambda_N) = Z^{-1} \prod_{i<j} |\lambda_i - \lambda_j| \exp[-\frac{1}{2}\sum_i \lambda_i^2]$$

- Transform Van der Monde determinant with Orthogonal Polynomials, compute $\rho(\lambda)$ by integrating over $N-1$ variables, take the large N limit $\to \rho(\lambda) = \sqrt{4-\lambda^2}/2\pi$

- Use the analogy with the partition function of a charged gaz with logarithmic interactions, confined by a parabolic potential. At equilibrium, force on each particle is zero:

$$\lambda = \int \mathrm{d}\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}$$

Four-ways to the semi-circle

Tricomi's equation, solved by Wigner's semi-circle

- Note: Dyson's Brownian motion: add a small Gaussian matrix and use second order perturbation theory and rescale to keep a fixed variance:

$$d\lambda_i = [-\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}]dt + dB_t$$

Four-ways to the semi-circle

- 2. From a recursion relation on the Green function $\mathbf{G}(z) = (z\mathbf{1} - \mathbf{H})^{-1}$

- Start from an $N \times N$ sym. matrix with IID entries and add a row and a column of IID elements.

- Expand twice the inverse of a matrix in terms of minors. One easily gets:

$$\frac{1}{G_{00}^{N+1}(z)} = z - H_{00} - \sum_{ij}^{N} H_{0i} H_{0j} G_{ij}^{N}(z)$$

- Find a similar recursion for G_{0i}^{N+1} which shows that off diagonal elements are $O(1/\sqrt{N})$ whereas diagonal elements are of order one.

Four-ways to the semi-circle

- Hence:

$$\frac{1}{G_{00}^{N+1}(z)} \approx z - \sum_{i}^{N} H_{0i}^2 G_{ii}^N(z) + O(1/\sqrt{N})$$

- Since H_{0i} and G_{ii} are independent, one can use the law of large numbers to get, for large N:

$$\overline{G}^{-1} = z - \overline{G}^{-1} \to \overline{G} = \frac{1}{2}(z \pm \sqrt{z^2 - 4})$$

Four-ways to the semi-circle

- 3. The REPLICA method

- Use a Gaussian integral representation of the inverse:

$$\mathbf{A}_{ii}^{-1} = \frac{\int [\prod_j d\phi_j] \phi_i^2 \exp[-\frac{1}{2}\sum_{jk} \phi_j A_{kj} \phi_k]}{\int [\prod_j d\phi_j] \exp[-\frac{1}{2}\sum_{jk} \phi_j A_{kj} \phi_k]}$$

The "Replica Trick" is to write this as:

$$\mathbf{A}_{ii}^{-1} = \lim_{n \to 0} \int [\prod_{ja} d\phi_j^a] \frac{1}{n} \sum_{a=1}^{n} \phi_i^{a2} \exp[-\frac{1}{2} \sum_{a=1}^{n} \sum_{jk} \phi_j^a A_{kj} \phi_k^a]$$

- Use this with $\mathbf{A} = z\mathbf{1} - \mathbf{H}$, trace over i and average over Gaussian H_{ij}, leading to:

$$G(z) = \lim_{n \to 0} \frac{2\partial}{n\partial z} \int [\prod_{ja} d\phi_j^a] \exp[-\frac{z}{2} \sum_a^n \sum_j \phi_i^{a2} + \frac{1}{4N} \sum_{a,b}^n \sum_{ij} \phi_i^a \phi_i^b \phi_j^a \phi_j^b]$$

Four-ways to the semi-circle

- ‘Replica sym.’ ansatz: Introduce two ‘order parameters’

$$q_0(z) = \frac{1}{N}\sum_i \phi_i^a \phi_i^a, \qquad q_1(z) = \frac{1}{N}\sum_i \phi_i^a \phi_i^b,$$

  enforced by two δ functions, expressed in Fourier transform. Integration over ϕ becomes Gaussian again.

- The remaining integral finally reads:

$$\int dq_0\, dq_1\, d\lambda\, d\mu\, \exp[NnF(q_0, q_1, \lambda, \mu)]$$

  with:

$$F(q_0, q_1, \lambda, \mu) = (\lambda + \frac{z}{2})q_0 + \mu(n-1)q_1 - \frac{1}{2}\ln\lambda - \frac{1}{2n}\ln\frac{\lambda + n\mu}{\lambda} + \frac{q_0^2}{4} + \frac{(n-1)q_1^2}{4}$$

Four-ways to the semi-circle

- Work at finite n and let $N \to \infty$: saddle point calculation leading to the following equation:

$$\lambda^* + \frac{z}{2} + \frac{q_0^*}{2}, \quad \lambda^* q_0^* = 1, \longrightarrow G(z) = q_0^* = \frac{1}{2}(z \pm \sqrt{z^2 - 4})$$

CAPITAL FUND MANAGEMENT

Four-ways to the semi-circle

- 4. Free convolution (more below)

- If $\mathbf{H_1}$ and $\mathbf{H_2}$ are two IID large ($N \to \infty$) Gaussian matrices, then:

$$R_{H_1+H_2}(z) = R_{H_1}(z) + R_{H_2}(z)$$

- But: $\mathbf{H_1} + \mathbf{H_2} =_L \sqrt{2}\mathbf{H}$ – Hence:

$$R_{\sqrt{2}H}(z) = \sqrt{2}R_H(\sqrt{2}z) = 2R_H(z) \longrightarrow R_H(z) = az$$

- Generalized CLT

Four-ways to the semi-circle: pros and cons

- 1. Rigorous, access to multi-point correlations but specific (Gaussian ensemble) and relatively heavy

- 2. Makes explicit the CLT character of the semi-circle, can be extended to Lévy variables, no access to multi-point correlations

- 3. Non rigorous but very flexible, leads to solution for more complicated RM ensembles, can be extended to multi-point correlations

- 4. Elegant and powerful (see below) but no access to multi-point correlations and structure of eigenvectors

Other classic RMT results

- 1. Structure of eigenvectors

- The GOE is invariant under rotations, hence there cannot be any localisation of eigenvectors. Therefore, the inverse participation ratio (Hirfindahl index) of any eigenvector is zero:

$$\sum_i w_i^{\alpha 2} = \frac{1}{N}, \qquad w_i^\alpha = |\langle i|\alpha\rangle|^2$$

- More precisely, for a given α,

$$P(w) = N\,\mathrm{exp}[-Nw]$$

(Porter-Thomas distribution)

Other classic RMT results

- 2. Finite N results

- Note; The density of states is *self-averaging*

- Convergence of the averaged density of states for Gaussian elements:

  $$|E[\rho_N] - \rho_\infty| \le \kappa N^{-2/3}$$

  Note: comes from the edge scaling spill-over:

  $$\rho_N(\lambda = 2 + \epsilon) = N^{-2/3} f(\epsilon N^{-2/3})$$

  which itself can be guessed from: $\int_{2-\epsilon}^{2} du \sqrt{2-u} = \frac{1}{N} \propto \epsilon^{3/2}$.

- For a fixed realization:

  $$|\rho_N - \rho_\infty| = \xi N^{-2/5}$$

Other classic RMT results

- Can be extended to non Gaussian elements provided high enough moments exist.
- More precise results about the largest eigenvalue: Tracy-Widom, see below.

Other classic RMT results

- 3. Universal correlations

- Universal level repulsion: degeneracies are of co-dimension 1 – For example for a 2×2 matrix:

$$\Delta = 0 \quad \text{when} \quad (\mathrm{H}_{11} - \mathrm{H}_{22})^2 + \mathrm{H}_{12}^2 = 0$$

  This implies, for the level spacing distribution:

$$P(s) \sim_{s \to 0} s$$

- Wigner surmise (for a 2×2 Gaussian matrix):

$$P(s) \propto s \exp[-s^2]$$

  extremely good fit for the exact large N result, and universal (resists change of global densities, etc.) – for example, true

Other classic RMT results

for the Marcenko-Pastur problem, or for quantized classically chaotic systems

- Universal two-level density on a local scale (after rescaling such that $\rho(\lambda) = 1$):

$$\rho(\lambda, \lambda + \frac{u}{N}) = 1 - \frac{\sin \pi u}{\pi u}$$

- Incompressibility

$$\langle n^2(\Delta) \rangle - \langle n(\Delta) \rangle^2 \sim \frac{2}{\pi^2} \ln \Delta \ll \Delta$$

Other classic RMT results

- 4. Lévy matrices

- H_{ij}: IID with power-law tails, exponent $\mu < 2$ such that the variance diverges, rescaled by $N^{1/\mu}$

- $\rho(\lambda)$ can be exactly computed and has no edge [for a rigorous proof, see Ben-Arous-Guionnet]

- $\rho(\lambda) \sim_{|\lambda| \to \infty} |\lambda|^{-1-\mu}$

- Interesting structure of eigenvectors (no rotational symmetry) – transition between localized and extended

- Structure of correlations and level spacing unknown