4. Extreme eigenvalues

J.P Bouchaud

CAPITAL FUND MANAGEMENT

http://www.cfm.fr

The Tracy-Widom problem

- Edge and extreme eigenvalues

- Max of IID variables: trivial problem – Fréchet, Gumbel and Weibull

- Max of correlated variables: a few known non trivial cases, such as the max of eigenvalues of GOE, GUE

- Tracy-Widom result:

$$\lambda_{\text{max}} = 2 + \frac{\xi}{N^{2/3}}, \qquad F_{TW}(\xi)$$

where $F_{TW}(\xi)$ is expressed in terms of Painlev solutions, $\ln F_{TW}(\xi) \sim_{\pm\infty} -|\xi|^{3/2,3}$

Rank one perturbations

- Universality of TW?

- TW result expected to breakdown when the distribution of matrix elements is too broad

- An auxiliary problem: what is the largest eigenvalue of $\mathbf{H} + \mathbf{\Lambda}$, where $\mathbf{H}$ is Wigner and $\mathbf{\Lambda}$ a rank-one perturbation with eigenvalue S ?

- Several methods: free, direct perturbation, replicas

Rank one perturbations

- R-transform method

- Green function for $\mathbf{\Lambda}$:

$$G(z) = \frac{N-1}{Nz} + \frac{1}{N}\frac{1}{z-S} \longrightarrow B(z) \approx \frac{1}{z} + \frac{1}{N}\frac{S}{1-Sz}$$

  Sum of R-transforms:

$$R_{H+\Lambda} = z + \frac{1}{N}\frac{S}{1-Sz} \longrightarrow z \approx G + \frac{1}{G} + \frac{1}{N}\frac{S}{1-SG}$$

- Isolated eigenvalue out of the Wigner sea if $SG_W(z) = 1$ has a non trival solution, leading to

$$z = \lambda_{\max} = S + \frac{1}{S} \quad (S > 1); \qquad \lambda_{\max} = 2 \quad (S \leq 1)$$

Rank one perturbations

- But: No info on the structure of the largest eigenvector

Extension: Wigner with one Large Element

- Symmetric H_{ij} Gaussian with variance $1/N$ except one pair $H_{\alpha\beta} = H_{\beta\alpha} = S$

- Rank two perturbation with eigenvalues $\pm S$

- Extension of the above result: two eigenvalues $\pm(S + 1/S)$ pop out of the Wigner sea when $S > 1$

- Different derivation based on Statistical Mechanics

$$\mathcal{H} = -\frac{1}{2} \sum_{i,j=1}^{N} H_{ij} s_i s_j + \frac{z}{2} \sum_{i=1}^{N} s_i^2$$

where s_i are soft spins verifying the spherical constraint $\sum_{i=1}^{N} \langle s_i^2 \rangle = N$. Solution with 'replica trick'.

Wigner Problem with One Large Element

- Zero temperature solution: $\lambda_{\mathrm{max}} = 2$ or $\lambda_{\mathrm{max}} = S + 1/S$.
  - First solution corresponds to usual Tracy-Widom statistics ($N^{-2/3}$) with a delocalized eigenvector (Porter-Thomas).
  - Second solution has Gaussian fluctuations ($1/\sqrt{N}$), localized on α and β with weight $w_\alpha = w_\beta = (1 - 1/S^2)/2$.

Wigner Problem with Fat Tails

- Matrix elements IID with distribution

$$P(H) \sim \frac{A^\mu}{|H|^{1+\mu}} \text{ with } A \sim O(1/\sqrt{N}).$$

- Largest element (out of $N^2/2$) is such that $H_{\max}$ is distributed with Fréchet of order $N^{2/\mu-1/2}$. From the above:
  - If $\mu > 4$: $H_{\max} \ll 1$, one recover Tracy-Widom.
  - If $\mu < 4$: $H_{\max} \gg 1$, $\lambda_{\max} = H_{\max}$: Fréchet distribution.
  - If $\mu = 4$: $H_{\max} \sim O(1)$, $\lambda_{\max} = 2$ or $\lambda_{\max} = H_{\max} + 1/H_{\max}$

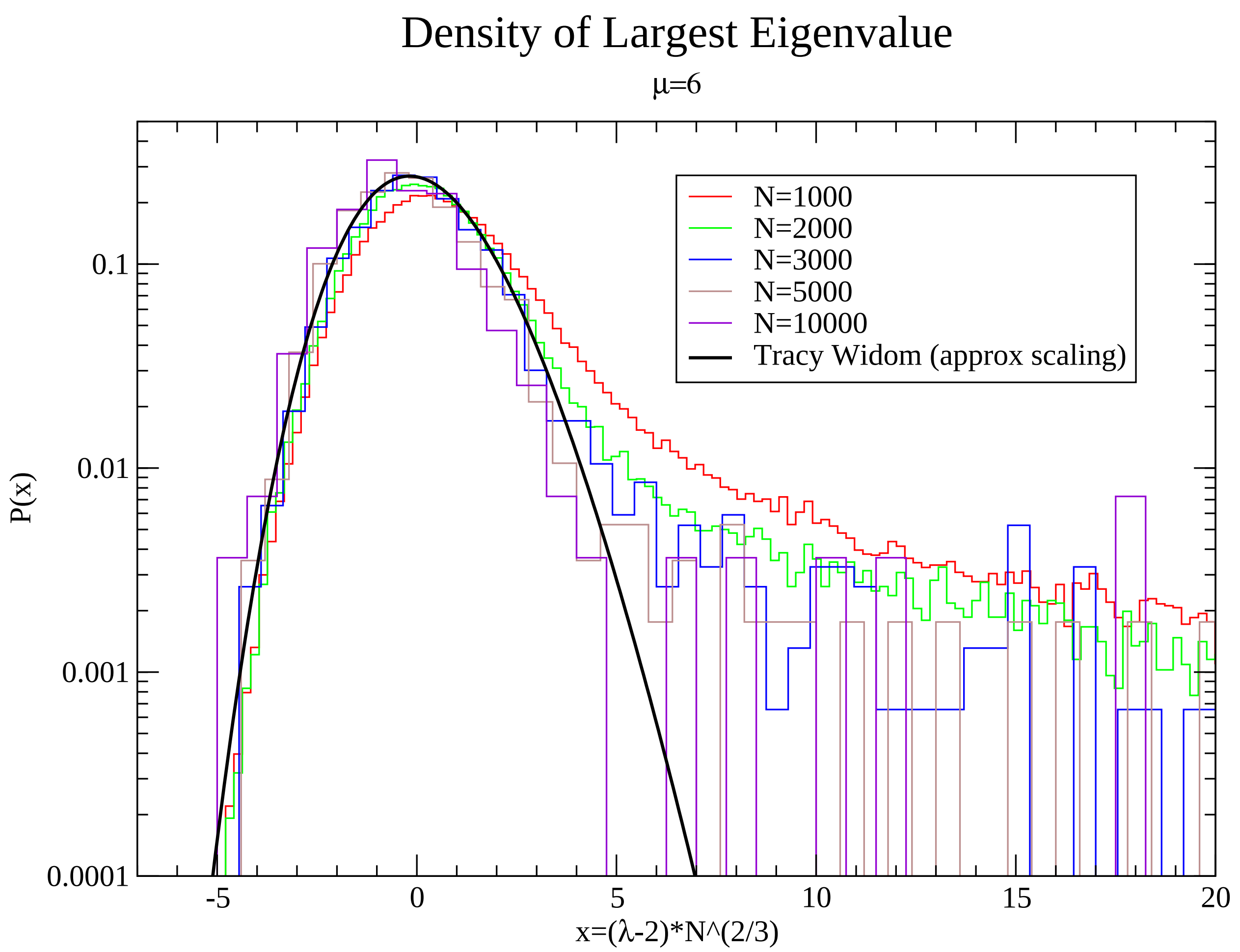
Wigner Problem with Fat Tails

- Largest Eigenvalue statistics
  - $\mu > 4$: $\lambda_{\max} - 2 \sim N^{-2/3}$ with a Tracy-Widom distribution
  - $2 < \mu < 4$: $\lambda_{\max} \sim N^{\frac{2}{\mu}-\frac{1}{2}}$ with a *Fréchet* distribution (although the density goes to zero when $\lambda > 2$!!)
  - $\mu = 4$: $\lambda_{\max} \geq 2$ but remains $O(1)$, with a new distribution:

$$P(s) = w\delta(s-1) + (1-w)\Theta(s-1)F(s) \quad \lambda_{\max} = s + \frac{1}{s}$$

- Note: The case $\mu > 4$ still has a power-law tail for finite N

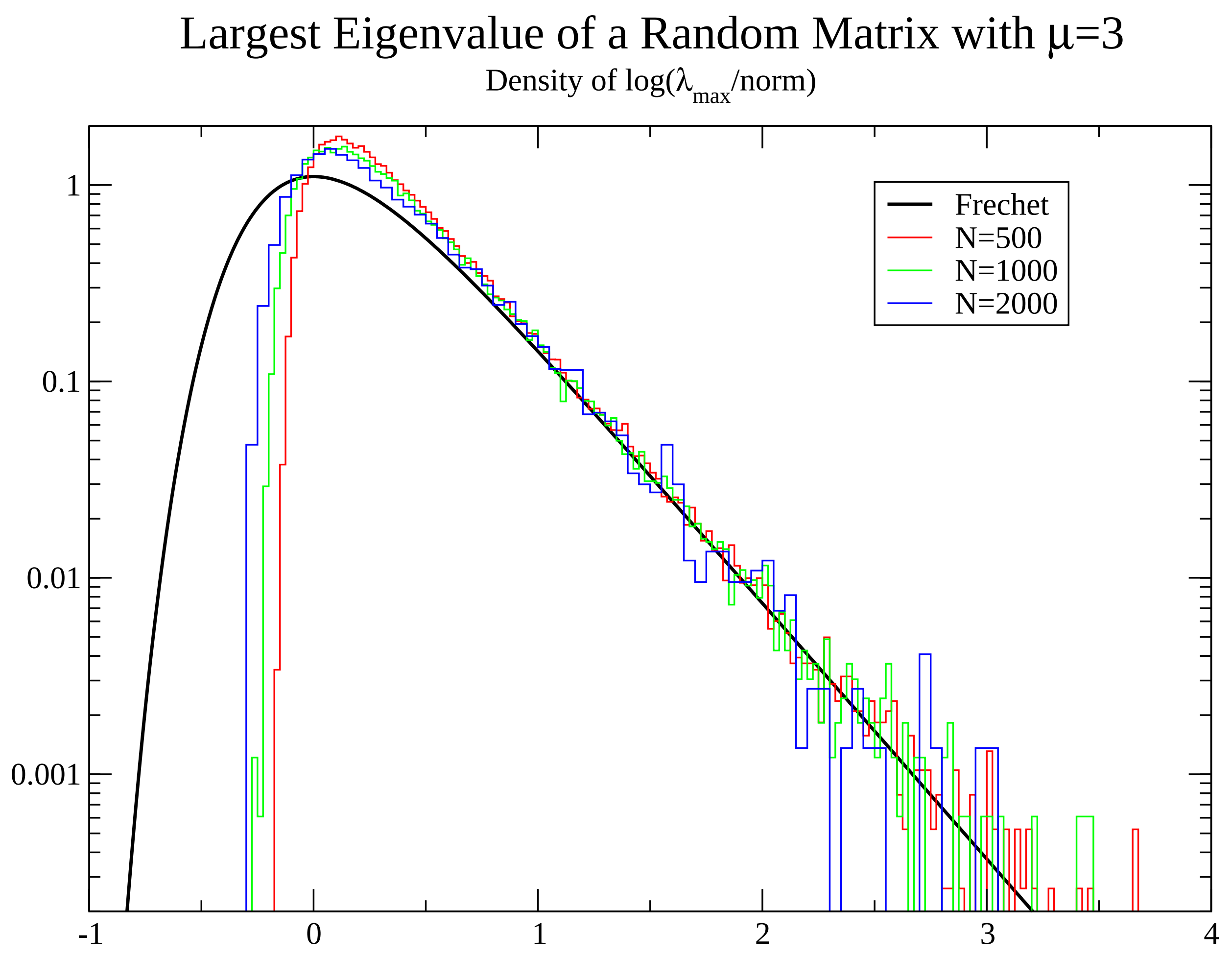
Density for $\mu = 6$



CAPITAL FUND MANAGEMENT

J.Ph. Bouchaud

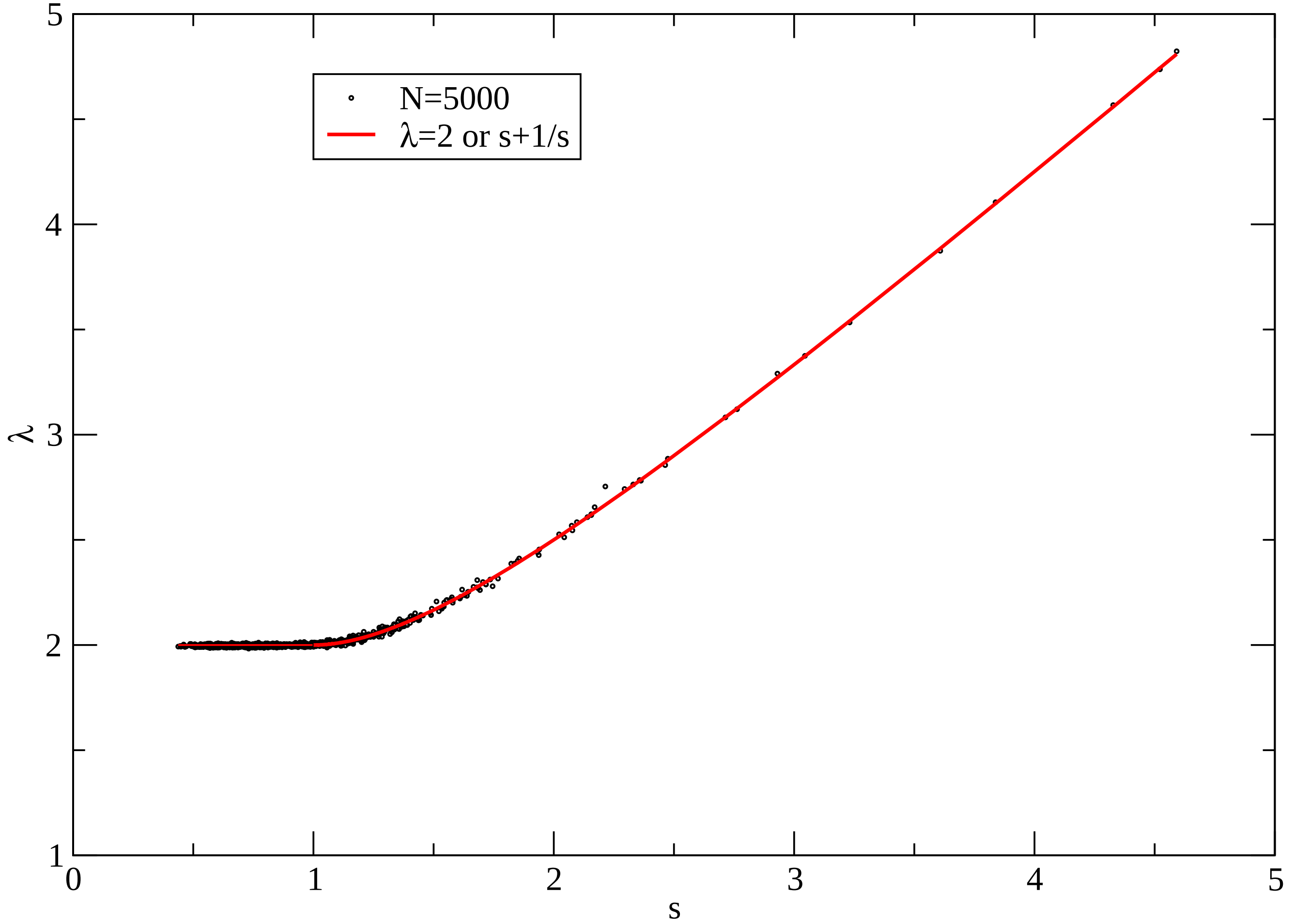
Density for $\mu = 3$



CAPITAL FUND MANAGEMENT

J.Ph. Bouchaud

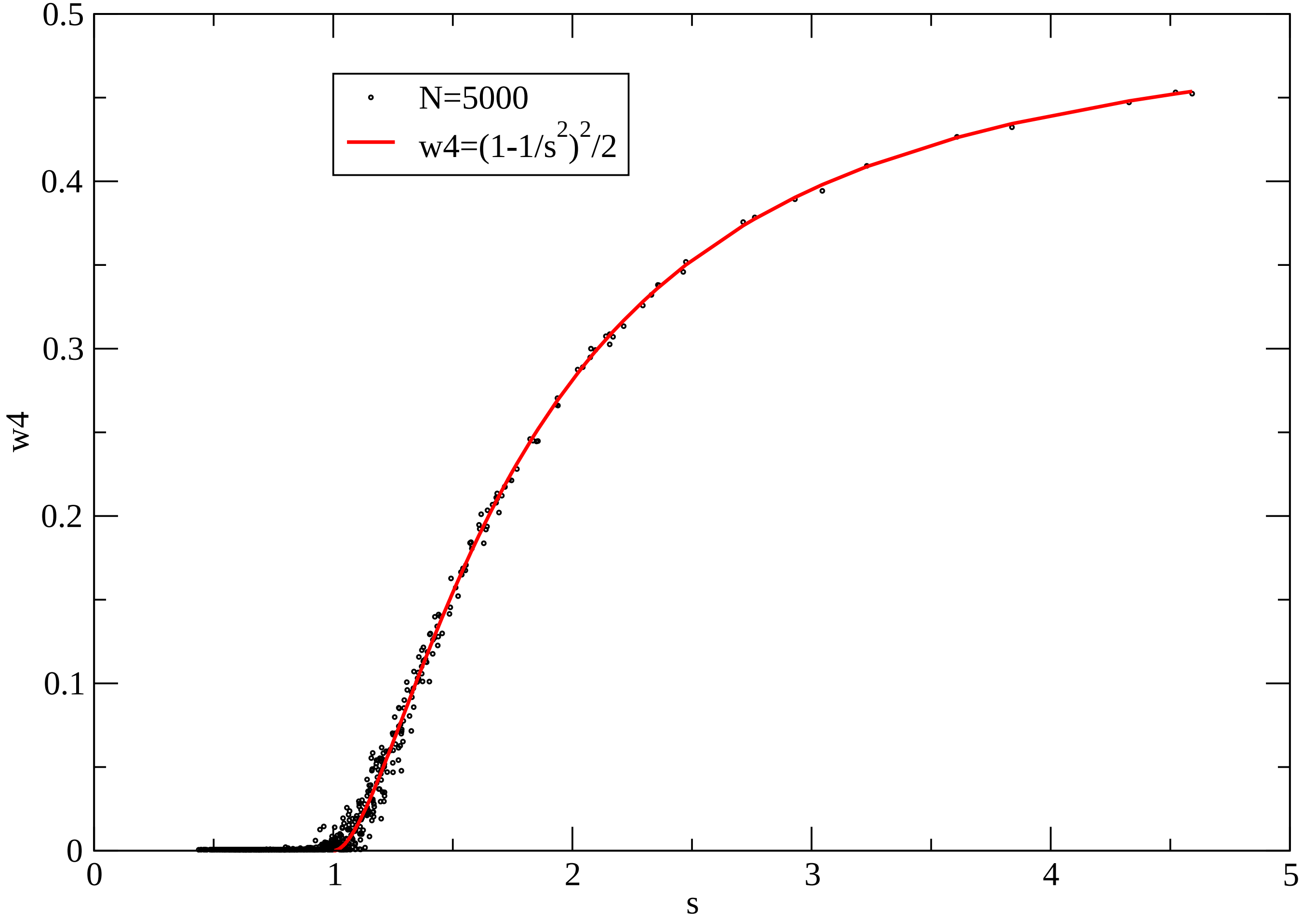
Largest Eigenvalue vs Largest Element ($\mu = 4$)



CAPITAL FUND MANAGEMENT

J.Ph. Bouchaud

Inverse Participation Ratio vs Largest Element



CAPITAL FUND MANAGEMENT

J.Ph. Bouchaud

Wishart Problem with a Large Element

- Take H_{ij} standard Wishart matrix, except for $H_{11} = S$ – i.e. one day one stock jumps

$$W = \sum_{k>1}^{T} H_{i,k} H_{j,k} + H_{i,1} H_{j,1}$$

- Solve using R transform.

$$\lambda_{\max}(S) = \left(\frac{1}{Q} + S^2\right)\left(1 + \frac{1}{S^2}\right) \qquad S \geq 1/Q^{1/4} \qquad (1)$$

- For $S < 1/Q^{1/4}$ one has $\lambda_{\max} = 1 + 1/Q + 1/(2\sqrt{Q})$ (M-P solution).

- Same classification as for the Wigner case around $\mu = 4$

Wishart Problem with a Large Element

- Note: different from the Student ensemble: one day, all stocks jump

Dynamics of the top eigenvector of EMA matrices

- Specific dynamics of large top eigenvalue and eigenvector: Ornstein-Uhlenbeck processes (on the unit sphere for $\mathbf{V}^1$)

- The angle obeys the following SDE:

$$d\theta \approx -\frac{\epsilon}{2} \sin 2\theta dt + \zeta_t \, dW_t$$

with

$$\zeta_t^2 \approx \epsilon^2 \left[\frac{1}{2} \sin^2 2\theta_t + \frac{\lambda_1}{\lambda_0} \cos^2 2\theta_t \right]$$

- Eigenvector dynamics:

$$\left\langle \langle \psi_{0t+\tau} | \psi_{0t} \rangle \right\rangle \approx E(\cos(\theta_t - \theta_{t+\tau})) \approx 1 - \epsilon \frac{\lambda_1}{\lambda_0} (1 - \exp(-\epsilon\tau))$$

The variogram of the top eigenvector