A Direct Proof of the Bichteler-Dellacherie Theorem and Connections to Arbitrage

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(joint with Mathias Beiglböck and Walter Schachermayer)

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Introduction

We have a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ satisfying the usual conditions and a real-valued, càdlàg, adapted process $S = (S_t)_{0 \leq t \leq T}$.

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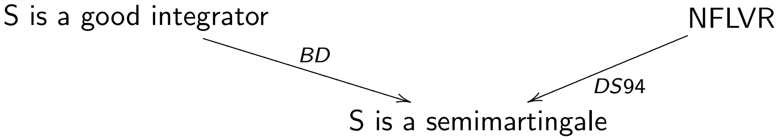

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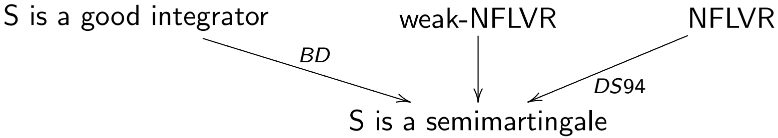

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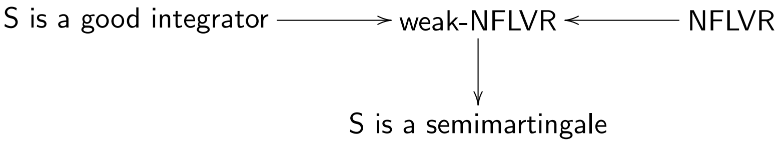

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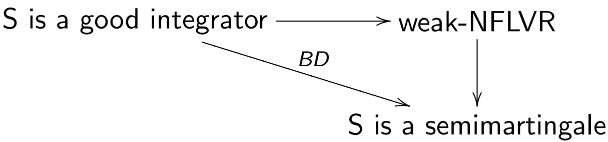

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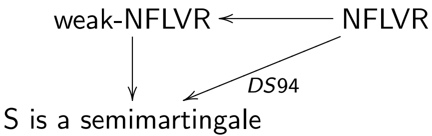

Good Integrator

- A simple integrand is a stochastic process of the form

$$H_t = \sum_{j=1}^{n} h_j \mathbb{1}_{]\tau_{j-1},\tau_j]}(t)$$

where $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n = T$ are stopping times and $h_j \in L^\infty(\Omega, \mathcal{F}_{\tau_{j-1}}, \mathbb{P})$. Denote by $\mathcal{SI}$ the vector space of simple integrands.

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- For each S, we may well-define the integration operator

$$I : \mathcal{SI} \to L^0(\Omega, \mathcal{F}, \mathbb{P})$$

$$\sum_{j=1}^{n} h_j \mathbb{1}_{]\tau_{j-1},\tau_j]} \mapsto \sum_{j=1}^{n} h_j (S_{\tau_j} - S_{\tau_{j-1}}) =: (H \cdot S)_T.$$

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- S is a good integrator if I is continuous i.e. if $\|H^n\|_\infty \to 0$, then $(H^n \cdot S)_T \to 0$ in probability.

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Theorem (Main Theorem)

Let S be locally bounded process. TFAE:

- *S satisfies NFLVRLI.*
- *S is a semimartingale.*

Sketch of Proof

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- Apply discrete Doob-Meyer to obtain $S^n = M^n + A^n$, where $(M^n_{\frac{j}{2^n}})_{j=0}^{2^n}$ is a martingale and $(A^n_{\frac{j}{2^n}})_{j=0}^{2^n}$ is predictable.

Sketch of Proof Continued...

Lemma

Assume NFLVRLI. For $\varepsilon > 0$, there exist a constant $C > 0$ and a sequence of $\{\frac{j}{2^n}\}_{j=1}^{2^n} \cup \{\infty\}$-valued stopping times $(\varrho_n)_{n=1}^{\infty}$ such that $\mathbb{P}(\varrho_n < \infty) < \varepsilon$ and

$$TV(A^{n,\varrho_n}) = \sum_{j=1}^{2^n(\varrho_n \wedge 1)} \left| A^n_{\frac{j}{2^n}} - A^n_{\frac{j-1}{2^n}} \right| \leq C, \tag{1}$$

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$H_t^n = -2 \sum_{j=1}^{2^n} S_{\frac{j-1}{2^n}} \mathbb{1}_{]\frac{j-1}{2^n}, \frac{j}{2^n}]}(t) \Rightarrow \|M_1^{n,\varrho_n}\|^2_{L^2(\Omega)} \leq (H^n \cdot S)_T.$

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$H_t^n = \sum_{j=1}^{2^n} \operatorname{sign}\left(A^n_{\frac{j}{2^n}} - A^n_{\frac{j-1}{2^n}}\right) \mathbb{1}_{]\frac{j-1}{2^n}, \frac{j}{2^n}]}(t) \Rightarrow TV(A^n) \leq (H^n \cdot S)_T.$

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Let $(f_n)_{n\geq 1}$ be a sequence of measurable functions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\sup_{n\geq 1} \|f_n\|_2 < \infty$. Then, there exist functions $g_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$ such that $(g_n)_{n\geq 1}$ converges almost surely and in $\|.\|_{L^2(\Omega)}$.

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- $R_T^n = \mathbb{1}_{[0,\varrho_n]} \to R_T$ using Komlos.
- R_T is a good enough substitute for ϱ.
- Using Komlos again, $A^n \to A$ and $M^n \to M$ on $[0, \varrho]$.