A functional differential equation approach to BSDEs

Matteo Casserini*
joint work with Gechun Liang†

*Department of Mathematics
ETH Zürich

†Oxford-Man Institute

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Outline

Motivation

- $(Y_t)_{t\geq 0}$ a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with known terminal value $Y_T = \xi \in \mathcal{F}_T$.
- Doob-Meyer decomposition:

$$Y_t = M_t - V_t,$$

M martingale, V cont. adapted process of finite variation.

- If V_T is integrable, then:

$$M_t = M(V,\xi)_t = E[\xi + V_T|\mathcal{F}_t] \quad \forall\, t \in [0,T],$$
$$Y_t = Y(V,\xi)_t = E[\xi + V_T|\mathcal{F}_t] - V_t \quad \forall\, t \in [0,T]. \quad (1.1)$$

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Setting

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ filtered probability space satisfying the usual conditions

- $\mathcal{C}([0,T],\mathbb{R}^d) := \{V : \Omega \times [0,T] \to \mathbb{R}^d | V$ continuous and adapted, $E[\max_j \sup_t |V_t^j|^2] < \infty\}$

- $\mathcal{C}_0([0,T],\mathbb{R}^d) := \mathcal{C}([0,T],\mathbb{R}^d) \cap \{V | V_0 = 0\}$

- $\mathcal{M}^2([0,T],\mathbb{R}^d) := \{M : \Omega \times [0,T] \to \mathbb{R}^d | M$ square integrable martingale on $[0,T]\}$

Setting

- $\|V\|_{\mathcal{C}[0,T]} := \sqrt{\sum_j E[\sup_t |V_t^j|^2]}$
- $\mathcal{S}([0,T],\mathbb{R}^d) := \mathcal{C}([0,T],\mathbb{R}^d) \oplus \mathcal{M}^2([0,T],\mathbb{R}^d)$
- $\mathcal{H}^2([0,T],\mathbb{R}^m) := \{Z : \Omega \times [0,T] \to \mathbb{R}^m | Z \text{ predictable}, \|Z\|^2_{\mathcal{H}^2[0,T]} := \sum_j E[\int_0^T |Z_t^j|^2 dt] < \infty\}$
- L nonlinear functional from $\mathcal{M}^2([0,T],\mathbb{R}^d)$ to $\mathcal{H}^2([0,T],\mathbb{R}^m)$ [or $\mathcal{C}([0,T],\mathbb{R}^m)$].

Backward stochastic dynamics

We will consider the backward stochastic dynamics

$$\begin{cases} dY_t &= -f(t, Y_t, L(M)_t)dt + dM_t, \quad t \in [0, T], \\ Y_T &= \xi \in \mathcal{F}_T, \end{cases} \tag{1.2}$$

where $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$.

A solution to (1.2) is a pair of adapted processes (Y, M) satisfying the integral formulation of (1.2) and such that M is a square-integrable martingale.

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Examples for L

Example (1)

- *B m-dim. BM, $(\mathcal{F}_t)_{t\geq 0}$ corresponding augmented filtration*
- *$L : \mathcal{M}^2([0,T],\mathbb{R}) \to \mathcal{H}^2([0,T],\mathbb{R}^m)$ defined via the Itô representation theorem, i.e.*

$$M_t = E[M_t] + \sum_{j=1}^{m} \int_0^t L(M)_s^j dB_s^j$$

- *Backward stochastic dynamics* (1.2) $\Rightarrow$ *classical BSDEs*

Examples for L

Example (2)

- *B m-dim. BM, $(\mathcal{F}_t)_{t\geq 0}$ with usual assumptions*
- *$L : \mathcal{M}^2([0,T],\mathbb{R}) \to \mathcal{H}^2([0,T],\mathbb{R}^m)$ defined via the orthogonal decomposition w.r.t. B, i.e.*

$$M_t = E[M_t] + \sum_{j=1}^{m} \int_0^t L(M)_s^j dB_s^j + M_t'$$

Examples for L

Example (3)

- $(\mathcal{F}_t)_{t\geq 0}$ *quasi-left continuous*
- *For* $M \in \mathcal{M}^2([0,T],\mathbb{R})$ *consider the decomposition*

$$M = M^c + M^d$$

  M^c *continuous martingale null at 0,* M^d *purely discontinuous martingale*
- $L : \mathcal{M}^2([0,T],\mathbb{R}) \to \mathcal{C}([0,T],\mathbb{R})$ *defined by*

$$L(M)_t := \sqrt{\langle M^c \rangle_t}$$

Functional differential equation for V

Question

How can we find a solution of (1.2)?

Define $\mathbb{L} : \mathcal{C}_0([0,T],\mathbb{R}^d) \to \mathcal{C}_0([0,T],\mathbb{R}^d)$ by

$$\mathbb{L}(V)_t := \int_0^t f(s, Y(V)_s, L(M(V))_s)ds,$$

where $M(V)_t := E[\xi + V_T|\mathcal{F}_t]$ and $Y(V)_t := M(V)_t - V_t$.

Assume that $\mathbb{L}$ has a fixed point $\widehat{V} \in \mathcal{C}_0([0,T],\mathbb{R}^d)$. Then it is easy to check that $(Y(\widehat{V}), M(\widehat{V}))$ is a solution of (1.2).

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Existence of local solutions

Assumptions

- $\xi \in L^2(\mathcal{F}_T)$
- f has linear growth in t, y and z and is Lipschitz in y and z:

$$|f(t,y,z)| \leq C_1(1+t+|y|+|z|),$$
$$|f(t,y,z)-f(t,y',z')| \leq C_1(|y-y'|+|z-z'|) \quad \forall\ t,y,z.$$

Existence of local solutions

Assumptions

- The mapping L is bounded and Lipschitz continuous, i.e.

$$\|L(M)\|_{\mathcal{H}^2[0,T]} \leq C_2 \|M\|_{\mathcal{C}[0,T]},$$

$$\|L(M) - L(M')\|_{\mathcal{H}^2[0,T]} \leq C_2 \|M - M'\|_{\mathcal{C}[0,T]} \quad \forall\, M, M' \in \mathcal{M}^2$$

(resp. with $\|\cdot\|_{\mathcal{C}[0,T]}$ on the l.h.s.)

Existence of local solutions

Proposition

There is a constant $\tau \in (0,1]$ *(depending on* C_1*,* C_2 *and the dimension* d*, but independent of* ξ*) such that, for* $T \leq \tau$*,* $\mathbb{L}$ *admits a unique fixed point* V *on* $\mathcal{C}_0([0,T],\mathbb{R}^d)$*.*

Remark

When $(\mathcal{F}_t)_{t\geq 0}$ *is the augmented filtration of a BM and* L *is given by Itô representation, then* $\mathbb{L}$ *admits a unique fixed point for any* $T > 0$*.*

Existence of local solutions

Proposition

There is a constant $\tau \in (0,1]$ (depending on C_1, C_2 and the dimension d, but independent of ξ) such that, for $T \leq \tau$, $\mathbb{L}$ admits a unique fixed point V on $\mathcal{C}_0([0,T], \mathbb{R}^d)$.

Remark

When $(\mathcal{F}_t)_{t\geq 0}$ is the augmented filtration of a BM and L is given by Itô representation, then $\mathbb{L}$ admits a unique fixed point for any $T > 0$.

Existence of local solutions

As a consequence, we obtain a local solution for the equation (1.2):

Theorem

For $T \leq \tau$ and under the above assumptions, the backward stochastic dynamics (1.2) *have a unique solution* (Y, M), *which may be expressed in terms of the solution of the fixed point equation* $\mathbb{L}(V) = V$.

Global solution

- **Next step:** Extend local solutions to global ones

- Additional assumptions on L needed!!!

- For $[T_2, T_1] \subset [0, T]$, define the restriction $L_{[T_2,T_1]}$ from $\mathcal{M}^2([T_2, T_1], \mathbb{R}^d)$ to $\mathcal{H}^2([T_2, T_1], \mathbb{R}^m)$ [resp. $\mathcal{C}([T_2, T_1], \mathbb{R}^m)$] by

$$L_{[T_2,T_1]}(N)_t := L(\widetilde{N})_t, \quad N \in \mathcal{M}^2([T_2, T_1], \mathbb{R}^d),$$

where $\widetilde{N}_t := E[N_{T_1}|\mathcal{F}_t]$, $t \in [0, T]$, is the extension of N to $\mathcal{M}^2([0, T], \mathbb{R}^d)$.

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Global solution

Assumptions

- For $0 \leq T_2 < T_1 \leq T$ and $M \in \mathcal{M}^2([0,T],\mathbb{R}^d)$,

$$L(M) = L_{[T_2,T_1]}(\widehat{M}) \text{ on } (T_2,T_1),$$

  where $\widehat{M} = M\big|_{[T_2,T_1]}$.
- For $0 \leq T_2 < T_1 \leq T$ and $N \in \mathcal{M}^2([T_2,T_1],\mathbb{R}^d)$,

$$L_{[T_2,T_1]}(N - N_{T_2}) = L_{[T_2,T_1]}(N) \text{ on } (T_2,T_1).$$

Global solution

Consider a subdivision of $[0, T]$ in k subintervals $[T_j, T_{j-1}]$:

$$0 = T_k < \cdots < T_1 < T_0 = T,$$

so that $T_{j-1} - T_j \leq \tau$.

Theorem

Under the above additional assumptions, the backward stochastic dynamics (1.2) *have a unique solution* (Y, M) *for any* $T > 0$.

Numerical approximation

Goal

Numerical solution of

$$V_0 = 0, \quad dV_t = f(t, Y(V)_t, L(M(V))_t)dt \quad \text{on } [0, T] \qquad (2.1)$$

- We restrict to classical BSDEs (Example (1)), and assume that $\xi = \phi(W_T)$ for a Lipschitz function ϕ.

$$V_0 = 0, \quad dV_t = f(t, Y(V)_t, Z(V)_t)dt \quad \text{on } [0, T], \qquad (2.2)$$

$$Z(V)_t := D_t M(V)_T = D_t(\phi(W_T) + V_T).$$

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Time discretization

- $\pi := (t_0, t_1, \cdots, t_n)$ a partition of $[0, T]$ with mesh $|\pi|$
  $\Delta t_i = t_i - t_{i-1}$, $\Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$ for $1 \leq i \leq n$
- Discretization of (2.2):

$$V_0^\pi = 0, \; V_{t_i}^\pi = V_{t_{i-1}}^\pi + f(t_{i-1}, Y(V^\pi)_{t_{i-1}}, Z(V^\pi)_{t_{i-1}})\Delta t_i \tag{2.3}$$

$$Y(V^\pi)_{t_{i-1}} = E[\phi(W_T) + V_T^\pi | \mathcal{F}_{t_{i-1}}] - V_{t_{i-1}}^\pi,$$

$$Z(V^\pi)_{t_{i-1}} = \frac{1}{\Delta t_i} E\left[(\phi(W_T) + V_T^\pi)(\Delta W_{t_i})^T | \mathcal{F}_{t_{i-1}}\right].$$

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Time discretization

Extend V^π to $[0, T]$ by linear interpolation.

Theorem

Under the assumption that

$$|f(t,y,z) - f(t',y',z')| \le C_1\left(\sqrt{|t-t'|} + |y-y'| + |z-z'|\right),$$

there is a constant C_3, depending only on the constants and dimensions involved, such that

$$\sup_{0\le t\le T} E[|V_t - V_t^\pi|^2] \le C_3|\pi|.$$

Picard iteration of the discretized equation

- The discretized equation (2.3) can't be solved explicitly (dependence on the terminal value).
- Continuous time results $\Rightarrow$ approximate (2.3) via a Picard iteration procedure.
- **Advantage:** Avoid the nesting of conditional expectations (arising in most numerical approaches to BSDEs), thus reducing the amplificaton of the error.

Picard iteration of the discretized equation

Define the Picard approximations $V^{p,\pi}$ of V^{π} recursively by $V^{0,\pi} \equiv 0$ and, for $p \geq 1$ and $1 \leq i \leq n$,

$$\begin{aligned} V_0^{p,\pi} &= 0, \\ V_{t_i}^{p,\pi} &= V_{t_{i-1}}^{p,\pi} + f(t_{i-1}, Y(V^{p-1,\pi})_{t_{i-1}}, Z(V^{p-1,\pi})_{t_{i-1}})\Delta t_i. \end{aligned} \tag{2.4}$$

Picard iteration of the discretized equation

Theorem

Under above assumptions, there are constants C_4 and C_5, depending only on the constants and dimensions involved, such that

$$\max_{0\leq i\leq n} E[|V_{t_i}^{p,\pi} - V_{t_i}^{\pi}|^2] \leq C_4 \left(\frac{1}{2} + C_5|\pi|\right)^p .$$

Picard iteration of the discretized equation

By collecting the above results, we can get:

Theorem

Under above assumptions, there is a constant C, depending only on the Lipschitz constants involved and the dimension of the problem, such that

$$\sup_{0\leq t\leq T} E[|V_t - V_t^{p,\pi}|^2] \leq C\left(|\pi| + \left(\frac{1}{2} + C|\pi|\right)^p\right).$$

Possible future directions

- Extension to fully coupled FBSDEs (Delarue, ...)
- Extend the numerical approximation to other types of functionals L (Example 3)
- Extension to BSDEs of quadratic growth (Kobylanski, Briand-Hu, Morlais, Tevzadze, ...)

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Thank you for your attention!