On refined volatility smile expansion in the Heston model

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Heston Model

- Dynamics

$$dS_t = S_t \sqrt{V_t} dW_t, \qquad S_0 = 1,$$
$$dV_t = (a + bV_t)\, dt + c\sqrt{V_t} dZ_t, \qquad V_0 = v_0 > 0,$$

- Correlated Brownian motions

$$d\langle W, Z \rangle_t = \rho dt, \qquad \rho \in [-1, 1]$$

- Parameters

$$a \geq 0, b \leq 0, c > 0$$

Density and smile asymptotics

- Consider a fixed maturity $T > 0$.
- $D_T :=$ density of S_T.
- How heavy are the tails?

$$D_T(x) \sim \ ? \qquad (x \to 0, \infty)$$

- Implied Black-Scholes volatility ($k = \log K$ is the log-strike)

$$\sigma_{BS}^2(k, T) \sim \ ? \qquad (k \to \pm\infty)$$

Known results

- Leading term of smile asymptotics: Lee's moment formula. Andersen, Piterbarg (2007); Benaim, Friz (2008)
- Drăgulescu, Yakovenko (2002): Stationary variance regime. Leading growth order of distribution function of S_T, by (non-rigorous) saddle-point argument
- Gulisashvili-Stein (2009): Precise density asymptotics for uncorrelated Heston model

Main results (right tail), SG et al. 2010

- Density asymptotics for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-3/4+a/c^2} \left(1+O((\log x)^{-1/2})\right)$$

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS}(k,T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

(φ arbitrary function tending to ∞)

Interpretation of smile expansion

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS}(k, T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

- β_1 does not depend on $\sqrt{v_0}$
- β_2 depends linearly on $\sqrt{v_0}$
- Changes of $\sqrt{v_0}$ have second-order effects
- Increase $\sqrt{v_0}$: parallel shift, slope not affected
- Changes in mean-reversion level $\bar{v} = -a/b$ seen only in β_3

General remarks

- Constants depend on: critical moment, critical slope, critical curvature
- Critical moment etc. defined in a model-free manner
- Closed form of Fourier (Mellin) transform not needed
- Work only with affine principles (Riccati equations)

Lee's moment formula (2004)

- Model-free result
- Relates critical moment to implied volatility

$$s^* := \sup\{s : E[S_T^s] < \infty\}$$

$$s^* =: \frac{1}{2\beta_1^2} + \frac{\beta_1^2}{8} + \frac{1}{2}$$

$$\limsup_{k \to \infty} \frac{\sigma_{BS}(k, T)\sqrt{T}}{\sqrt{k}} = \beta_1$$

- Refinements by Benaim, Friz (2008), Gulisashvili (2009)

Heston Model: Mgf of log-spot X_t

- Moment generating function

$$E[e^{sX_t}] = \exp(\phi(s,t) + v_0\psi(s,t))$$

- Riccati equations

$$\begin{aligned} \partial_t\phi &= F(s,\psi),\ \ \phi(0) = 0, \\ \partial_t\psi &= R(s,\psi),\ \ \psi(0) = 0 \end{aligned}$$

$$\begin{aligned} F(s,v) &= av, \\ R(s,v) &= \frac{1}{2}(s^2 - s) + \frac{1}{2}c^2v^2 + bv + s\rho cv \end{aligned}$$

- Explicit solution possible, but cumbersome expression

Moment explosion

- Critical moment for time T

$$s^* := \sup\{s \geq 1 : E[S_T^s] < \infty\}$$

- Explosion time for moment of order s

$$T^*(s) = \sup\{t \geq 0 : E[S_t^s] < \infty\}$$

- Critical slope, critical curvature:

$$\sigma := -\partial_s T^*|_{s^*} \geq 0 \qquad \text{and} \qquad \kappa := \partial_s^2 T^*|_{s^*}$$

Explicit Explosion time for the Heston model

- Explosion time for moment of order s

$$T^*(s) = \frac{2}{\sqrt{-\Delta(s)}}\left(\arctan\frac{\sqrt{-\Delta(s)}}{s\rho c + b} + \pi\right),$$

$$\Delta(s) := (s\rho c + b)^2 - c^2\left(s^2 - s\right)$$

- Critical moment s^*: Find numerically from

$$T^*(s^*) = T.$$

Mellin (Fourier) inversion

- Mellin transform of spot: $M(u) = E[e^{(u-1)X_T}]$
- Analytic in a complex strip
- Density of S_T by Mellin inversion:

$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) du.$$

- Valid for contour in analyticity strip of the Mellin transform
- Justification: exponential decay of $M(u)$ at $\pm i\infty$.

Analyticity and growth

- Mellin transform analytic in a strip

$$u_- < \Re(u) < u^* = s^* + 1$$

- Leading order of density for $x \to \infty$

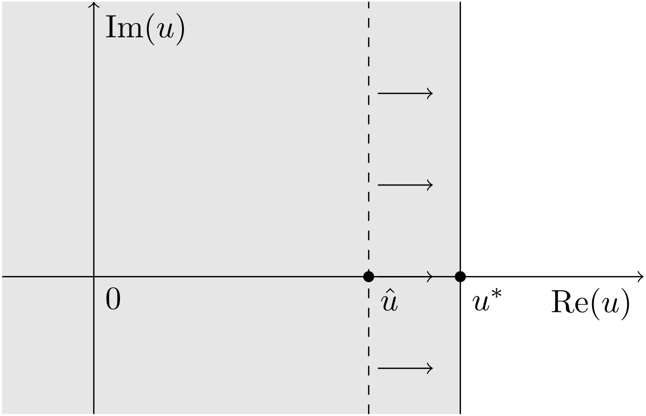
$$x^{-u^*-\varepsilon} \ll D_T(x) \ll x^{-u^*+\varepsilon},$$

  depends on *location* of singularity
- Refinement: lower order factors depend on *type* of singularity

Saddle point method

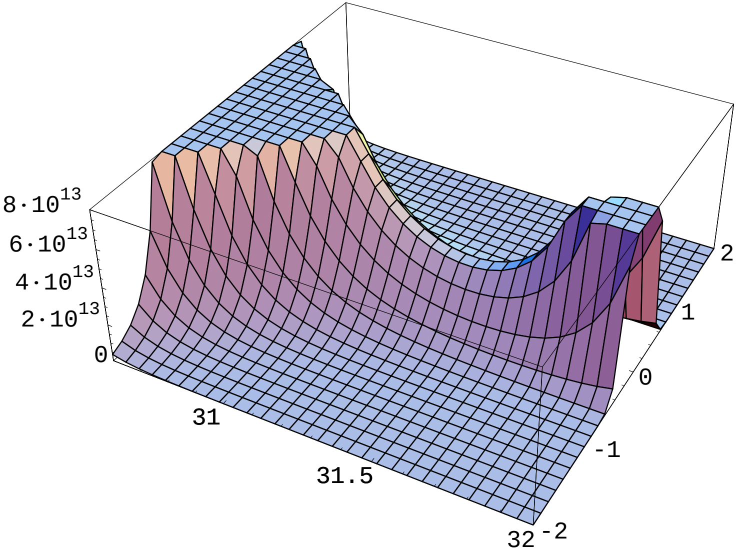
- Recall:
$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) du$$
- Shift contour to the right, close to the singularity.
- Let it pass through a saddle point of the integrand.
- For large x, the integral is concentrated around the saddle.
- Local expansion of integrand yields expansion of whole integral.
- (Laplace, Riemann, Debye...)

New integration contour



- Contour runs through saddle point $\hat{u} = \hat{u}(x)$
- Moves to the right as $x \to \infty$

The surface $|x^{-u}M(u)|$



Asymptotics of ψ and ϕ near critical moment

- Recall $M(u) = \exp(\phi(u-1,t) + v_0\psi(u-1,t))$
- For $u \to u^*$ we have (with $\beta := \sqrt{2v_0}/c\sqrt{\sigma}$)

$$\psi(u-1,T) = \frac{\beta^2}{u^*-u} + const + O(u^*-u),$$
$$\phi(u-1,T) = \frac{2a}{c^2}\log\frac{1}{u^*-u} + const + O(u^*-u)$$

- Found from Riccati equations

Saddle point method

- Finding the saddle point: $0 =$ derivative of integrand
- Use only first order expansion:

$$0 = \frac{\partial}{\partial u} x^{-u} \exp\left(\frac{\beta^2}{u^* - u}\right)$$

- Approximate saddle point at

$$\hat{u}(x) = u^* - \beta / \sqrt{\log x}$$

New integration contour

- Contour depends on x:

$$u = \hat{u}(x) + iy, \qquad -\infty < y < \infty$$

- Divide contour into three parts:

$$|y| < (\log x)^{-\alpha} \quad \text{(central part)},$$

  upper tail, lower tail (symmetric)
- Uniform local expansion at saddle point $\Rightarrow$ need large α
- Tails negligible $\Rightarrow$ need small α
- Can take $\frac{2}{3} < \alpha < \frac{3}{4}$

Local expansion

- Recall Mellin transform

$$M(u) = \exp(\phi(u-1,t) + v_0\psi(u-1,t))$$

- Determine singular expansions of ϕ and ψ from Riccati equations
- Abbreviation $L := \log x$
- Local expansion of the integrand:

$$x^{-u}M(u) = Cx^{-u^*}\exp\left(2\beta L^{1/2} + \tfrac{a}{c^2}\log L - \beta^{-1}L^{3/2}y^2 + o(1)\right)$$

Local expansion

- Gaussian integral

$$
\begin{aligned}
&\int_{-L^{-\alpha}}^{L^{-\alpha}} \exp(-\beta^{-1} L^{3/2} y^2) dy \\
&= \beta^{1/2} L^{-3/4} \int_{-\beta^{-1/2} L^{3/4-\alpha}}^{\beta^{-1/2} L^{3/4-\alpha}} \exp(-w^2) dw \\
&\sim \beta^{1/2} L^{-3/4} \int_{-\infty}^{\infty} \exp(-w^2) dw = \sqrt{\pi} \beta^{1/2} L^{-3/4}
\end{aligned}
$$

Tail estimate

- Finding saddle point + local expansion fairly routine
- Problem: Verify concentration
- Needs some insight into behaviour of function away from saddle point
- Show exponential decay by ODE comparison

Result of saddle point method

- Density asymptotics for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2\sqrt{\log x}} (\log x)^{-3/4+a/c^2} \left(1+O((\log x)^{-1/2})\right)$$

- Constants in terms of critical moment and critical slope:

$$A_3 = u^* = s^* + 1 \qquad \text{and} \qquad A_2 = 2\frac{\sqrt{2v_0}}{c\sqrt{\sigma}}$$

- Easily extended to full asymptotic expansion

Explicit expression for constant factor

- From closed form of ϕ and ψ:

$$A_1 = \frac{1}{2\sqrt{\pi}} (2v_0)^{1/4-a/c^2} \, c^{2a/c^2-1/2} \sigma^{-a/c^2-1/4}$$

$$\times \exp\left(-v_0\left(\frac{b+s^*\rho c}{c^2}+\frac{\kappa}{c^2\sigma^2}\right)-\frac{aT}{c^2}(b+c\rho s^*)\right)$$

$$\times \left(\frac{2\sqrt{b^2+2bc\rho s^*+c^2s^*(1-(1-\rho^2)s^*)}}{c^2s^*(s^*-1)\sinh\frac{1}{2}\sqrt{b^2+2bc\rho s^*+c^2s^*(1-(1-\rho^2)s^*)}}\right)^{2a/c^2}$$

Call prices and Smile asymptotics

- Gulisashvili (2009): Assumes that density of spot varies regularly at infinity

$$D_T(x) = x^{-\gamma} h(x),$$

  h varies slowly at infinity, $\gamma > 2$
- Expansions of call prices and implied volatility
- Similarly for left tail

Smile asymptotics

- Implied volatility for log-strike $k \to \infty$

$$\sigma_{BS}(k,T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

- Constants

$$\begin{aligned}
\beta_1 &= \sqrt{2}\left(\sqrt{A_3 - 1} - \sqrt{A_3 - 2}\right), \\
\beta_2 &= \frac{A_2}{\sqrt{2}}\left(\frac{1}{\sqrt{A_3 - 2}} - \frac{1}{\sqrt{A_3 - 1}}\right), \\
\beta_3 &= \frac{1}{\sqrt{2}}\left(\frac{1}{4} - \frac{a}{c^2}\right)\left(\frac{1}{\sqrt{A_3 - 1}} - \frac{1}{\sqrt{A_3 - 2}}\right)
\end{aligned}$$

Call prices

- Call price for strike $K \to \infty$

$$C(K) = \frac{A_1}{(-A_3+1)(-A_3+2)} K^{-A_3+2} e^{A_2\sqrt{\log K}} (\log K)^{-\frac{3}{4}+\frac{a}{c^2}} \times \left(1 + O\left((\log K)^{-\frac{1}{4}}\right)\right)$$

Smile asymptotics

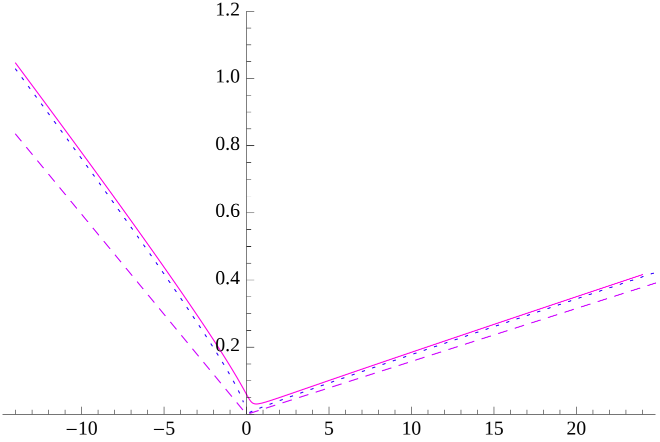


Figure: Implied variance $\sigma(k,1)^2$ in terms of log-strikes compared to the first order (dashed) and third order (dotted) approximations.

References

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