Joint Dynamics using Asymptotic Methods

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Equity Markets

Cross Smile Prediction

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GROUPE BPCE

Outline of Presentation

1. Problem

2. Model Description

- Local Volatility Component
- Stochastic Volatility Component

3. Cross Smile Estimation

- Spread Pricing
- Smile Calculation

4. Back testing

5. Summary & Conclusions

1 Problem

Financial Problem

USDJPY and AUDUSD are two liquid currency pairs (ATM vols known, Smile known)

AUDJPY is less liquid (ATM vols known)

What is the smile of AUDJPY?

Mathematical Problem

$S_{1,t}, S_{2,t}$ **Marginal Laws known**

What is the law of $\dfrac{S_{1,t}}{S_{2,t}}$ **?**

Approach

Use **Risk Neutral** Approach to take on board all observed data (photography)

Use **Historical Data** to incorporate evolution information (dynamic)

Base analysis on **rich** models (Multi LSVLC)

Use **efficient numerical** techniques to perform all calculations (asymptotics)

2.1 Model Description

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Model

Each currency pair is a mixture of local volatility and stochastic volatility

Stochastic volatility introduces correlation between volatilities

Local volatility introduces level dependency to the cross

Literature I – on local correlation

A-M. Avenalleda, D. Boyer-Olson, J. Busca and P.Friz, « Reconstructing Volatility », October 2002

B-V. Durrleman + N. El-Karoui, « Basket Skew », April 2007

C-Bruno Dupire « Basket Skew Asymptotics » working paper 2004

D-X. Burtschell, J. Gregory and J-P. Laurent, « Beyond the Gaussian Copula: Stochastic and Local Correlation » , Working Paper, 2005

E-A. Langnau, « Introduction Into Local Correlation Modelling » , September 2009.

F-B. Jourdain, Mohamed Sbai "Coupling Index and stocks" 2009

Literature II – some comments

A-It gives the framework for calibrating baskets and numerical algorithms for short term asymptotics for pricing

B-C-It provides a good grasp of the phenomenology with model free approach

Good for the phenomenology

D-Simple idea to expand the dimension and obtain stochastic correlation at a cheap cost (is used for the local correlation model)

E-Simplest local volatility extension plus direct calibration formulae and model risk illustration through the chewing gum effect

F- Nice numerical method – particle method, specific to baskets

However,

As will be shown in the sequel, we need:

- **local volatility and local correlation**
- **Fast Calibration : flow business**
- **Precise formulae for pricing**

Local Volatility Component

$$\frac{dS_{1,t}}{S_{1,t}} = \sigma_1(S_{1,t}, t)dW_{1,t}$$

$$\frac{dS_{2,t}}{S_{2,t}} = \sigma_2(S_{2,t}, t)dW_{2,t}$$

$$< dW_{1,t}, dW_{2,t} >= \rho_{12}\left(\frac{S_{1,t}}{S_{2,t}}\right)dt$$

Equation (1)

Historical Data confirm link between correlation and cross level

Local Volatility Component (2)

Calibrated to
atm vol of
the cross

$$\ln\left(\frac{1+\rho_{12}\left(\frac{S_{1,t}}{S_{2,t}}\right)}{1-\rho_{12}\left(\frac{S_{1,t}}{S_{2,t}}\right)}\right)=a+b\ln\left(\frac{S_{1,t}}{S_{2,t}}\right)+c\ln^2\left(\frac{S_{1,t}}{S_{2,t}}\right) \quad \textbf{Eq. (2)}$$

Classical Fisher
Transform : avoid
boundary problems

Slope : trader's
input

Curvature :
trader's input

Stoch. Volatility Component

$$\frac{dS_{1,t}}{S_{1,t}} = \sigma_1 e^{\alpha_1 \tilde{W}_{1,t} - \frac{1}{2}\alpha_1{}^2 t} dW_{1,t}$$

$$\frac{dS_{2,t}}{S_{2,t}} = \sigma_2 e^{\alpha_2 \tilde{W}_{2,t} - \frac{1}{2}\alpha_2{}^2 t} dW_{2,t}$$

$$< dW_{1,t}, dW_{2,t} >= \rho_{12}^{S} dt$$

$$< d\tilde{W}_{1,t}, d\tilde{W}_{2,t} >= \rho_{12}^{\sigma} dt$$

$$< dW_{1,t}, d\tilde{W}_{2,t} >= \rho_{12}^{S,\sigma} dt$$

$$< d\tilde{W}_{1,t}, dW_{2,t} >= \rho_{12}^{\sigma,S} dt$$

Eq. (3)

Spot correlation is calibrated to the atm of the cross

Volvol correlation is a trader's input that can be estimated through historical data

Spot vol correlation has a very small impact

2.2 Mathematical Results

Results

A- Short Term Asymptotic for LSVLC

B- Multi Stochastic VoL Perturbation approach

C- Multi Local Volatility Using most likely path combined with gradient conditioning

D- LSVLC combination result

A-Ito and Short Term Asymptotic

Pricing European Options under the general Local Stochastic Volatility and Local Correlation

$$\frac{dS_{i,t}}{S_{i,t}} = \sigma_i dW^S_{i,t}, i = 1,\dots, n$$

$$\frac{d\sigma_i}{\sigma_i} = \alpha_i dW^{\sigma}_{i,t}$$

$$< dW^u_{i,t}, dW^v_{j,t} >= \rho^{u,v}_{i,j}\left(S_{1,t},\dots, S_{n,t}\right)dt ,\dots, u, v \in \{S, \sigma\}$$ **Eq. (4)**

We want to price the Basket European options “linear”

$$B_T = \sum_{i=1}^{n} S_i(T)$$ **Eq. (5)**

A-Ito and Short Term Asymptotic

Notations

$$\sigma_B^2 = \sum_{i,j=1}^{n} \rho_{i,j}\sigma_i\sigma_j\omega_i\omega_j$$

$$\omega_i = \frac{S_i}{\sum_{j=1}^{n} S_j}$$

$$\beta_{i,j} = \frac{\rho_{i,j}\sigma_i\sigma_j\omega_i\omega_j}{\sum_{i,j=1}^{n} \rho_{i,j}\sigma_i\sigma_j\omega_i\omega_j}$$

$$\beta_i = \sum_{j=1}^{n} \beta_{i,j} = \sum_{j=1}^{n} \beta_{j,i}$$

Approach

- **Use Ito on special variable**
- **Take limit when time goes to zero**

A-Ito and Short Term Asymptotic

Result

$$X_t = B_T^{\frac{1}{\sigma_B}}$$

$$dX_t = \frac{\sum_{i=1}^{n} \sigma_i \omega_i dW_{i,t}}{\sigma_B} - \frac{1}{2} \ln X_t \frac{d\sigma_B^2}{\sigma_B^2} + \theta_t dt$$

$$\frac{d\sigma_B^2}{\sigma_B^2} = \sum_{i,j=1}^{n} \underbrace{\beta_{i,j} \frac{d\rho_{i,j}}{\rho_{i,j}}}_{\text{correlation dynamic}} + 2\sum_{i=1}^{n} \underbrace{\beta_i \frac{d\sigma_i}{\sigma_i}}_{\text{volatility dynamic}} + 2\sum_{i=1}^{n} \underbrace{\beta_i \frac{d\omega_i}{\omega_i}}_{\text{weight's variability}}$$

Three terms contributing to the distortion from a log normal

- (1) Weights variability
- (2) Each underlying own distortion
- (3) Correlation skew

A-Case I : Pat Hagan formula recovered

We look at a one stoch vol model - keep one underlying :

$$X_t = B_T^{\frac{1}{\sigma_B}}$$

$$\frac{dX_t}{X_t} = \sigma(X_t)dZ_t$$

$$\sigma^2(X_t) = 1 + \alpha^2 \ln^2(X_t) - 2\rho_{S,\sigma}\alpha \ln(X_t)$$

This becomes a local volatility model for which the implied volatility is given by the classical BBF formula in [6]

Recover easily the Pat Hagan formula cf [7]

A-Case II : Sum of log-normals is not a log normal

We keep one term coming from the weights variability:

$$X_t = B_T^{\frac{1}{\sigma_B}}$$

$$\frac{dX_t}{X_t} = \sigma(X_t)dZ_t$$

$$\sigma^2(X_t) =$$

$$\sum_{i,j=1}^{n} \frac{w_i \sigma_i w_j \sigma_j \rho_{i,j}^{S,S}}{\sigma_B^2}$$

$$-2\ln(X_t)\sum_{i,j=1}^{n}(\beta_i - w_i)\frac{\sigma_i w_j \sigma_j \rho_{i,j}^{S,S}}{\sigma_B}$$

$$+\ln^2(X_t)\sum_{i,j=1}^{n}(\beta_i - w_i)(\beta_j - w_j)\sigma_i \sigma_j \rho_{i,j}^{S,S}$$

- **We have a skew**
- **We have a curvature**
- **The distribution that is generated is not a log-normal**

A-Case III : Multi Stoch vol and no local correlation nor local volatility

We keep contribution from each underlying smile

We neglect the variability of the weights (in practice it is negligible)

$$X_t = B_T^{\frac{1}{\sigma_B}}$$

$$\frac{dX_t}{X_t} = \sigma(X_t)dZ_t$$

$$\sigma^2(X_t) = 1 - 2\ln(X_t)\sum_{i,j=1}^{n}\frac{\sigma_i}{\sigma_B}w_i\beta_j\alpha_j\rho_{i,j}^{S,\sigma} + \ln^2(X_t)\sum_{i,j=1}^{n}\beta_i\alpha_i\beta_j\alpha_j\rho_{i,j}^{\sigma,\sigma}$$

A-Case IV : Multi Stoch vol asymptotic implied volatility calculation

Moment match the two coefficients of the log(X) expansions

Use the Pat Hagan formula

$$\tilde{\alpha}^2 = \sum_{i,j=1}^{n} \beta_i \alpha_i \beta_j \alpha_j \rho_{i,j}^{\sigma,\sigma}$$

$$\tilde{\rho}\tilde{\alpha} = \sum_{i,j=1}^{n} \frac{\sigma_i}{\sigma_B} w_i \beta_i \alpha_i \rho_{i,j}^{S,\sigma}$$

Moment Match

$$\Sigma_{BS}(T,K) = \Sigma(T,B_0) f\left(\frac{\ln\left(\frac{B_0}{K}\right)}{\Sigma(T,B_0)} \right)$$

$$f(x) = \frac{\tilde{\alpha}x}{\ln\left(\frac{\sqrt{\tilde{\alpha}^2 x^2 - 2\tilde{\alpha}\tilde{\rho}x + 1} - \tilde{\rho} + \tilde{\alpha}x}{1-\tilde{\rho}} \right)}$$

Pat Hagan Formula cf[7]

A-Case V : Local Correlation Model

No stochastic volatility

We assume that the local correlation is given by the following formula

$$\frac{d\rho_{i,j}}{\rho_{i,j}} = -(1-\delta_{i-j})\lambda_{i,j}\ln\left(X_t\right)dZ_t$$

We obtain the following dynamic

$$X_t = B_T^{\frac{1}{\sigma_B}}$$

$$\frac{dX_t}{X_t} = \frac{\sum_{i=1}^{n}\sigma_i\omega_i dW_{i,t}}{\sigma_B} - \frac{1}{2}\ln^2 X_t \sum_{i,j=1}^{n}\beta_{i,j}\lambda_{i,j}dZ_t + \theta_t dt$$

$$\frac{dX_t}{X_t} = \sigma\left(X_t\right)dB_t$$

$$\sigma^2\left(X_t\right) = 1 + \frac{1}{4}\ln^4 X_t\left(\sum_{i,j=1}^{n}\beta_{i,j}\lambda_{i,j}\right)$$

B-Multi Stoch. Volatility European

Pricing European Options under multi asset Stochastic Volatility can be performed using perturbation techniques

$$\begin{cases} \dfrac{dS_i}{S_i} = \mu_i dt + \sigma_i dW_i^s \\ d\sigma_i = \varepsilon \alpha_i \sigma_i dW_i^\sigma \end{cases}$$

Eq. (6)

$$< dW_i^s, dW_j^\sigma >= \sqrt{\varepsilon}\rho_{i,j}^{s,\sigma} dt, < dW_i^s, dW_j^s >= \rho_{i,j}^s dt, < dW_i^\sigma, dW_j^\sigma >= \rho_{i,j}^\sigma dt$$

We want to price the European payoff f

$$u = E\left(f\left(S_1(T),..., S_n(T)\right) | S_1(t) = s_1,..., S_n(t) = s_n, \sigma_1(t) = \sigma_1,..., \sigma_n(t) = \sigma_n\right)$$

B-Multi Stoch. Volatility (2)

It is based on the Black Scholes price (eps=0) and its greeks

Result

$$\begin{cases} \partial_t u_0 + \sum_i \left(\mu_i - \frac{1}{2}\sigma_i^2 \right) \partial_{x_i} u_0 + \frac{1}{2} \sum_{i,j} \rho^s_{i,j} \sigma_i \sigma_j \partial_{x_i x_j} u_0 = 0 \\ u_0(T) = f \end{cases}$$

Eq. (7)

$$\begin{aligned} u &= u_0 && \textit{Black \& Scholes} \\ &+ (T-t)\left\{ \sum_{i,j} \frac{1}{12} \alpha_i \alpha_j \sigma_i(t) \sigma_j(t) \rho^{\sigma,\sigma}_{i,j} \frac{\partial u_0}{\partial(\sigma_i \sigma_j)} \right\} && \textbf{cross Varga} \\ &+ (T-t)\left\{ \sum_{i,j} \frac{1}{6} \alpha_i \alpha_j \sigma_i(t) \sigma_j(t) \rho^{\sigma,\sigma}_{i,j} \frac{\partial^2 u_0}{\partial \sigma_i \partial \sigma_j} \right\} && \textbf{cross Vomma} \\ &+ (T-t)\left\{ \sum_{i,j} \frac{1}{2} \sigma_i(t) \alpha_j \sigma_j(t) \rho^{S,\sigma}_{i,j} S_i \frac{\partial^2 u_0}{\partial S_i \partial \sigma_j} \right\} && \textbf{cross Vanna} \end{aligned}$$

Eq. (8)

B-Proof in 1D

Dynamic in 1D is given by:

$$\frac{dS_t^{\varepsilon}}{S_t^{\varepsilon}} = \mu_t dt + \sigma_t dW_t^1$$
$$d\sigma_t = \varepsilon\eta(t,\sigma_t)dt + \sqrt{\varepsilon}\alpha(\sigma_t,t)dW_t^2$$

Eq. (9)

Option's price satisfies $C(t,s,\varepsilon)$

$$C_t + \mu_t s C_s + \frac{1}{2}\sigma^2 s^2 C_{ss} + \varepsilon\left(\eta(t,\sigma)C_\sigma + \frac{\alpha^2(t,\sigma)}{2}C_{\sigma\sigma} + \sigma\rho_t s\alpha(t,\sigma)\right) = 0$$
$$C(T,s) = \phi(s)$$

Eq. (10)

B-Proof in 1 D : order 0

Equation order 0 is Black & Scholes:

$$C_t^0 + \mu_t s C_s^0 + \frac{1}{2}\sigma_0^2 s^2 C_{ss}^0 = 0$$
$$C^0(T,s) = \phi(s)$$

Eq. (11)

Change of variables

$$\phi_{\exp}(x) = \phi(S_0 e^x)$$

$$C_t + (\mu_t - \frac{\sigma^2}{2})C_x + \frac{1}{2}\sigma^2 C_{xx} + \varepsilon\left(\eta(t,\sigma)C_\sigma + \frac{\alpha^2(t,\sigma)}{2}C_{\sigma\sigma} + \sigma\rho_t\alpha(t,\sigma)C_{x\sigma}\right) = 0$$
$$C(T,s) = \phi_{\exp}(s)$$

Eq. (12)

B-Proof in 1 D : order 1

Compute derivative w.r.t to epsilon:

$$v^{\varepsilon}(t,x) = \partial_{\varepsilon} C(t,x,\varepsilon)$$

Equation becomes:

$$v_t^{\varepsilon} + (\mu_t - \frac{\sigma^2}{2})v_x^{\varepsilon} + \frac{1}{2}\sigma^2 v_{xx}^{\varepsilon} + \varepsilon\left(\eta(t,\sigma)v_{\sigma}^{\varepsilon} + \frac{\alpha^2(t,\sigma)}{2}v_{\sigma\sigma}^{\varepsilon} + \sigma\rho_t\alpha(t,\sigma)v_{x\sigma}^{\varepsilon}\right) =$$

$$-\left(\eta(t,\sigma)C_{\sigma} + \frac{\alpha^2(t,\sigma)}{2}C_{\sigma\sigma} + \sigma\rho_t\alpha(t,\sigma)C_{x\sigma}\right)$$

$$v^{\varepsilon}(T,x) = 0$$

Eq. (13)

B-Proof in 1 D : Math Toolbox

(1) Lemma (magical lemma):

- Let Xt be a martingale and let P(t, Xt) a pricing function then :

Eq. (14)

$$\frac{\partial^n P}{\partial x^n}(t, X_t) = E\left(\frac{\partial^n P}{\partial x^n}(T, X_T) \middle| X_t\right) \quad \forall n$$

Feymann-Kac

if X_t satifies the following SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

Then the following value function

$$u(t, x) = E_{X_t = x}\left(f(X_T) + \int_t^T g(X_s)ds \right)$$

Satisfies The Feymann - Kac equation

$$u_t + au_x + \frac{1}{2}bu_{xx} = g$$

$$u_T = f$$

Eq. (15)

B-Proof in 1 D : order 1

Keep only order 1 in eps:

$$v_t + (\mu_t - \frac{\sigma_0^2}{2})v_x + \frac{1}{2}\sigma_0 v_{xx} = -\left(\eta(t,\sigma_0)C^0_\sigma + \frac{\alpha^2(t,\sigma_0)}{2}C^0_{\sigma\sigma} + \sigma_0\rho_t\alpha(t,\sigma_0)C^0_{x\sigma}\right) \qquad \textbf{Eq. (16)}$$

$$v(T,x) = 0$$

Use Feymann-Kac:

$$v(t,x) = E\left[\int_t^T (\eta(\theta,\sigma_0)C^0_\sigma(\theta,X_\theta) + \frac{1}{2}\alpha^2(\theta,\sigma_0)C^0_{\sigma\sigma}(\theta,X_\theta) + \sigma_0\rho_\theta\alpha^2(\theta,\sigma_0)C^0_{x\sigma}(\theta,X_\theta))d\theta \middle| X_t = x\right] \qquad \textbf{Eq. (17)}$$

To use lemma we need to transform vol derivatives into x derivatives

B-Proof in 1 D : order 1

Following Black&Scholes relations hold:

$$C^0_{\sigma}(t,x) = \sigma_0 (T-t)(C^0_{xx}(t,x) - C^0_{x}(t,x))$$

$$C^0_{x\sigma}(t,x) = \sigma_0 (T-t)(C^0_{xxx}(t,x) - C^0_{xx}(t,x))$$

Eq. (18)

$$C^0_{\sigma\sigma}(t,x) = (T-t)\left(C^0_{xx}(t,x) - C^0_{x}(t,x)\right) + \sigma_0^2 (T-t)^2 (C^0_{xxxx}(t,x) - 2C^0_{xxx}(t,x) + C^0_{xx}(t,x))$$

B-Proof in 1 D : order 1

Therefore:

$$E\left[\int_t^T \eta(\theta,\sigma_0)C_\sigma^0(\theta,X_\theta)d\theta\right]=\sigma_0\left(\int_t^T (T-\theta)\eta(\theta,\sigma_0)d\theta\right)(C_{xx}^0(t,x)-C_x^0(t,x))=\frac{1}{(T-t)}\left(\int_t^T (T-\theta)\eta(\theta,\sigma_0)d\theta\right)C_\sigma^0(t,x)$$

$$E\left[\int_t^T \rho_\theta\alpha(\theta,\sigma_0)C_{x\sigma}^0(\theta,X_\theta)d\theta\right]=\sigma_0\left(\int_t^T (T-\theta)\eta(\theta,\sigma_0)d\theta\right)(C_{xxx}^0(t,x)-C_{xx}^0(t,x))$$

$$=\frac{1}{(T-t)}\left(\int_t^T (T-\theta)\rho_\theta\alpha(\theta,\sigma_0)d\theta\right)C_{x\sigma}^0(t,x)$$

Eq. (19)

$$E\left[\int_t^T \alpha^2(\theta,\sigma_0)C_{\sigma\sigma}^0(\theta,X_\theta)d\theta\right]=\left(\int_t^T (T-\theta)\alpha^2(\theta,\sigma_0)d\theta\right)(C_{xx}^0(t,x)-C_x^0(t,x))$$

$$+\sigma_0^2\left(\int_t^T (T-\theta)^2\alpha^2(\theta,\sigma_0)d\theta\right)(C_{xxxx}^0(t,x)-2C_{xxx}^0(t,x)+C_{xx}^0(t,x))$$

B-Proof in 1 D : conclusion

finally:

$$C(t,x,\varepsilon) = C^0(t,x) + \varepsilon(VegaFactor\ C^0_\sigma(t,x) + VannaFacto\ rC^0_{x\sigma}(t,x) + Vo\lg aFactorC^0_{\sigma\sigma}(t,x))$$

$$VegaFactor = \frac{1}{(T-t)}\left(\int_t^T (T-\theta)\eta(\theta,\sigma_0)d\theta\right) + \frac{1}{2(T-t)\sigma_0}\left(\int_t^T (T-\theta)\alpha^2(\theta,\sigma_0)d\theta\right) - \frac{1}{2(T-t)^2\sigma_0}\left(\int_t^T (T-\theta)^2\alpha^2(\theta,\sigma_0)d\theta\right)$$

$$Vo\lg aFactor = \frac{1}{2(T-t)^2}\int_t^T (T-\theta)^2\alpha^2(\theta,\sigma_0)d\theta$$

$$VannaFactor = \frac{s}{(T-t)}\int_t^T (T-\theta)\sigma_0\rho_\theta\alpha(\theta,\sigma_0)d\theta$$

Eq. (20)

C-Multi European under Local Volatility Local Correlation (1)

Pricing European Options under multi local volatility local correlation model can be performed using perturbation techniques

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i(t, S_i) dW_i$$

$$< dW_i, dW_j > = \rho_{i,j}(t, S_1, \ldots, S_n) dt$$

Eq. (21)

We want to price the European payoff f

$$u = E\left(f\left(S_1(T), \ldots, S_n(T)\right) \mid S_1(t) = s_1, \ldots, S_n(t) = s_n \right)$$

C-Multi European under Local Volatility Local Correlation (2)

Using a simple perturbation analysis and Feymann Kac, we obtain the following result – just like before in the stochastic volatility case:

$$u_{LVLC} = u_{BS} + E_{BS}\left(\frac{1}{2}\sum_{i,j}^{n}\int_0^T \left(\rho_{i,j}(t,S_1,\dots,S_n)\sigma_i(t,S_i)\sigma_j(t,S_j) - \rho_{i,j}^{BS}\sigma_i^{BS}\sigma_j^{BS}\right)\frac{\partial^2 u_{BS}}{\partial S_i \partial S_j}dt \right) + O\left(\varepsilon^2\right)$$

Eq. (22)

Yet formulae are not practical – many integrals to be computed

We do not have the magical lemma ☹

C- Specific work for linear payoffs (3)

Consider a linear payoff

- **Basket with positive weights**
- **Spread options**

$$\psi = \left(\underbrace{\sum_{i=1,n} w_i S_i(T)}_{B_T} - k \right)^+ \quad \textbf{Eq. (23)}$$

Under the general dynamic

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i(t, S_i) dW_i$$

$$< dW_i, dW_j >= \rho_{i,j}(t, S_1, \ldots, S_n) dt$$

C-Most Likely Path Pricing under Multi Local Volatility (4)

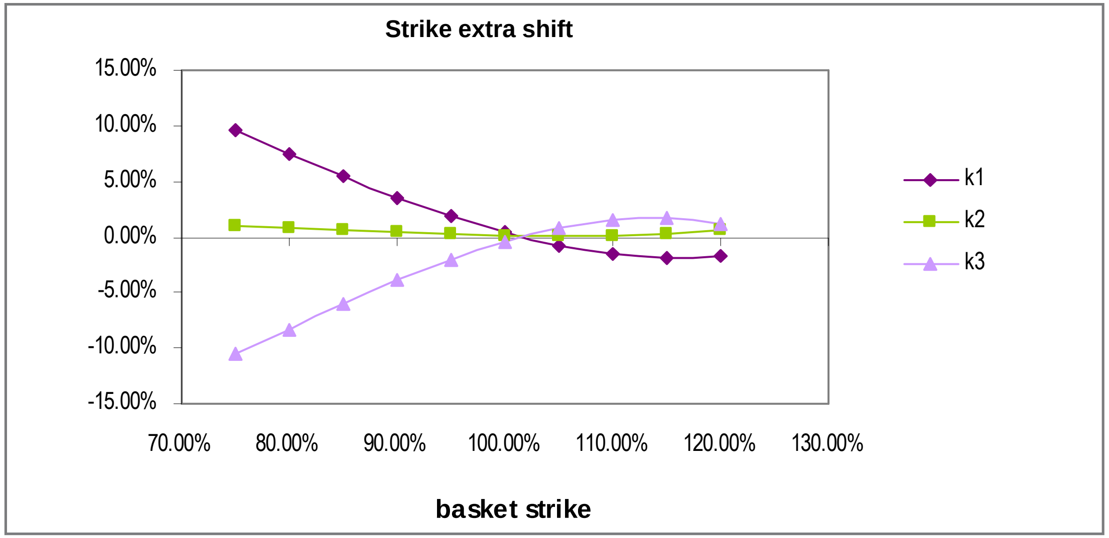
Model Reduction – using Gradient Conditionning (Curran)

$$k_i = E\left(S_i(T) \mid B(T) = k\right) \cong E\left(S_i(T) \mid Z = z^*\right)$$

$$E\left(B(T) \mid Z = z^*\right) = k$$

Eq. (24)

Asset	Forward	Vol. ATM	Slope	Weight
1	1.	0.20	-0.30	0.333
2	1.	0.25	-0.30	0.333
3	1.	0.30	-0.30	0.333



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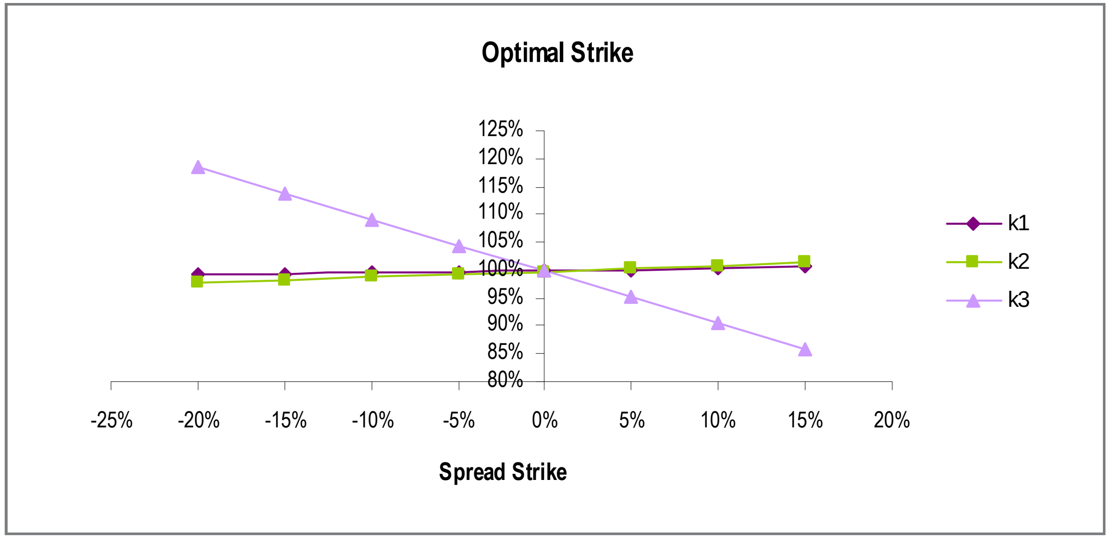
C-Most Likely Path Pricing under Multi Local Volatility (5)

Model Reduction – works as well for spreads

Asset/Value	Forward	Vol. ATM	Slope	Weight
1	1.	0.20	-0.30	0.50
2	1.	0.25	-0.30	0.50
3	1.	0.30	-0.30	-1.00



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C-Most Likely Path Pricing under Multi Local Volatility (6)

Local Volatility Model becomes like when pricing : ψ

A simpler model – A multi Black Scholes Model :

Same methodology as in [3] and [4] :

$$\frac{dS_{\mathbf{i}}}{S_{\mathbf{i}}} = \mu_i dt + \sigma_i * dW_i$$

$$< dW_i, dW_j >= \rho_{i,j}\left(t, k_1, \ldots, k_n\right)dt$$

Eq. (25)

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C-From Specific to Generic(7)

Differentiating twice and integrating with exp(ik) we obtain the moment generating function for the joint distribution - trick in [6]

$$\int_{-\infty}^{\infty} \frac{\partial^2 \psi}{\partial k^2} \exp(ik)\,dk = E\left(\exp\left(i\sum_{j=1}^{n} w_j S_j(T)\right)\right) \qquad \text{Eq. (26)}$$

We can recover the joint density (Fourier inversion) and be able to price all European payoffs ☺

Numerically tractable in low dimensions 3 to 4 ☹

D-Multi Local Vol & Stoch. Volatility

Using Perturbation techniques under the general model

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i f_i(t, S_i) dW_i^s$$

$$d\sigma_i = \varepsilon \sigma_i dW_i^\sigma$$

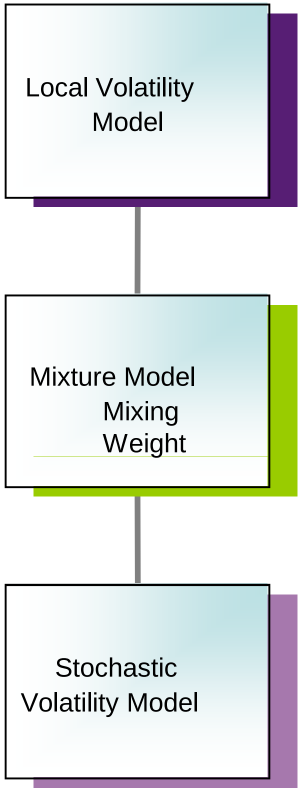
Eq. (27)

We have the following Pricing approximating results

$$\text{Price} = \text{Price Local Vol} + \varepsilon^2 (\text{Price Stoch Vol - Price Local Vol Local Correl})$$

Eq. (28)

Summary



65% from Barrier Business

Relies on fast approximation

Allows to Price Correlation Products

Determines Cross Smile

3 Cross Smile Estimation

Pricing Calls on the cross

Pricing a call on the cross

$$\left(\frac{S_{2,t}}{S_{1,t}} - k \right)^{+}$$

Is equivalent to pricing a spread option on the two currency pairs

$$\left(S_{2,t} - kS_{1,t} \right)^{+}$$

Warning: only true after change of numeraire

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Pricing Example: AUDJPY

Pricing a call on the cross

$$\left(\frac{JPY}{AUD} - k\right)^{+} \text{ under JPY measure}$$

Is equivalent to pricing a spread option on the two currency pairs

$$\left(\frac{USD}{AUD} - k\,\frac{USD}{JPY}\right)^{+} \text{ under USD measure}$$

Calculating Smile

From call prices we back out implied volatilities

From Implied Volatilities we back out smile characteristics

4 Back Testing

Method

Collect complete data of :

- **AUDUSD, USDJPY and AUDJPY**

Apply Model with different inputs

Compare Predicted smiles with observed ones of AUDJPY

Historical Data

From 18/11/02 to 17/11/04

Data for 1y cross smile

spot			fwd			discount factors		
AUDUSD	USDJPY	AUDJPY	AUDUSD	USDJPY	AUDJPY	USD	AUD	JPY
0.56235	121.02	68.04	0.54498	119.11	64.91257	0.984225738	0.953849	0.99935
0.55895	122.19	68.28	0.54165	120.29	65.15508	0.98412732	0.953849	0.99935
0.56055	122.67	68.78	0.54345	120.74	65.61615	0.983438672	0.953801	0.99935
0.56285	122.65	69.04	0.54563	120.645	65.82753	0.983242004	0.953324	0.99935
0.56375	122.835	69.25	0.5465	120.805	66.01993	0.983340333	0.9528	0.99935

Not liquid data

Vols			RR25			ST25		
AUDUSD	USDJPY	AUDJPY	AUDUSD	USDJPY	AUDJPY	AUDUSD	USDJPY	AUDJPY
9.60%	9.35%	11.10%	-0.10%	-0.50%	0.55%	0.31%	0.37%	0.37%
9.60%	9.35%	10.90%	-0.10%	-0.40%	0.55%	0.31%	0.37%	0.37%
9.55%	9.30%	10.90%	-0.12%	-0.35%	0.55%	0.31%	0.37%	0.37%
9.60%	9.30%	11.05%	-0.12%	-0.30%	0.55%	0.31%	0.37%	0.37%
9.60%	9.25%	11.00%	-0.10%	-0.30%	0.55%	0.31%	0.37%	0.37%

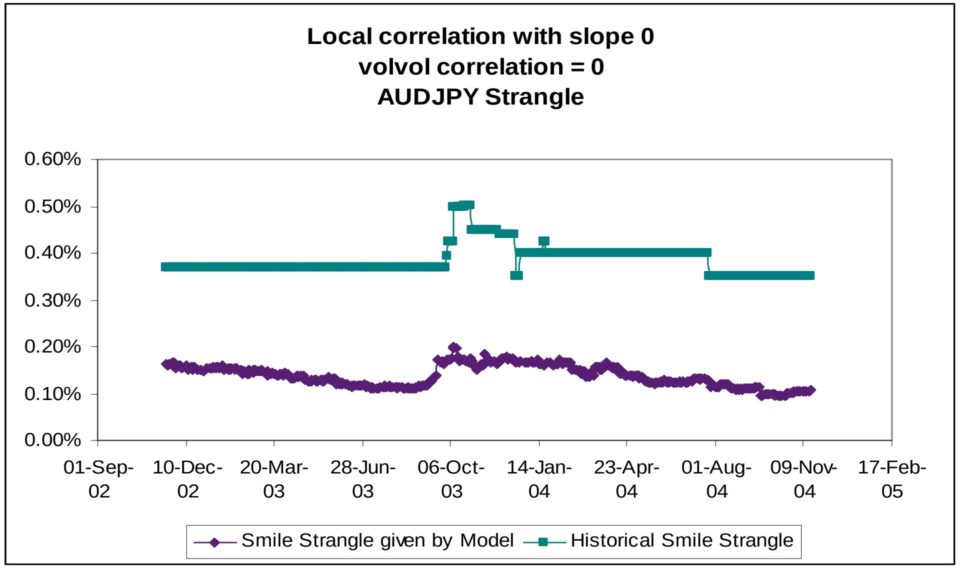
Tests

We consider 4 different cases:

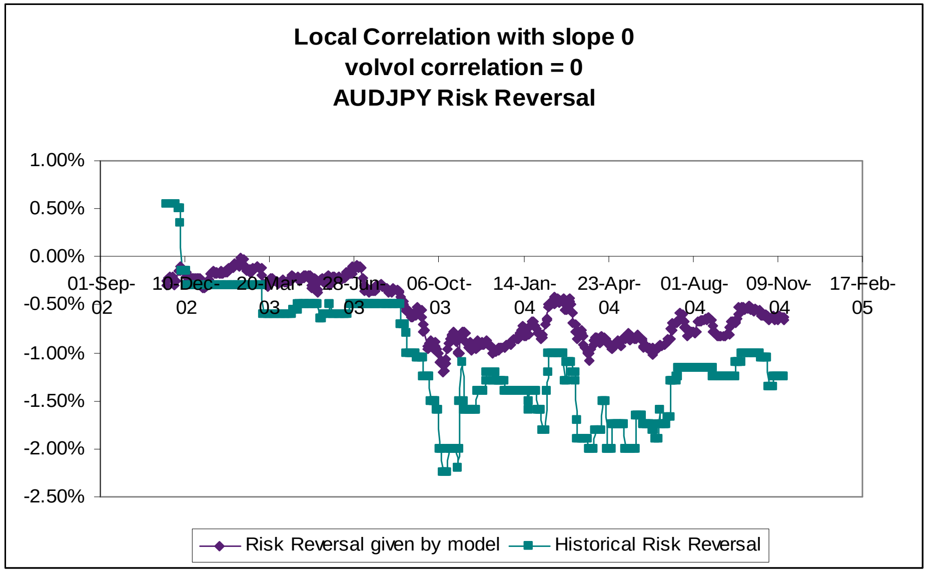
- **Slope corr=0, curve corr=0,volvolcorr=0**
- **Slope corr=2, curve corr=0,volvolcorr=0**
- **Slope corr=2, curve corr=0,volvolcorr=0.5**
- **Slope corr=1.8, curve corr=-28,volvolcorr=0.5**

Test 1

Strangle

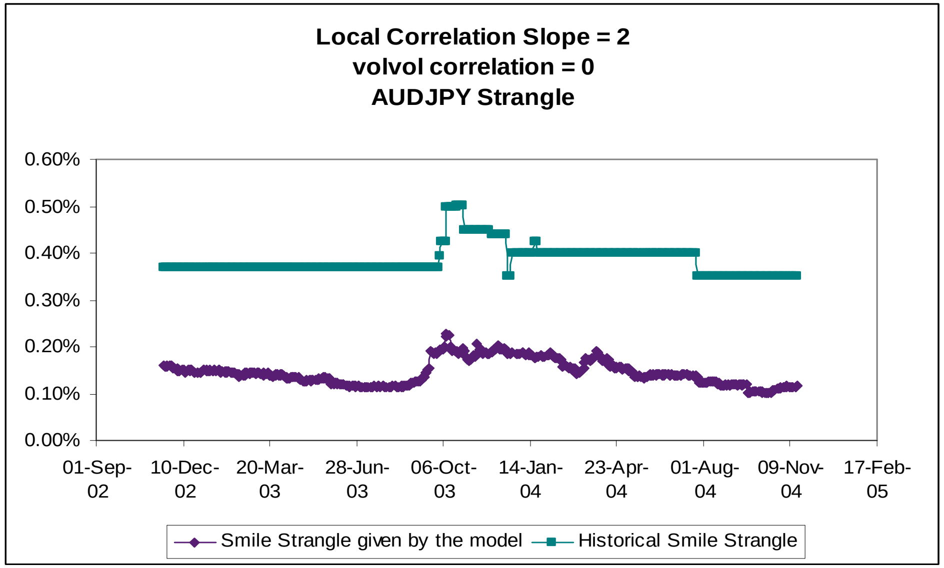


Risk Reversal

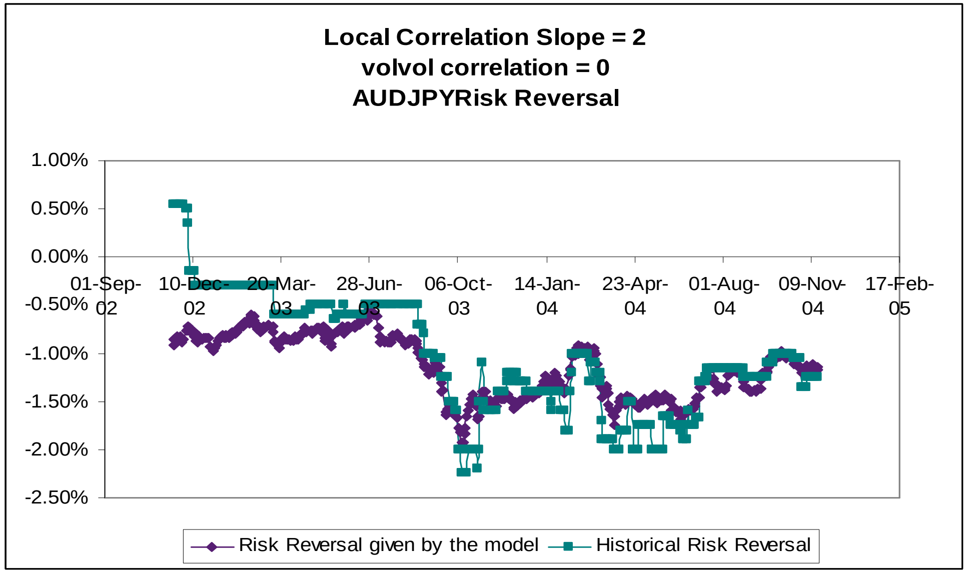


Test 2

Strangle



Risk Reversal



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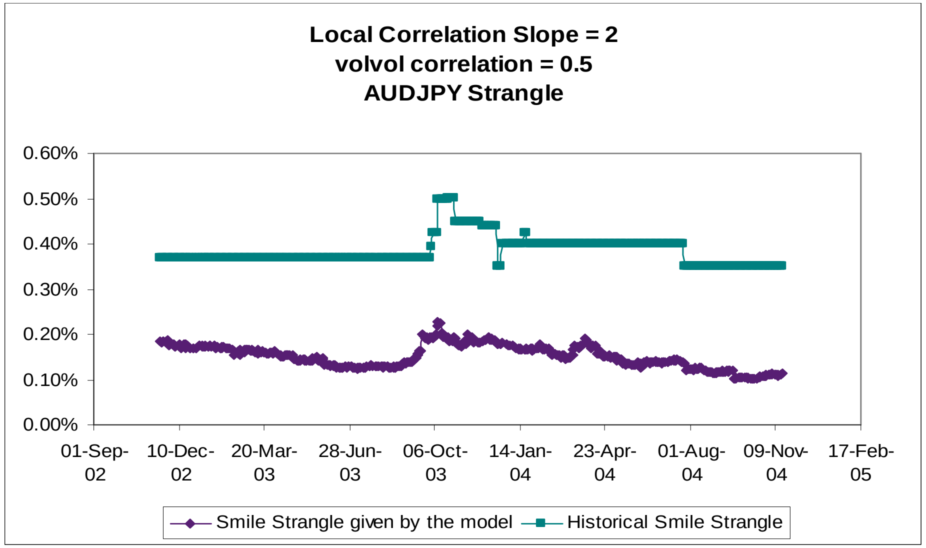
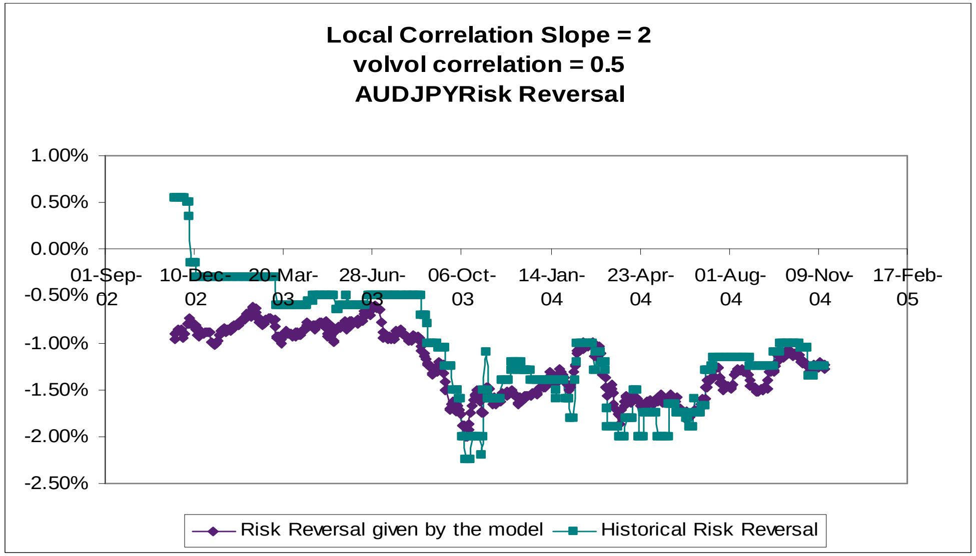
Test 3

Strangle

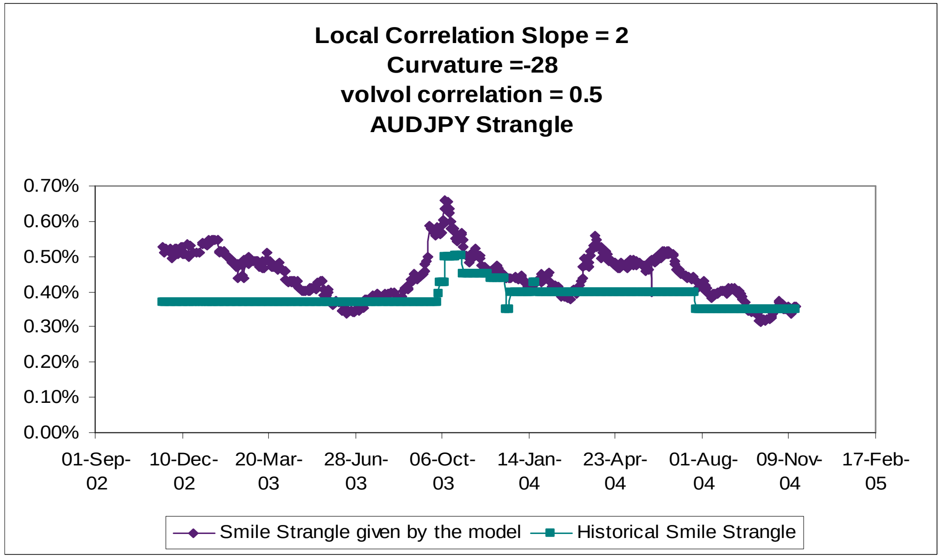


Risk Reversal

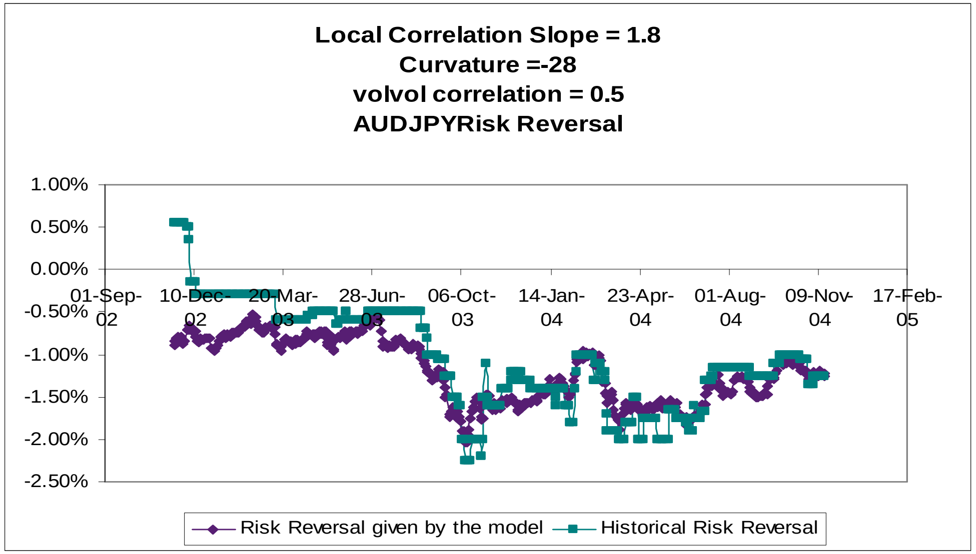


Test 4

Strangle



Risk Reversal



Summary

A new model for cross smile estimation is produced

Uses a mixture of local vol and stoch. Vol

Introduces volvol correlation and local correlation

Playing on the parameters offers flexibility to predict market levels

It is based on efficient numerical techniques

Questions

Thank you for your attention

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