# Conditional Density Method in the Computation of the Delta with Application to Power Market

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Third SMAI European Summer School in Financial Mathematics, 2010

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- In this presentation, we are concerned about the derivative of expectation functionals of the form

$$F(x) = \mathbb{E}\left[f(x+Y)\right] , \tag{1}$$

which can be represented as

$$F'(x) = \mathbb{E}\left[f(x+Y)\pi\right] . \tag{2}$$

- The random variable $\pi$ is called a sensitivity weight.

# Density method

- The density method was introduced by Broadie and Glasserman (1996).
- If $Y$ has a differentiable density $p_Y$, then

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_{\mathbb{R}} f(y)p_Y(y-x)\,dy \\ &= \int_{\mathbb{R}} f(y)(-1)p'_Y(y-x)\,dy \\ &= \mathbb{E}\left[f(x+Y)(-\partial_y \ln p_Y(Y))\right], \end{aligned}$$

- Note no differentiability of $f$.

- Another case which is important in our analysis will be the situation where we have two independent random variables $Y$ and $Z$.
  Consider the functional

$$F(x) = \mathbb{E}\left[f(x + Y + Z)\right] .$$

- In this case we can use the density method with respect to $Y + Z$ which leads to

$$F'(x) \quad = \mathbb{E}\left[f(x + Y + Z)(-\partial_{y+z} \ln p_{Y+Z}(Y + Z))\right] .$$



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# Conditional density method

Conditioning on $Z$, leads to

$$
\begin{aligned}
F'(x) &= \frac{d}{dx}\mathbb{E}\Big[\mathbb{E}[x+Y+Z|Z]\Big] \\
&= \frac{d}{dx}\int_{\mathbb{R}} \mathbb{E}\left[f(y-Z)\right] p_Y(y-x)\,dy \\
&= \mathbb{E}\left[f(x+Y+Z)(-\partial_y \ln p_Y(Y))\right].
\end{aligned}
$$

Otherwise, conditioning on $Y$, leads to

$$F'(x) = \mathbb{E}\left[f(x+Y+Z)(-\partial_z \ln p_Z(Z))\right].$$

We essentially consider a Lévy process $(X_t)_{t\geq 0}$ of the form

$$X_t = at + bW_t + N_t + \lim_{\varepsilon \to 0} \widetilde{N}_t^{\varepsilon}, \tag{3}$$

where the convergence is almost sure and uniform in $t \in [0, T]$. Here $a, b \in \mathbb{R}$,

$$N_t = \sum_{s \in [0,t]} \triangle X_s \mathbf{1}_{\{|\triangle X_s| \geq 1\}}$$

is a compound Poisson process,

$$\widetilde{N}_t^{\varepsilon} = \sum_{s \in [0,t]} \triangle X_s \mathbf{1}_{\{\varepsilon \leq |\triangle X_s| < 1\}} - t \int_{\varepsilon \leq |z| < 1} zQ(dz)$$

is the compensated Poisson process and $Q$ is the Lévy measure of $(X_t)_{t\geq 0}$.

The components $(W_t)_{t\geq 0}$, $(N_t)_{t\geq 0}$ and $(\widetilde{N}_t^{\varepsilon})_{t\geq 0}$ are independent of each other.

## Proposition

*Consider a process* $(X_t^{\varepsilon})_{t\geq 0}$ *defined by*

$$X_t^{\varepsilon} = at + bW_t + \sigma(\varepsilon)B_t + N_t + \widetilde{N}_t^{\varepsilon}, \tag{4}$$

*where* $(B_t)_{t\geq 0}$ *is a Brownian motion independent of the other components. Let* $(X_t)_{t\geq 0}$ *be as defined in (3). Then*

$$\lim_{\varepsilon\to 0} X_t^{\varepsilon} = X_t \qquad \textit{the limit is in } L^1 \textit{ and } \mathbb{P} - \textit{a.s.}$$

**Lemma**

*Under the condition* $u\widehat{f}(u) \in L^1(\mathbb{R})$ *we have*

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial x}\mathbb{E}[f(x+X_t^\varepsilon)] = \frac{\partial}{\partial x}\mathbb{E}[f(x+X_t)]\,.$$

# Example: NIG-distribution

Take T=1. We assume that $X(1)$ is an NIG-Lévy process, with parameters $\alpha$, $\beta$, $\delta$ and $\mu$, the density is

$$p_{\mathrm{NIG}}(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}}.$$

Here, $K_\lambda$ is the modified Bessel function of the second order with parameter $\lambda$.
Notice that there is no Brownian motion component in the NIG process.

We apply the density method to find a sensitivity weight $\pi$:

$$\pi = -\beta + \frac{X(1)-\mu}{\delta^2 + (X(1)-\mu)^2} \left\{ 1 - \frac{\alpha\sqrt{\delta^2 + (X(1)-\mu)^2} K_1'\left(\alpha\sqrt{\delta^2 + (x(1)-\mu)^2}\right)}{K_1\left(\alpha\sqrt{\delta^2 + (x(1)-\mu)^2}\right)} \right\}.$$

- The computation of the two modified Bessel functions $K_1$ and $K_2$ in order to calculate the sensitivity weight $\pi$ makes the Monte Carlo simulation technique very inefficient.
- An alternative will then be to use an approximation of $X(t)$ and derive the sensitivity weight with respect to this process. In this case we find from the calculations above

$$\pi = \frac{B(1)}{\sigma(\varepsilon)}.$$

- This will be obviously much simpler to compute.

- We consider a multi-factor spot price process $S(t)$ as follows

$$S(t) = g(t, X_1(t), ..., X_n(t)).$$

- We consider a payoff function $h : \mathbb{R} \to \mathbb{R}$ and we suppose that there exist a function $f$ and a differentiable function $\zeta$ such that

$$h(S(T)) = f\left(X_1(T) + \zeta(S(0)), X_2(T), \dots, X_n(T)\right). \quad (5)$$

- The arbitrage-free price of a european option written on the spot price $S$ is defined as

$$C(S(0)) = e^{-rT}\mathbb{E}\left[f\left(X_1(T) + \zeta(S(0)), X_2(T), \dots, X_n(T)\right)\right], \quad (6)$$

where $r$ is the risk-free instantaneous interest rate.

- The aim is to study the delta in such a model.

Denote by $p_1$ the density of $X_1(T)$, which we suppose to be known.

## Proposition

*Assume that there exists an integrable function $u$ on $\mathbb{R}$ such that*

$$|\mathbb{E}\left[f(x, X_2(T), \ldots, X_n(T))\right]| p_1(x - \zeta(S(0))) \leq u(x)\,. \tag{7}$$

*Then*

$$\frac{\partial C}{\partial S(0)} = -\zeta'(S(0))e^{-rT}\mathbb{E}\left[h(S(T))\partial \ln p_1(X_1(T))\right]\,.$$

# Example 1

We start with considering the two-factor model of Schwartz and Smith (2000) applied as a model for the oil price dynamics. It takes the form

$$S(t) = S(0)\exp(X(t) + Y(t)), \tag{8}$$

where

$$dX(t) = (\theta - \alpha X(t))\,dt + \sigma\,dW(t), \tag{9}$$

and

$$dY(t) = \mu\,dt + \eta\,d\widetilde{W}(t), \tag{10}$$

with independent $\widetilde{W}$ and $W$.

We represent the payoff as

$$h(S(T)) = h(\exp(\ln(S(0)) + X(T) + Y(T))).$$

Thus

$$f(x + \zeta(S(0)), y) = h(\exp(\ln(S(0)) + x + y))$$

with $\zeta(s) = \ln s$. The mean of $X(t)$ is

$$X(0)\exp(-\alpha t) + \theta(1 - \exp(-\alpha t))/\alpha$$

and the variance is

$$\sigma^2/2\alpha(1 - \exp(-2\alpha t)).$$

Therefore, it holds that

$$\partial \ln p_1(x) = -\frac{1}{\frac{\sigma^2}{2\alpha}(1-e^{-2\alpha T})}\left(x - X(0)e^{-\alpha T} - \frac{\theta}{\alpha}(1-e^{-\alpha T})\right).$$

Hence, the delta is

$$\begin{aligned}\frac{\partial C}{\partial S(0)} &= \frac{e^{-rT}2\alpha}{S(0)\sigma^2(1-e^{-2\alpha T})} \\ &\quad \mathbb{E}\left[h(S(T))\left(X_1(T) - X(0)e^{-\alpha T} - \frac{\theta}{\alpha}(1-e^{-\alpha T})\right)\right].\end{aligned}$$

# Example 2

- We consider the additive model for the electricity spot price proposed by Benth, Kallsen, and Meyer-Brandis (2007) given by

$$S(t) = X(t) + Y(t), \quad S(0) > 0.$$

- The process $Y(t)$ is given by

$$Y(t) = -\lambda_2 Y(t)dt + dL_2(t), \quad Y(0) = 0,$$

  where $L_2$ is a compound Poisson process.
- The process $X(t)$ is a $\Gamma(a, b)$-OU process. Namely, it is a Lévy process following the dynamics

$$dX(t) = -\lambda_1 X(t)dt + dL_1(t), \quad X(0) = S(0),$$

  where $L_1(t)$ is a subordinator, admitting a stationary distribution which is here $\Gamma(a, b)$.

- The solution takes the form

$$X(t) = e^{-\lambda_1 t} X(0) + \int_0^t e^{\lambda_1 (s-t)} dL_1(s),$$

- We observe that the stochastic integral

$$\int_0^t e^{\lambda_1 (s-t)} dL_1(s) = X(t) - e^{-\lambda_1 t} X(0)$$

  has an asymptotic distribution being $\Gamma(a, b)$ when $t$ goes to infinity, since $e^{-\lambda_1 t} X(0)$ goes to 0 when $t$ goes to infinity.

- Denoting by $Z$ a random variable which is $\Gamma(a, b)$-distributed, we consider

$$\widetilde{S}(t) = e^{-\lambda_1 t} X(0) + Z + Y(t),$$

  which is asymptotically equal in distribution to $S(t)$.

- Consider the payoff of a call option

$$h(S(T)) = \max(S(T) - K, 0).$$

- Therefore, the expression for the delta is given by

$$\frac{\partial C}{\partial S(0)} \approx e^{-rT}\mathbb{E}\Big[h(e^{-\lambda_1 T}X(0)+Z+Y(T))e^{-\lambda_1 T}(b-\frac{a-1}{Z})\Big].$$

- To make a numerical example which is relevant for energy markets, we note that Benth, Kiesel, and Nazarova (2009) showed empirically that the spot model fitted the Phelix Base electricity price index at the European Power Exchange (EEX) very well. We used the data from that paper.

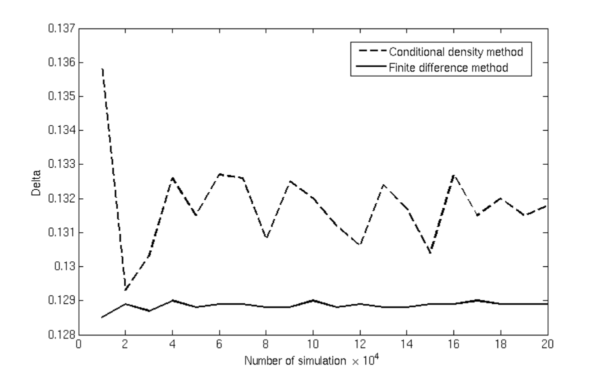


Figure: Simulation of a delta for a call option

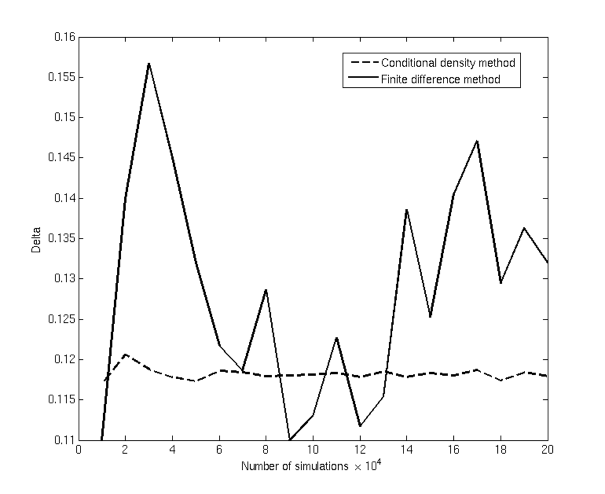


Figure: Simulation of a delta for a digital option

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