Utility Maximization with Additive Habits: Optimal Consumption and Wealth

Roman Muraviev

Department of Mathematics
ETH Zurich

26.08.10
THIRD SMAI EUROPEAN SUMMER SCHOOL IN FINANCIAL MATHEMATICS

Consumption: Empirically Observed Features

- *Consumption* is increasing in *Wealth*.
- *Investments* are increasing in *Wealth*.
- *Consumption* is concave in *Wealth*.
- *Consumption* exhibits patterns of *Habit Formation*.

Habit Formation

- Preference functional with additive habits:

$$\sum_{t=0}^{T} E\left[u_t\left(c_t - \sum_{s=0}^{t-1}\beta_s^{(t)}c_s\right)\right]$$

- When measuring satisfaction from consumption, habits determined by past consumption are incorporated.
- Current high level of consumption forces addiction to high level of consumption in the future.

Related Literature

- Abel (1990)
- Constantinides (1990)
- Detemple and Zapatero (1991, 1992)
- Chan and Kogan (2002)
- Detemple and Karatzas (2003)
- Karatzas and Zitkovic (2003)
- Malamud and Trubowitz (2007)
- Englezos and Karatzas (2009)

The Financial Market - Settings I

- Finite probability space (Ω, G, P) with filtration

$$G_0 := \{\phi, \Omega\} \subseteq ... \subseteq G_T := G$$

$$L^2(G_0) \cong R \subseteq L^2(G_1) \subseteq ... \subseteq L^2(G_T)$$

- Market: N risky securities and one period risk-free bonds.
- $S_t = (1, S_t^1, ..., S_t^N)$ - adapted positive **price processes**.
  $d_t = (r_t, d_t^1, ..., d_t^N)$ - adapted **dividend processes**.
  $r_t > 0$ - predictable *interest rate* process.
- $\pi_t = (\phi_t, \pi_t^1, ..., \pi_t^N)$ - adapted **portfolio process**.
  Assumption: $\pi_{-1} = \pi_T = 0$.

The Financial Market - Settings II

- **Investment** process corresponding to π:

$$I_t^{\pi} := \pi_t \cdot S_t$$

- **Financial Wealth** process corresponding to π:

$$W_t^{\pi} := \pi_{t-1} \cdot (S_t + d_t)$$

- **No Arbitrage:** $\exists$ positive adapted SPD $(R_t)_{t=0,\ldots,T}$:

$$S_{t-1} \cdot R_{t-1} = E\left[(S_t + d_t) \cdot R_t | G_{t-1}\right]$$

- **Remark:** For every portfolio π and every SPD $(R_t)_{t=0,\ldots,T}$:

$$I_{t-1}^{\pi} \cdot R_{t-1} = E\left[W_t^{\pi} \cdot R_t | G_{t-1}\right]$$

The Aggregate State Price Density

- The **Financial Wealth Space** in period $0 \leq t \leq T$:

$$L_t = \left\{ W_t^\pi = \pi_{t-1} \cdot (S_t + d_t) | \pi_{t-1} \in L^2(G_{t-1}) \right\}$$

$$L^2(G_{t-1}) \subseteq L_t \subseteq L^2(G_t)$$

- Orthogonal projection of $L^2(G_T)$ onto L_t:

$$P_L^t : L^2(G_T) \to L_t$$

- **Theorem** (Malamud and Trubowitz 2007): $\exists!$ **Aggregate SPD** $M_0 = 1, M_1, ..., M_T$, s.t. $M_t \in L_t$, and moreover

$$M_t = \prod_{\tau=0}^{t-1} P_L^{\tau+1} \left[\frac{R_{\tau+1}}{R_\tau} \right]$$

for every positive SPD R_t.

- **Assumption**: We consider only markets with $M_t \neq 0$.

Random Endowment and Consumption

- Agent with **random endowments**: adapted process $\epsilon_t \geq 0$.
- Agent's **consumption**: $c_t = \epsilon_t + W_t^\pi - I_t^\pi \geq 0$.
- **Utility maximization problem**:

$$\sup_{c_0,\dots,c_T} \sum_{t=0}^{T} E\left[u_t\left(c_t - \sum_{s=0}^{t-1} \beta_s^{(t)} c_s\right)\right]$$

Habits: Positive random variables: $\beta_s^{(t)} \in L^2(G_s)$, $s < t$.

- **Inada conditions**: $u_t : [0, +\infty) \to R$, C^2-smooth, $u_t'(x) > 0$, $u_t''(x) < 0$, $u_t'(0) = \infty$ and $u_t'(\infty) = 0$.

First Order Conditions

- **Theorem**: $\exists !$ positive *optimal consumption* stream c_t^* and *wealth process* W_t^* solving the system of equations:

$$P_L^t \left[R_t(c)\right] = \frac{M_t}{M_{t-1}} \cdot R_{t-1}(c),$$

and

$$c_t = \epsilon_t + W_t - E\Big[\frac{M_{t+1}}{M_t} W_{t+1} \big| G_t\Big],$$

for all $t = 0, ..., T$; where

$$R_t(c) := u_t'\Big(c_t - \sum_{\tau=0}^{t-1} \beta_\tau^{(t)} c_\tau\Big) - \sum_{s=t+1}^{T} E\Big[\beta_t^{(s)} \cdot u_s'\Big(c_s - \sum_{\tau'=0}^{s-1} \beta_{\tau'}^{(s)} c_{\tau'}\Big) \Big| G_t\Big] > 0$$

is a **positive SPD**.

Example: Complete Market with $u_t(x) = \frac{x^{1-\gamma}}{1-\gamma}$

- **Theorem** :

$$c_t^* = A_t \cdot W_t^* + B_t \quad ; \quad c_0^* = A_o \cdot \epsilon_0 + B_0$$

$$W_t^* = F_t \cdot c_{t-1}^* + G_t$$

$$\sum_{s=0}^{T} E\left[M_s \cdot c_s^*\right] = \sum_{s=0}^{T} E\left[M_s \cdot \epsilon_s\right]$$

where $0 < A_t < 1$ and $F_t > 0$.

- **Conclusion:** The optimal consumption stream c_t^* and the investment is a linear increasing function of the wealth W_t^* with a slope < 1.

Results in Incomplete Markets

- Generally no explicit solution due to P_t^L.
- Monotonicity of the consumption and investments

$$0 < \frac{\partial c_t^*}{\partial W_t^*} \leq 1$$

- Concavity

$$\frac{\partial^2 c_t^*}{\partial^2 W_t^*} \leq 0$$

- Asymptotic behavior and estimates

$$\lim_{W_t^* \to \infty} \frac{c_t^*}{W_t^*} \quad ; \quad f(W_t^*) \leq c^* \leq g(W_t^*)$$

Dynamic Programming Fails Showing Monotonicity

- Value Function:

$$V_t\left(\epsilon_0, \pi_0, \dots, \pi_{t-1}\right) = \sup_{\pi_t}\{u(c_t - \sum_{k=0}^{t-1} \beta_k^{(t)} c_k) - E[V_{t+1}(\epsilon_0, \pi_0, \dots, \pi_t)|\mathcal{G}_t]\}$$

- **Proposition**:

$$V_0'(\epsilon_0) = u'(c_0(\epsilon_0)) + E\left[\frac{\partial V_1}{\partial \epsilon_0}(\epsilon_0, \pi_o(\epsilon_0))\right],$$

- **No habits**: $\beta_0^{(s)} = 0$ implies $c_0'(\epsilon_0) \geq 0$, but $c_0'(\epsilon_0) \leq 1$ is **unclear**.
- **Habits**: For $\beta_0^{(s)} > 0$ both properties are **unclear**.

Incomplete Markets with a Constant Interest Rate

- Aggregate SPD density satisfies

$$E\left[M_t | G_{t-1}\right] = (1+r)^{-1} \cdot M_{t-1}$$

- **Lemma**: The First order conditions are:

$$P_L^t \left[u_t'\left(c_t - \sum_{\tau=0}^{t-1} \beta_\tau^{(t)} c_\tau\right)\right] \cdot \widetilde{M}_{t-1} = u_{t-1}' \left(c_{t-1} - \sum_{\tau=0}^{t-2} \beta_\tau^{(t)} c_\tau \right) \cdot \widetilde{M}_t$$

  for all $t = 1, ..., T$.

- **Theorem**: We have

$$1 \geq \frac{\partial c_t^*}{\partial W_t^*} > 0 \quad ; \quad 1 \geq \frac{\partial c_0^*}{\partial \epsilon_0} > 0$$

Idiosyncratic Incompleteness I: Definitions

- Market is complete w.r.t $(F_t)_{0\le t\le T}$;
  Incompleteness: $(G_t)_{0\le t\le T}$, s.t., $\epsilon_t \in L^2(G_t)$ and $F_t \subseteq G_t$.
  Source of incompleteness is the random endowment:
- **Assumption** $(\star)$: For every $X \in L^2(F_t)$, we have

$$E\left[X|F_{t-1}\right] = E\left[X|G_{t-1}\right]$$

- **Proposition**: $L_t = L^2\left(\sigma\{G_{t-1}, F_t\}\right)$.
- **Proposition**: Under assumption $(\star)$, the aggregate SPD satisfies $M_t \in L^2(F_t)$.

Idiosyncratic Incompleteness II: Monotonicity

- **Theorem**: For Idiosyncratic incomplete market, we have

$$0 < \frac{\partial c_t^*}{\partial W_t^*} \leq C(\beta, M) \leq 1 \quad ; \quad 0 < \frac{\partial c_0^*}{\partial \epsilon_0} \leq C(\beta, M) \leq 1$$

$$0 < \frac{\partial W_t^*}{\partial c_{t-1}^*}$$

- **Theorem**: Assume that only risk free bonds are available for trading with a constant interest rate r, and $\beta_{t-1}^{(t)} = \beta$, and $\beta_s^{(t)} = 0$ for all $s < t-1$, then

$$\frac{\partial c_t^*}{\partial W_t^*} \leq \frac{1-\beta(1+r)^{-1}}{1-\beta^{T-t+1}(1+r)^{-T+t-1}} \, ; \; \frac{\partial c_0^*}{\partial \epsilon_0} \leq \frac{1-\beta(1+r)^{-1}}{1-\beta^{T+1}(1+r)^{-T-1}}$$

Low Degree of Habits

- **Theorem (no habits)**: Assume that $\beta = 0$, then

$$1 \geq \frac{\partial c_t^*}{\partial W_t^*} > 0 \quad ; \quad 1 \geq \frac{\partial c_0^*}{\partial \epsilon_0} > 0$$

- **Stability Theorem**: For an arbitrary incomplete market, $\exists$ $\beta^* := \beta^*(M_1, ..., M_T)$, s.t.,

$$0 < \frac{\partial c_t^*(\beta, W^*)}{\partial W_t^*} \leq 1 \quad ; \quad 0 < \frac{\partial c_0^*(\beta, \epsilon_0)}{\partial \epsilon_0} \leq 1$$

for all $0 \leq \beta \leq \beta^*$.

Why only CRRA utilities?

- *Assumption*: $u_t(x) = u(x), \forall\ 0 \leq t \leq T$.
- Desirable: concavity of c_t^* w.r.t W_t^* for all markets and habits.
- Simplest utility maximization problem:

$$\sup_{\pi_o} u\big(\epsilon_0 - \pi_0\big) + u\big(\pi_0 \cdot \big(1 + r\big)\big)$$

- Solution: $c_0(\epsilon_0) := \epsilon_0 - \pi_0(\epsilon_0)$, where

$$\pi_0(\epsilon_0) + (u')^{-1}\big((1+r) \cdot u\big(1 + (1+r) \cdot \pi_0(\epsilon_0)\big)\big) = \epsilon_0$$

- **Theorem**: $c_0(\epsilon_0)$ is a concave function $\forall r > 0$, iff

$$u(x) = C \cdot x^{1-\gamma}$$

Incomplete Markets of Type $\mathcal{C}$

- **Definition**: Incomplete Market is of type **class** $\mathcal{C}$, if

$$P_L^t = E\left[\,\cdot\,|H_t\right]$$

.

- **Theorem**: For any incomplete market of type $\mathcal{C}$ with constant interest rate, we have

$$\frac{\partial^2 c_t^*}{\partial^2 W_t^*} \leq 0 \quad ; \quad \frac{\partial^2 c_0^*}{\partial^2 \epsilon_0} \leq 0$$

- **Theorem**: For Idiosyncratic incomplete markets, we have

$$\frac{\partial^2 c_t^*}{\partial^2 W_t^*} \leq 0 \quad ; \quad \frac{\partial^2 c_0^*}{\partial^2 \epsilon_0} \leq 0$$

Convergence

- We assume $u_t(x) = \frac{x^{1-\gamma}}{1-\gamma}$ $\forall\, 0 \leq t \leq T$.
- **Lemma:** For CRRA utility with habits and *no random endowment* ($\epsilon_t = 0$ for all $t > 0$), we have

$$c_t^{NE} = X_t \cdot W_t^{NE}$$

$$W_t^{NE} = Y_t \cdot c_{t-1}^{NE}$$

  where X_t is some adapted positive process and $Y_t \in L_t$.
- **Theorem** For any incomplete market with constant interest rate, the optimal consumption stream c_t^* satisfies

$$\lim_{W_t^* \to \infty} \frac{c_t^*}{W_t^*} = X_t \quad ; \quad \lim_{W_t^* \to \infty} \frac{c_0^*}{\epsilon_0} = X_0$$

$$\lim_{W_t^* \to \infty} \frac{W_t^*}{c_{t-1}^*} = Y_t$$

Estimates

- **Theorem:** For Idiosyncratic incomplete markets and markets of type $\mathcal{C}$ with constant interest rates, we have

$$X_t \cdot W_t^* - A_t \leq c_t^* \leq X_t \cdot W_t^* + A_t$$

$$X_{0\cdot} \cdot \epsilon_0 - A_0 \leq c_0^* \leq X_0 \cdot \epsilon_0 + A_0$$

where A_t is some adapted process.

- **Corollary**: The rate of convergence is linear.