Hedging under arbitrage

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Motivation

- Given: a frictionless market of stocks with continuous Markovian dynamics.
- **If there does not exist an equivalent local martingale measure can we have the concept of hedging?**
- Answer: Yes, if a square-integrable "market price of risk" exists.
- **If there exists an equivalent local martingale measure and a stock price process is a "strict local martingale" what is the cheapest way to hold this stock at time T?**
- Answer: Delta-hedging.
- **How can we compute hedging prices?**
- Answer: PDE techniques, (non-)equivalent changes of measures
- Techniques: Itô's formula, PDE techniques to prove smoothness of hedging prices, Föllmer measure

Two generic examples

- Reciprocal of the three-dimensional Bessel process (NFLVR):

$$d\tilde{S}(t) = -\tilde{S}^2(t)dW(t)$$

- Three-dimensional Bessel process:

$$dS(t) = \frac{1}{S(t)}dt + dW(t)$$

Strict local martingales

- A stochastic process $X(\cdot)$ is a *local martingale* if there exists a sequence of stopping times (τ_n) with $\lim_{n\to\infty} \tau_n = \infty$ such that $X^{\tau_n}(\cdot)$ is a martingale.
- Here, in our context, a local martingale is a nonnegative stochastic process $X(\cdot)$ which does not have a drift:

$$dX(t) = X(t)\textbf{something}dW(t).$$

- Strict local martingales (local martingales, which are not martingales) do only appear in continuous time.
- Nonnegative local martingales are supermartingales.

We assume a Markovian market model.

- Our time is finite: $T < \infty$. Interest rates are zero.
- The stocks $S(\cdot) = (S_1(\cdot), \dots, S_d(\cdot))^{\mathsf{T}}$ follow

$$dS_i(t) = S_i(t)\left(\mu_i(t, S(t))dt + \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW_k(t)\right)$$

  with some measurability and integrability conditions.
- $\rightarrow$ Markovian
- but not necessarily complete ($K > d$ allowed).
- The covariance process is defined as

$$a_{i,j}(t, S(t)) := \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))\sigma_{j,k}(t, S(t)).$$

- The underlying filtration is denoted by $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t \le T}$.

An important guy: the market price of risk.

- A *market price of risk* is an $\mathbb{R}^K$-valued process $\theta(\cdot)$ satisfying

$$\mu(t, S(t)) = \sigma(t, S(t))\theta(t).$$

- We assume it exists and

$$\int_0^T \|\theta(t)\|^2 dt < \infty.$$

- The market price of risk is not necessarily unique.
- We will always use a Markovian version of the form $\theta(t, S(t))$. (needs argument!)

Related is the stochastic discount factor.

- The *stochastic discount factor* corresponding to θ is denoted by

$$Z^\theta(t) := \exp\left(-\int_0^t \theta^\mathsf{T}(u, S(u))dW(u) - \frac{1}{2}\int_0^t \|\theta(u, S(u))\|^2 du\right).$$

- It has dynamics

$$dZ^\theta(t) = -\theta^\mathsf{T}(t, S(t))Z^\theta(t)dW(t).$$

- If $Z^\theta(\cdot)$ is a martingale, that is, if $E[Z^\theta(T)] = 1$, then it defines a risk-neutral measure $\mathbb{Q}$ with $d\mathbb{Q} = Z^\theta(T)d\mathbb{P}$.
- Otherwise, $Z^\theta(\cdot)$ is a strict local martingale and classical arbitrage is possible.
- From Itô's rule, we have

$$d\left(Z^\theta(t)S_i(t)\right) = Z^\theta(t)S_i(t)\sum_{k=1}^K \left(\sigma_{i,k}(t, S(t)) - \theta_k(t, S(t))\right) dW_k(t$$

Everything an investor cares about: how and how much?

- We call *trading strategy* the number of shares held by an investor: $\eta(t) = (\eta_1(t), \ldots, \eta_d(t))^{\mathsf{T}}$
- We assume that $\eta(\cdot)$ is progressively measurable with respect to $\mathbb{F}$ and self-financing.
- The corresponding wealth process $V^{v,\eta}(\cdot)$ for an investor with initial wealth $V^{v,\eta}(0) = v$ has dynamics

$$dV^{v,\eta}(t) = \sum_{i=1}^{d} \eta_i(t) dS_i(t).$$

- We restrict ourselves to trading strategies which satisfy $V^{1,\eta}(t) \geq 0$

The terminal payoff

- Let $p : \mathbb{R}_+^d \to [0, \infty)$ denote a measurable function.
- The investor wants to have the payoff $p(S(T))$ at time T.
- For example,
  - market portfolio: $\tilde{p}(s) = \sum_{i=1}^d s_i$
  - money market: $p^0(s) = 1$
  - stock: $p^1(s) = s_1$
  - call: $p^C(s) = (s_1 - L)^+$ for some $L \in \mathbb{R}$.
- We define a candidate for the hedging price as

$$h^p(t, s) := \mathbb{E}^{t,s}\left[\tilde{Z}^\theta(T) p(S(T))\right],$$

where $\tilde{Z}^\theta(T) = Z^\theta(T)/Z^\theta(t)$ and $S(t) = s$ under the expectation operator $\mathbb{E}^{t,s}$.

Prerequisites

- We shall call $(t, s) \in [0, T] \times \mathbb{R}_+^d$ a *point of support for* $S(\cdot)$ if there exists some $\omega \in \Omega$ such that $S(t, \omega) = s$.
- We have assumed Markovian stock price dynamics such that $S(t)$ is $\mathbb{R}^d$-valued, unique and stays in the positive orthant and a square-integrable Markovian market price of risk $\theta(t, S(t))$.
- We have defined

$$h^p(t, s) := \mathbb{E}^{t,s}\left[\tilde{Z}^\theta(T)p(S(T))\right],$$

where $\tilde{Z}^\theta(T) = Z^\theta(T)/Z^\theta(t)$ and $S(t) = s$ under the expectation operator $\mathbb{E}^{t,s}$.
- In particular,

$$h^p(T, s) := p(s).$$

A first result: non path-dependent European claims

Assume that we have a contingent claim of the form $p(S(T)) \geq 0$ and that for all points of support (t,s) for $S(\cdot)$ with $t \in [0,T)$ we have $h^p \in C^{1,2}(\mathcal{U}_{t,s})$ for some neighborhood $\mathcal{U}_{t,s}$ of (t,s). Then, with $\eta_i^p(t,s) := D_i h^p(t,s)$ and $v^p := h^p(0, S(0))$, we get

$$V^{v^p,\eta^p}(t) = h^p(t, S(t)).$$

The strategy η^p is optimal in the sense that for any $\tilde{v} > 0$ and for any strategy $\tilde{\eta}$ whose associated wealth process is nonnegative and satisfies $V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T))$, we have $\tilde{v} \geq v^p$. Furthermore, h^p solves the PDE

$$\frac{\partial}{\partial t} h^p(t,s) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} s_i s_j a_{i,j}(t,s) D^2_{i,j} h^p(t,s) = 0$$

at all points of support (t,s) for $S(\cdot)$ with $t \in [0,T)$.

The proof relies on Itô's formula.

- Define the martingale $N^p(\cdot)$ as

$$N^p(t) := \mathbb{E}[Z^\theta(T)p(S(T))|\mathcal{F}(t)] = Z^\theta(t)h^p(t, S(t)).$$

- Use a localized version of Itô's formula to get the dynamics of $N^p(\cdot)$. Since it is a martingale, its dt term must disappear which yields the PDE.
- Then, another application of Itô's formula yields

$$dh^p(t, S(t)) = \sum_{i=1}^{d} D_i h^p(t, S(t))dS_i(t) = dV^{v^p,\eta^p}(t).$$

- This yields directly $V^{v^p,\eta^p}(\cdot) \equiv h^p(\cdot, S(\cdot))$.

Proof (continued)

- Next, we prove optimality.
- Assume we have some initial wealth $\tilde{v} > 0$ and some strategy $\tilde{\eta}$ with nonnegative associated wealth process such that $V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T))$ is satisfied.
- Then, $Z^{\theta}(\cdot)V^{\tilde{v},\tilde{\eta}}(\cdot)$ is a supermartingale.
- This implies

$$
\begin{aligned}
\tilde{v} &\geq \mathbb{E}[Z^{\theta}(T)V^{\tilde{v},\tilde{\eta}}(T)] \geq \mathbb{E}[Z^{\theta}(T)p(S(T))] \\
&= \mathbb{E}[Z^{\theta}(T)V^{v^{p},\eta^{p}}(T)] = v^{p}
\end{aligned}
$$

Non-uniqueness of PDE

- Usually,

$$\frac{\partial}{\partial t}v(t,s)+\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}s_i s_j a_{i,j}(t,s)D^2_{i,j}v(t,s)=0$$

  does not have a unique solution.
- However, if h^p is sufficiently differentiable, it can be characterized as the minimal nonnegative solution of the PDE.
- This follows as in the proof of optimality. If $\tilde{h}$ is another nonnegative solution of the PDE with $\tilde{h}(T,s)=p(s)$, then $Z^\theta(\cdot)\tilde{h}(\cdot,S(\cdot))$ is a supermartingale.

Corollary: Modified put-call parity

For any $L \in \mathbb{R}$ we have the modified put-call parity for the call- and put-options $(S_1(T) - L)^+$ and $(L - S_1(T))^+$, respectively, with strike price L:

$$\mathbb{E}^{t,s}\left[\tilde{Z}^\theta(T)(L - S_1(T))^+\right] + h^{p^1}(t,s) = \mathbb{E}^{t,s}\left[\tilde{Z}^\theta(T)(S_1(T) - L)^+\right] + Lh^{p^0}(t,s),$$

where $p^0(\cdot) \equiv 1$ denotes the payoff of one monetary unit and $p^1(s) = s_1$ the price of the first stock for all $s \in \mathbb{R}_+^d$.

A technical definition

We shall call a function $f : [0, T] \times \mathbb{R}_+^d \to \mathbb{R}$ *locally Lipschitz and bounded* on $\mathbb{R}_+^d$ if for all $s \in \mathbb{R}_+^d$ the function $t \to f(t, s)$ is right-continuous with left limits and for all $M > 0$ there exists some $C(M) < \infty$ such that for all $t \in [0, T]$.

$$\sup_{\substack{\frac{1}{M} \le \|y\|, \|z\| \le M \\ y \neq z}} \frac{|f(t, y) - f(t, z)|}{\|y - z\|} + \sup_{\frac{1}{M} \le \|y\| \le M} |f(t, y)| \le C(M).$$

Sufficient conditions for the differentiability of h^p.

(A1) The functions θ_k and $\sigma_{i,k}$ are for all $i = 1, \dots, d$ and $k = 1, \dots, K$ locally Lipschitz and bounded.

(A2) For all points of support (t, s) for $S(\cdot)$ with $t \in [0, T)$ there exist some $C > 0$ and some neighborhood $\mathcal{U}$ of (t, s) such that

$$\sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}(u, y)\xi_i\xi_j \geq C\|\xi\|^2$$

for all $\xi \in \mathbb{R}^d$ and $(u, y) \in \mathcal{U}$.

(A3) The payoff function p is chosen so that for all points of support (t, s) for $S(\cdot)$ there exist some $C > 0$ and some neighborhood $\mathcal{U}$ of (t, s) such that $h^p(u, y) \leq C$ for all $(u, y) \in \mathcal{U}$.

We will proceed in three steps to show that these conditions imply smoothness of h^p.

Step 1: Stochastic flows

We define $X^{t,s,z}(\cdot) := (S^{t,s\mathsf{T}}(\cdot), z\tilde{Z}^{\phi,t,s}(\cdot))^{\mathsf{T}}$.

Take $(t,s) \in [0,T] \times \mathbb{R}_+^d$ a point of support for $S(\cdot)$. Then under Assumption (A1) [locally Lipschitz and bounded] we have for all sequences $(t_k, s_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}(t_k, s_k) = (t,s)$ that

$$\lim_{k\to\infty} \sup_{u\in[t,T]} \|X^{t_k,s_k,1}(u) - X^{t,s,1}(u)\| = 0$$

almost surely.

In particular, for $K(\omega)$ sufficiently large we have that $X^{t_k,s_k,1}(u,\omega)$ is strictly positive and $\mathbb{R}_+^{d+1}$-valued for all $k > K(\omega)$ and $u \in [t,T]$.

Step 2: Schauder estimates

Fix a point $(t, s) \in [0, T) \times \mathbb{R}_+^d$ and a neighborhood $\mathcal{U}$ of (t, s). Suppose Assumptions (A1) and (A2) [locally Lipschitz and bounded, non-degenerate a] hold.

Let $(f_k)_{k \in \mathbb{N}}$ denote a sequence of solutions of the Black-Scholes PDE on $\mathcal{U}$, uniformly bounded under the supremum norm on $\mathcal{U}$. If $\lim_{k \to \infty} f_k(t, s) = f(t, s)$ on $\mathcal{U}$ for some function $f : \mathcal{U} \to \mathbb{R}$, then f solves also the PDE on some neighborhood $\tilde{\mathcal{U}}$ of (t, s). In particular, $f \in C^{1,2}(\tilde{\mathcal{U}})$.

- Janson and Tysk (2006), Tysk and Ekström (2009)
- Interior Schauder estimates by Knerr (1980) together with Arzelà-Ascoli type of arguments

Step 3: Putting everything together

Under Assumptions (A1)-(A3) [locally Lipschitz and bounded, non-degenerate a, locally boundedness of h^p] there exists for all points of support (t, s) for $S(\cdot)$ with $t \in [0, T)$ some neighborhood $\mathcal{U}$ of (t, s) such that the function h^p is in $C^{1,2}(\mathcal{U})$.

- Define $\tilde{p}(s_1, \ldots, s_d, z) := zp(s_1, \ldots, s_d)$.
- Define $\tilde{p}^M(\cdot) := \tilde{p}(\cdot)\mathbf{1}_{\{\tilde{p}(\cdot) \leq M\}}$ for some $M > 0$
- Approximate by sequence of continuous functions $\tilde{p}^{M,m}$ such that $\tilde{p}^{M,m} \leq 2M$ for all $m \in \mathbb{N}$.

Proof (continuation)

- The corresponding expectations are defined as

$$\tilde{h}^{p,M}(u,y) := \mathbb{E}^{u,y}[\tilde{p}^M(S_1(T),\dots,S_d(T),\tilde{Z}^\theta(T))]$$

  for all $(u,y) \in \tilde{\mathcal{U}}$ for some neighborhood $\tilde{\mathcal{U}}$ of (t,s) and equivalently $\tilde{h}^{p,M,m}$.
- We have continuity of $\tilde{h}^{p,M,m}$ for large m due to the bounded convergence theorem.
- A result from Jansen and Tysk (2006) yields that under Assumption (A2) [non-degenerate a] $\tilde{h}^{p,M,m}$ is a solution of the PDE.
- Then, by Step 2 firstly, $\tilde{h}^{p,M}$ and secondly, h^p also solve the PDE.

We can change the measure to compute h^p

- There exists not always an equivalent local martingale measure.
- However, after making some technical assumptions on the probability space and the filtration we can construct a new measure $\mathbb{Q}$ which corresponds to a "removal of the stock price drift".
- Based on the work of Föllmer and Meyer and along the lines of Delbaen and Schachermayer.

Theorem: Under a new measure $\mathbb{Q}$ the drifts disappear.

There exists a measure $\mathbb{Q}$ such that $\mathbb{P} \ll \mathbb{Q}$. More precisely, for all nonnegative $\mathcal{F}(T)$-measurable random variables Y we have

$$\mathbb{E}^{\mathbb{P}}[Z^\theta(T)Y] = \mathbb{E}^{\mathbb{Q}}\left[Y\mathbf{1}_{\left\{\frac{1}{Z^\theta(T)}>0\right\}}\right].$$

Under this measure $\mathbb{Q}$, the stock price processes follow

$$dS_i(t) = S_i(t)\sum_{k=1}^{K}\sigma_{i,k}(t, S(t))\, d\widetilde{W}_k(t)$$

up to time $\tau^\theta := \inf\{t \in [0, T] : 1/Z^\theta(t) = 0\}$. Here,

$$\widetilde{W}_k(t \wedge \tau^\theta) := W_k(t \wedge \tau^\theta) + \int_0^{t\wedge\tau^\theta} \theta_k(u, S(u))du$$

is a K-dimensional $\mathbb{Q}$-Brownian motion stopped at time τ^θ.

What happens in between time 0 and time T: Bayes' rule.

For all nonnegative $\mathcal{F}(T)$-measurable random variables Y the representation

$$\mathbb{E}^{\mathbb{Q}}\left[Y\mathbf{1}_{\{1/Z^\theta(T)>0\}} \middle| \mathcal{F}(t)\right] = \mathbb{E}^{\mathbb{P}}[Z^\theta(T)Y|\mathcal{F}(t)]\frac{1}{Z^\theta(t)}\mathbf{1}_{\{1/Z^\theta(t)>0\}}$$

holds $\mathbb{Q}$-almost surely (and thus $\mathbb{P}$-almost surely) for all $t \in [0, T]$.

The class of Bessel processes with drift provides interesting arbitrage opportunities.

- We begin with defining an auxiliary stochastic process $X(\cdot)$ as

$$dX(t) = \left(\frac{1}{X(t)} - c\right) dt + dW(t)$$

  with $W(\cdot)$ denoting a Brownian motion and $c \geq 0$ a constant.
- $X(t)$ is for all $t \geq 0$ strictly positive since $X(\cdot)$ is a Bessel process under an equivalent measure.
- The stock price process is now defined via

$$dS(t) = \frac{1}{X(t)}dt + dW(t) = S(t)\left(\frac{1}{S^2(t) - S(t)ct}dt + \frac{1}{S(t)}dW(t)\right)$$

  with $S(0) = X(0) > 0$.

After a change of measure, the Bessel process becomes Brownian motion.

- As a reminder:

$$dS(t) = \frac{1}{S(t) - ct} dt + dW(t).$$

- We have $S(t) \geq X(t) > 0$ for all $t \geq 0$.
- The market price of risk is $\theta(t, s) = 1/(s - ct)$.
- Thus, the inverse stochastic discount factor $1/Z^\theta$ becomes zero exactly when $S(t)$ hits ct.
- Removing the drift with a change of measure as before makes $S(\cdot)$ a Brownian motion (up to the first hitting time of zero by $1/Z^\theta(\cdot)$) under $\mathbb{Q}$.

The optimal strategy for getting one dollar at time T can be explicitly computed.

- For $p(s) \equiv p^0(s) \equiv 1$ we get

$$h^{p^0}(t,s) = \mathbb{E}^{\mathbb{P}}\left[\frac{Z^\theta(T)}{Z^\theta(t)}\cdot 1 \middle| \mathcal{F}_t\right]\Big|_{S(t)=s} = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{1/Z^\theta(T)>0\}}|\mathcal{F}_t]|_{S(t)=s}$$

$$= \Phi\left(\frac{s-cT}{\sqrt{T-t}}\right) - \exp(2cs-2c^2t)\Phi\left(\frac{-s-cT+2ct}{\sqrt{T-t}}\right).$$

- This yields the optimal strategy

$$\eta^0(t,s) = \frac{2}{\sqrt{T-t}}\phi\left(\frac{s-cT}{\sqrt{T-t}}\right) - 2c\exp(2cs-2c^2t)\Phi\left(\frac{-s-cT}{\sqrt{T-}}\right.$$

- The hedging price h^p satisfies on all points $\{s>ct\}$ the PDE

$$\frac{\partial}{\partial t}h^p(t,s) + \frac{1}{2}D^2h^p(t,s) = 0.$$

Conclusion

- No equivalent local martingale measure needed to find an optimal hedging strategy based upon the familiar delta hedge.
- Sufficient conditions are derived for the necessary differentiability of expectations indexed over the initial market configuration.
- The dynamics of stochastic processes under a non-equivalent measure and a generalized Bayes' rule might be of interest themselves.
- We have computed some optimal trading strategies in standard examples for which so far only ad-hoc and not necessarily optimal strategies have been known.

Thank you!

Strict local martingales II

Assume $X(\cdot)$ is a nonnegative local martingale:

$$dX(t) = X(t)\textbf{something}\,dW(t).$$

- We always have $\mathbb{E}[X(T)] \leq X(0)$.
- If $\mathbb{E}[X(T)] = X(0)$ then $X(\cdot)$ is a (true) martingale.
- If "**something**" behaves nice (for example is bounded) then $X(\cdot)$ is a martingale.
- If $\mathbb{E}[X(T)] < X(0)$ then $X(\cdot)$ is a *strict local martingale*.

Role of Markovian market price of risk

Let $M \geq 0$ be a random variable measurable with respect to $\mathcal{F}^S(T)$. Let $\nu(\cdot)$ denote any MPR and $\theta(\cdot,\cdot)$ a Markovian MPR. Then, with

$$M^\nu(t) := \mathbb{E}\left[\frac{Z^\nu(T)}{Z^\nu(t)}M \middle| \mathcal{F}_t\right] \text{ and } M^\theta(t) := \mathbb{E}\left[\frac{Z^\theta(T)}{Z^\theta(t)}M \middle| \mathcal{F}_t\right]$$

for $t \in [0, T]$, we have $M^\nu(\cdot) \leq M^\theta(\cdot)$ almost surely.

Proof

- We define $c(\cdot) := \nu(\cdot) - \theta(\cdot, S(\cdot))$ and $c^n(\cdot) := c(\cdot)\mathbf{1}_{\{\|c(\cdot)\|\leq n\}}$
- Then,

$$\frac{Z^\nu(T)}{Z^\nu(t)} = \lim_{n\to\infty} \frac{Z^{c^n}(T)}{Z^{c^n}(t)} \cdot \exp\left(-\int_t^T \theta^\mathsf{T}(dW(u) + c^n(u)du) - \frac{1}{2}\int_t^T \|\theta\|^2 du\right).$$

- Since $c^n(\cdot)$ is bounded, $Z^{c^n}(\cdot)$ is a martingale.
- Fatou's lemma, Girsanov's theorem and Bayes' rule yield

$$M^\nu(t) \leq \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}^n}\left[\exp\left(-\int_t^T \theta^\mathsf{T} dW^n(u) - \frac{1}{2}\int_t^T \|\|^2 du\right) M \middle| \mathcal{F}_t\right.$$

- Since $\sigma(\cdot, S(\cdot))c^n(\cdot) \equiv 0$ the process $S(\cdot)$ has the same dynamics under $\mathbb{Q}^n$ as under $\mathbb{P}$.

Open problem

The last result might be related to the "Markovian selection results", as in Krylov (1973) and Ethier and Kurtz (1986).
There, the existence of a Markovian solution for a martingale problem is studied.
It is observed that a supremum over a set of expectations indexed by a family of distributions is attained and the maximizing distribution is a Markovian solution of the martingale problem.

Open problem

h^p can be characterized as the minimal nonnegative solution of the Cauchy problem

$$\frac{\partial}{\partial t}v(t,s) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} s_i s_j a_{i,j}(t,s) D^2_{i,j} v(t,s) = 0$$

$$v(T,s) = p(s)$$

Can an iterative method be constructed, which converges to the minimal solution of this PDE?

"Classical" Mathematical Finance I

- Reminder: $dZ^{\theta}(t) = -\theta^{\mathsf{T}}(t, S(t))Z^{\theta}(t)dW(t)$, where θ denotes the market price of risk.
- Assume: $Z^{\theta}(\cdot)$ is a true martingale.
- Then, there exists a *risk-neutral measure* $\mathbb{Q}$, under which $S(\cdot)$ has dynamics

$$dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t)) dW_k^{\mathbb{Q}}(t).$$

- Then,

$$h^p(t,s) = \mathbb{E}^{t,s}\left[\tilde{Z}^{\theta}(T)p(S(T))\right] = \mathbb{E}^{\mathbb{Q}^{t,s}}\left[p(S(T))\right].$$

- Below: Generalization to the situation where $Z^{\theta}(\cdot)$ is a strict local martingale and risk-neutral measure $\mathbb{Q}$ does not exist.

"Classical" Mathematical Finance II

- If we assume that the number of stocks d and the number of driving Brownian motions K is equal, that is, $d = K$, and σ has full rank, then the market is called *complete*.
- Then, by the Martingale Representation Theorem, there exists some strategy η such that

$$V^{v,\eta}(T) = p(S(T))$$

  for initial capital $v = h^p(0, S(0))$.
- That is, the contingent claim / payoff can be *hedged*.
- Often, one can use Itô's rule to compute

$$\eta_i(t) = D_i h^p(t, S(t)),$$

  which is called *delta hedge*.

"Classical" Mathematical Finance III

- Often, the hedging price h^p needs to be computed numerically.
- Theory behind it: *Feynman-Kac Theorem*
- It states that under some continuity and growth conditions on a and p, any solution $v : [0, T] \times \mathbb{R}_+^d \to \mathbb{R}$ of the Cauchy-Problem (*Black-Scholes PDE*)

$$\frac{\partial}{\partial t} v(t,s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t,s) D^2_{i,j} v(t,s) = 0$$

$$v(T,s) = p(s)$$

with polynomial growth can be represented as

$$v(t,s) = \mathbb{E}^{\mathbb{Q}^{t,s}}[p(S(T))] = h^p(t,s),$$

where $a(\cdot,\cdot) = \sigma(\cdot,\cdot)\sigma^{\mathsf{T}}(\cdot,\cdot)$ and $S(\cdot)$ has $\mathbb{Q}$-dynamics

$$dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t)) dW_k^{\mathbb{Q}}(t).$$

Feynman-Kac does not always work.

- We have seen, as long as
    - some growth and continuity conditions on σ and p are satisfied,
    - the risk-neutral measure $\mathbb{Q}$ exists,
    - h^p is of polynomial growth,
    - the Black-Scholes equation has a solution

  we know that the hedging price h^p is a solution.
- Growth conditions are often not satisfied, for example

$$d\tilde{S}(t) = -\tilde{S}^2(t)dW(t)$$

  with corresponding PDE

$$\frac{\partial}{\partial t}v(t,s) + \frac{1}{2}s^4 D^2 v(t,s) = 0.$$

- Then, $v_1(t,s) = s$ and $v_2(t,s) = 2s\Phi\left(\frac{1}{s\sqrt{T-t}}\right) - s$ are solutions of polynomial growth, satisfying $v(T,s) = s$ and $v(t,0) = 0$.

"Classical" Mathematical Finance IV

- Remember: We have assumed that there exists some θ which maps the volatility into the drift, that is $\sigma(\cdot,\cdot)\theta(\cdot,\cdot) = \mu(\cdot,\cdot)$.
- It can be shown that this assumption excludes "unbounded profit with bounded risk".
- Thus "making (a considerable) something out of almost nothing" is not possible.
- However, it is still possible to "certainly make something more out of something".
- The reason that the arbitrage is not scalable is due to the credit constraint (*admissibility*) $V^{1,\eta}(\cdot) \geq 0$.

Digression: Problems of the no-arbitrage assumption.

- A typical market participant can statistically detect whether a market price of risk θ exists or does not exist.
- However, there exists no statistical test to decide whether $Z^\theta(\cdot)$ is a true martingale or not (whether arbitrage exists or does not exist).
- Instead of starting from the normative assumption of no arbitrage, *Stochastic Portfolio Theory* takes a descriptive approach.
- One goal is to find models which provide realistic dynamics of the market weights $S_i(\cdot)/(S_i(\cdot) + \ldots S_d(\cdot))$.
- These models tend to violate the no-arbitrage assumption.

Stationarity of the market weights.

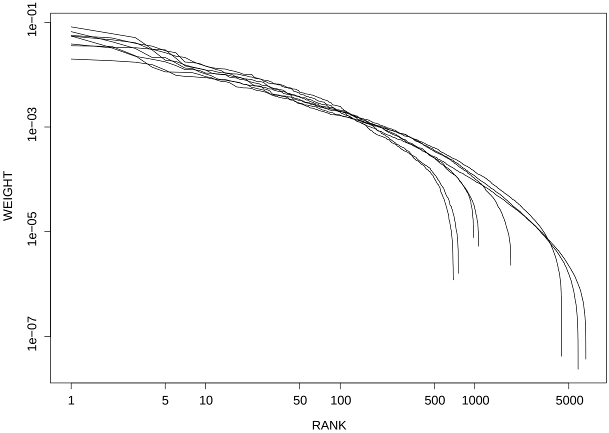

Figure: Market weights against ranks on logarithmic scale, 1929 - 1999, from Fernholz, *Stochastic Portfolio Theory*, page 95.